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Remark on regularity of weak solutions to the Navier-Stokes equations

Zdeněk Skalák, Petr Kučera

Abstract. Some results on regularity of weak solutions to the Navier-Stokes equations published recently in [3] follow easily from a classical theorem on compact operators. Further, weak solutions of the Navier-Stokes equations in the space $L^2(0, T, W^{1,3}(\Omega)^3)$ are regular.

Keywords: Navier-Stokes equations, weak solution, regularity

Classification: 35Q10, 76D05, 76F99

Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^2$-boundary $\partial \Omega$, let $T > 0$ and $Q_T = \Omega \times (0, T)$. We consider the Navier-Stokes initial-boundary value problem describing the evolution of the velocity $u(x, t)$ and the pressure $p(x, t)$ in $Q_T$:

\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p &= f, \\
\nabla \cdot u &= 0, \\
uq u &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
uq u |_{t=0} &= u_0,
\end{align*}

where $\nu > 0$ is the viscosity coefficient and $f$ is the external body force. The initial data $u_0$ should satisfy the compatibility conditions $u_0|_{\partial \Omega} = 0$ and $\nabla \cdot u_0 = 0$.

The definition and the proof of the existence of weak solutions of the equations (1)–(4) can be found for example in [3] or [6]. In general, it is unknown whether weak solutions are regular or not. Serrin ([5]) proved that if a weak solution $u$ of (1)–(4) belongs to $L^\alpha(0, T, L^q(\Omega))$ for $2/\alpha + 3/q = 1$ and $q \in (3, \infty]$ then $u$ is regular. Kozono ([3]) generalized this result to a certain class of functions characterized by means of local singularities in the weak-$L^3$ space. He further showed that there exists an absolute constant $\varepsilon > 0$ such that if $u$ is a weak solution of (1)–(4) in $L^\infty(0, T, L^3(\Omega)^3)$ and $\limsup_{t \to t^*} \|u(t)\|_{L^3(\Omega)} < \|u(t^*)\|_{L^3(\Omega)} + \varepsilon$, then $u$ is necessarily regular in $\Omega \times (t_*, t_* + \sigma)$ for some $\sigma > 0$. Let us mention here that the Kozono’s results were applied in [4] where partial regularity of weak solutions to the Navier-Stokes equations in the class $L^\infty(0, T, L^3(\Omega))$ was shown.

The main goal of this paper is to show that the results stated above can be easily derived from the following well known theorem on compact operators ([2]):
**Theorem A.** Let $X$, $Y$ be Banach spaces. Let $S$ be a one to one continuous linear operator from $X$ onto $Y$ and $K$ a linear compact operator from $X$ to $Y$. If $\text{Ker}(S + K) = 0$ then $(S + K)(X) = Y$.

Let $p > 1$. $L^p(\Omega)$ is the Lebesgue space with the norm $\| \cdot \|_p$. $C_0^\infty(\Omega)$ denotes the set of all infinitely differentiable vector-functions defined in $\Omega$, with a compact support in $\Omega$. $C_0^{\infty}(\Omega)$ is a subset of $C_0^\infty(\Omega)$ which contains only the divergence-free vector functions. $H$ is the closure of $C_0^{\infty}(\Omega)$ in $L^2(\Omega)^3$ with the scalar product $(\cdot, \cdot)$ and the norm $\| \cdot \|_2$. $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ ($m \in \mathbb{N}$) are the usual Sobolev spaces. $V$ denotes the completion of $C_0^{\infty}(\Omega)$ in the norm of $W_0^{1,2}(\Omega)^3$ with the scalar product $((u,v)) = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, dx$ and the norm $\| \cdot \|$. $P_H$ is the projection operator from $L^2(\Omega)^3$ onto $H$.

$L^p_w(\Omega)$ denotes the weak Lebesgue space over $\Omega$ with the quasi-norm $\| \cdot \|_{p,w}$ defined by $\|\phi\|_{p,w} = \sup_{R > 0} R \mu(\{ \phi \in \Omega; |\phi(x,t)| > R \})^{1/p}$, where $\mu$ is the Lebesgue measure. There is another equivalent norm to the above $\| \cdot \|_{p,w}$ (see [3]), so we may understand $L^p_w(\Omega)$ as a Banach space. Let us note that $L^p(\Omega) \subseteq L^p_w(\Omega)$ and $\|\phi\|_{p,w} \leq \|\phi\|_p$ for every $\phi \in L^p(\Omega)$.

Let $D(A) = \{ u \in V; \exists f \in H; ((u,v)) = (f,v) \ \forall \ v \in V \}$. $A$ is the Stokes operator from $D(A)$ onto $H$ defined for every $u \in D(A)$ by the equation $((u,v)) = (Au,v)$ $\forall \ v \in V$. $D(A)$ is endowed with the norm $\|u\|_{D(A)} = \|Au\|_2$ and $D(A) \hookrightarrow V$. Since $\Omega \in C^2$, $D(A) = W^{2,2}(\Omega)^3 \cap V$ and the norm $\|u\|_{D(A)}$ on $D(A)$ is equivalent to the norm induced by $W^{2,2}(\Omega)^3$ (see [6, Lemma 3.7]). We often use this fact throughout the paper. Let us define the Banach spaces $X = \{ u \in L^2(0,T,D(A)), u_t \in L^2(0,T,H) \}$ and $Y = L^2(0,T,H) \times V$ with $\|u\|_X = \|u\|_{L^2(0,T,D(A))} + \|u_t\|_{L^2(0,T,H)}$ and $\|(f,v_0)\|_Y = \|f\|_{L^2(0,T,H)} + \|v_0\|_V$.

Throughout the paper, we suppose that in (1)–(4) $f \in L^2(0,T,H)$ and $u_0 \in H$. For simplicity, we use the following notation: If $F$ is a space of real functions then $u \in F$ means that every component of $u$ is from $F$, e.g. $u \in W^{1,2}(\Omega)^3$ means in fact that $u \in W^{1,2}(\Omega)^3$. Similarly, $\|u\|_F$ means $\|u\|_{F^3}$.

**Proof of regularity results**

At first, we prove two basic propositions. The results mentioned in Introduction will then be their straightforward consequences.

**Proposition 1.** Let $u \in L^\alpha(0,T,L^q(\Omega))$ for $2/\alpha + 3/q \leq 1$ and $q \in (3, \infty)$. Then the operator $w \mapsto P_H(u \cdot \nabla w)$ is compact from $X$ to $L^2(0,T,H)$.

**Proof:** Firstly, suppose that $2/\alpha + 3/q < 1$ and $\alpha, q < \infty$. Using the Hölder inequality we have for almost every $t \in (0,T)$ and every $v \in H$: \[
\left| \int_\Omega u \cdot \nabla w \cdot v \, dx \right| \leq \|v\|_2 \|u\|_q \|\nabla w\|_{2q/(q-2)}.
\]
It follows further that
\[
\int_0^T \|u\|_{2q/(q-2)}^2 \|\nabla w\|_{2q/(q-2)}^2 \, dt \leq \|u\|_{L^\alpha(0,T,L^q(\Omega))}^2 \left( \int_0^T \|\nabla w\|_{2/(q-2)}^2 \, dt \right)^{(\alpha-2)/\alpha} \leq \]
\[
\|u\|_{L^\alpha(0,T,L^q(\Omega))}^2 \left( \int_0^T \|\nabla w\|_{2/(q-2)}^2 \, dt \right)^{(\alpha-2)/\alpha} \leq \]
\[
\|u\|_{L^\alpha(0,T,L^q(\Omega))}^2 \|w\|_{L^\infty(0,T,W^{1,2}(\Omega))}^{4/\alpha} \left( \int_0^T \|\nabla w\|_{2/(q-2)}^2 \, dt \right)^{(\alpha-2)/\alpha} \leq \]
\[
\|u\|_{L^\alpha(0,T,L^q(\Omega))}^2 \|w\|_{L^\infty(0,T,W^{1,2}(\Omega))}^{4/\alpha} \|w\|_{L^2(0,T,W^{1,2}(\Omega))}^{2(\alpha-2)/\alpha}.
\]
and, therefore,
\[
\|P_H(u \cdot \nabla w)\|_{L^2(0,T,H)} \leq \|u\|_{L^\alpha(0,T,L^q(\Omega))} \|w\|_{X}^{2/\alpha} \|w\|_{L^2(0,T,W^{1,2}(\Omega))}^{(\alpha-2)/\alpha} \leq \]

where we used the fact that $X$ is embedded continuously into $L^\infty(0,T,W^{1,2}(\Omega))$. Since $(2q-4)/(\alpha-2-q) < 6$ it follows e.g. from [5, Theorem 2.1, Chapter III] that the injection of $X$ into $L^2(0,T,W^{1,2}(\Omega))$ is compact. The proof now immediately follows from (5) and the definition of compact operators.

Secondly, let $u \in L^\alpha(0,T,L^\infty(\Omega))$, $\alpha > 2$. Then $|\int_\Omega u \cdot \nabla w \cdot v \, dx| \leq \|w\|_{H^1} \|u\|_\infty \|w\|_{W^{1,2}}$ for almost every $t \in (0,T)$ and every $v \in H$ and
\[
\int_0^T \|u\|_{L^\alpha(0,T,L^\infty(\Omega))} \|w\|_{W^{1,2}}^2 \, dt \leq \|u\|_{L^\alpha(0,T,L^\infty(\Omega))}^2 \left( \int_0^T \|w\|_{W^{1,2}}^2 \, dt \right)^{(\alpha-2)/\alpha} \leq \]
\[
\|u\|_{L^\alpha(0,T,L^\infty(\Omega))}^2 \|w\|_{L^\infty(0,T,W^{1,2}(\Omega))}^{4/\alpha} \|w\|_{W^{1,2}}^2 \, dt \right)^{(\alpha-2)/\alpha} \leq \]
\[
\|u\|_{L^\alpha(0,T,L^\infty(\Omega))}^2 \|w\|_{L^\infty(0,T,W^{1,2}(\Omega))}^{4/\alpha} \|w\|_{L^2(0,T,W^{1,2}(\Omega))}^{2(\alpha-2)/\alpha}.
\]

Therefore,
\[
\|P_H(u \cdot \nabla w)\|_{L^2(0,T,H)} \leq \|u\|_{L^\alpha(0,T,L^\infty(\Omega))} \|w\|_{X}^{2/\alpha} \|w\|_{L^2(0,T,W^{1,2}(\Omega))}^{(\alpha-2)/\alpha}.
\]

The injection of $X$ into $L^2(0,T,W^{1,2}(\Omega))$ is compact and the proof follows immediately from (6) and the definition of compact operators.

If $u \in L^\infty(0,T,L^q(\Omega))$ and $q > 3$ then the proof proceeds in the same way as in the previous paragraphs and we will skip it.
Finally, let \( u \in L^{\alpha}(0, T, L^q(\Omega)) \) for \( 2/\alpha + 3/q = 1, q \in (3, \infty] \). Let \( M_n = \{ t \in (0, T); \| u(t) \|_q > n \}, n \in \mathbb{N} \) and define \( u_n \) on \((0, T)\) as:

\[
\begin{align*}
  u_n(t) &= u(t) \quad \text{if} \quad t \notin M_n, \\
  u_n(t) &= 0 \quad \text{if} \quad t \in M_n.
\end{align*}
\]

Obviously, \( u_n \in L^\infty(0, T, L^q(\Omega)) \) and according to the previous paragraphs the operators \( w \mapsto P_H(u_n \cdot \nabla w) \) are compact from \( X \) to \( L^2(0, T, H) \). Further, the Lebesgue measure of \( M_n \) goes to zero for \( n \to \infty \) so that \( \| u - u_n \|_{L^\alpha(0, T, L^q(\Omega))} = (\int_{M_n} \| u \|^\alpha_q dt)^{1/\alpha} \to 0 \). Therefore, the operator \( w \mapsto P_H(u \cdot \nabla w) \) is compact from \( X \) to \( L^2(0, T, H) \) as a limit of compact operators \( w \mapsto P_H(u_n \cdot \nabla w) \) in the usual norm of the space of all linear bounded operators from \( X \) to \( L^2(0, T, H) \).

Let us consider the following Stokes equations with the perturbed convection term \( P_H(u \cdot \nabla w) \):

\[
\begin{align*}
  \omega_t + \nu A \omega + P_H(u \cdot \nabla w) &= f, \\
  \omega(0) &= \omega_0.
\end{align*}
\]

**Proposition 2.** Let \( 2/\alpha + 3/q = 1 \) with \( q \in (3, \infty] \). Then there exists \( \varepsilon > 0 \) with the following property: if \( u = u_0 + u_1 \) in \((0, T)\), \( u(t) \in V \) for almost every \( t \in (0, T) \), \( u_0 \in L^\infty(0, T, L^3(\Omega)) \), \( u_1 \in L^\alpha(0, T, L^q(\Omega)) \) and \( \text{sup}_{0 < t \leq T} \| u_0(t) \|_{3, w} < \varepsilon \), then for every \( \omega_0 \in V \) and \( f \in L^2(0, T, H) \) there exists a unique solution \( \omega \) of (7), (8) in \( X \).

**Proof:** The operator \( w \mapsto (\omega_t + \nu A \omega, \omega(0)) \) is a one to one continuous linear operator from \( X \) onto \( Y \). It is possible to prove (see also [3, Lemma 2.7]) that the operator \( w \mapsto P_H(u_0 \cdot \nabla w) \) is linear and bounded from \( X \) to \( L^2(0, T, H) \) with the norm less than \( C \| u_0 \|_{L^\infty(0, T, L^\alpha_\omega(\Omega))} \). Since the set of linear bounded one to one operators is open in the space of all linear bounded operators (using the usual topology) we get that the operator \( w \mapsto (\omega_t + \nu A \omega + P_H(u_0 \cdot \nabla w), \omega(0)) \) is a one to one operator from \( X \) onto \( Y \) for \( \varepsilon \) being sufficiently small. Finally, it follows from Proposition 1 that the operator \( w \mapsto P_H(u_1 \cdot \nabla w) \) is compact from \( X \) to \( L^2(0, T, H) \). Moreover, the operator \( w \mapsto (\omega_t + \nu A \omega + P_H(u \cdot \nabla w), \omega(0)) \) is one to one from \( X \) to \( Y \) and the proof follows immediately from Theorem A.

Now, we present proofs of the results stated in Introduction. The proofs are based on Propositions 1 and 2. Theorem 3 is a generalization of the famous Serrin’s result ([5]) on regularity of weak solutions in the subcritical case and was proved in [3]. Theorem 4 which is dealing with the partial regularity of weak solutions in the supercritical case \( L^\infty(0, T, L^3(\Omega)) \) was also proved in [3]. We present these theorems in a little more general way.
Theorem 4. There exists a positive constant $\varepsilon$ with the following property. If $u$ is a weak solution of (1)–(4) and there exists a non-negative $L^2$-function $M = M(t)$ on $(0, T)$ such that
\begin{equation}
\sup_{R \geq M(t)} R \mu \{ x \in \Omega; |u(x, t)| > R \}^{1/3} \leq \varepsilon
\end{equation}
for almost every $t \in (0, T)$, then $u$ is regular, that is $\frac{\partial u}{\partial t}, D_\alpha^0 u \in C(\Omega \times (0, T))$ for every multi-index $\alpha$ with $|\alpha| \leq 2$.

Proof: Due to the condition (9) $u$ can be easily decomposed as $u = u_0 + u_1$, where $u_0 \in L^\infty(0, T, L^3_w(\Omega))$, $u_1 \in L^2(0, T, L^\infty(\Omega))$ and $\sup_{0 < t < T} \|u_0(t)\|_{3,w} < \varepsilon$ (see [3]). Let $\sigma \in (0, T)$ be an arbitrary number. Since the weak solution $u \in L^2(0, T, V)$, there exists a $t_0 \in (0, \sigma)$ such that $u(t_0) \in V$. If $\varepsilon$ is sufficiently small it follows from Proposition 2 that there exists a unique solution $w \in X$ of (7), (8) on $(t_0, T)$ with $w(t_0) = u(t_0)$. It is easy to show that $u = w$ on $(t_0, T)$ and therefore $u \in X$ on $(t_0, T)$. Since $\sigma$ was chosen arbitrarily the theorem follows immediately using the results on interior regularity of weak solutions proved in [5]. \hfill \Box

Theorem 5. Let $u$ be a weak solution of (1)–(4) and $u \in L^2(0, T, W^{1,3}(\Omega))$. Then $\frac{\partial u}{\partial t}, D_\alpha^0 u \in C(\Omega \times (0, T))$ for every multi-index $\alpha$ with $|\alpha| \leq 2$.

Proof: Firstly, let us show that the operator $w \mapsto P_H(w \cdot \nabla u)$ is compact from $X$ to $L^2(0, T, H)$. Using the H"{o}lder inequality we have for almost every $t \in (0, T)$ and every $v \in H$:
\begin{equation}
\int_{\Omega} w \cdot \nabla u \cdot v \, dx \leq c\|v\|_2 \|w\|_{W^{1,2}(\Omega)} \|u\|_{W^{1,3}(\Omega)}.
\end{equation}
It follows easily as in the first paragraph of Proposition 1 that 
\[ \|P_H(w \cdot \nabla u)\|_{L^2(0,T,H)} \leq c\|w\|_X \|u\|_{L^2(0,T,W^{1,3}(\Omega))} \] 
so that \( w \mapsto P_H(w \cdot \nabla u) \) is a linear bounded operator from \( X \) to \( L^2(0,T,H) \). As in the last paragraph of Proposition 1 it is possible to construct \( u_n \in L^\infty(0,T,W^{1,3}(\Omega)) \) such that \( \|u - u_n\|_{L^2(0,T,W^{1,3}(\Omega))} \to 0 \) and the compactness of the operator \( w \mapsto P_H(w \cdot \nabla u) \) follows now from this and from the fact that the operators \( w \mapsto P_H(w \cdot \nabla u_n) \) are compact.

It follows from the standard estimates in Sobolev spaces, the Gronwall lemma and Theorem A that for every \( w_0 \in V \) and \( f \in L^2(0,T,H) \), the following problem has a unique solution \( w \in X \):

\[
\begin{align*}
w_t + \nu Aw + P_H(w \cdot \nabla u) &= f, \\
w(0) &= w_0.
\end{align*}
\]

The proof is concluded using the same arguments as in the proof of Theorem 3.

\[ \square \]

**Remark 6.** If e.g. \( f \in H \) (\( f \) independent of time) then in Theorem 3 and Theorem 5, resp. Theorem 4 \( u \) is analytic in time, in a neighborhood of the interval \((0,T)\), resp. \((a,b)\), as a \( D(A) \)-valued function (see [7]). It follows that \( u \in C^\infty(0,T,C(\bar{\Omega})) \), resp. \( u \in C^\infty(a,b,C(\bar{\Omega})) \). Therefore, \( u \) has no singular points in \( \Omega \times (0,T) \), resp. \( \Omega \times (a,b) \). Also, \( u(x,\cdot) \) is an infinitely differentiable function in \((0,T)\), resp. \((a,b)\), for every \( x \in \Omega \).

**Remark 7.** If \( \Omega \in C^{0,1} \) then the information from the Introduction — \( D(A) = W^{2,2}(\Omega)^3 \cap V \) and the norm \( \|u\|_{D(A)} \) on \( D(A) \) is equivalent to the norm induced by \( W^{2,2}(\Omega)^3 \) — cannot be used. We do not even know in this case whether \( D(A) \hookrightarrow W^{1,2+\varepsilon}(\Omega)^3 \) for a positive \( \varepsilon > 0 \) or not. What we only have here is that \( D(A) \hookrightarrow \hookrightarrow V \) and also \( X \hookrightarrow L^\infty(0,T,V) \). As a consequence, Propositions 1 and 2 can be proved only if \( u \in L^2(0,T,L^\infty(\Omega)) \) and the proofs of Theorems 3 and 4 fail totally. On the other hand, it is interesting that Theorem 5 can be stated and proved without any change.

**Remark 8.** If \( \Omega \) is the half-space or \( \mathbb{R}^3 \) (or possibly some other special unbounded domain) then we are able to obtain almost the same results as in the case of a bounded domain. Let us discuss it briefly. \( V \) denotes the completion of \( C_{0,\sigma}^\infty(\Omega) \) in the norm of \( W^{1,2}(\Omega)^3 \) with the scalar product \( ((u,v))_V = \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} + u_i v_i \right) \, dx \). \( D(A) \) is then defined as \( \{ u \in V; \exists f \in H; ((u,v))_V = (f,v) \ \forall v \in V \} \) and using the cut-off method it is possible to show that \( D(A) \hookrightarrow W^{2,2}(\Omega) \). It implies that \( X \hookrightarrow L^2(0,T,W^{2,2}(\Omega)) \) and, consequently, \( X \hookrightarrow \hookrightarrow L^2(0,T,W^{1,6-\varepsilon}(\bar{\Theta})) \) for every small \( \varepsilon > 0 \) and every smooth domain \( \Theta \subseteq \Omega \). As a result, Proposition 1 can be proved in a similar way as in the case of a bounded domain and Proposition 2 holds with only one change: the weak Lebesgue space \( L^3_w(\Omega) \) is replaced by
the Lebesgue space $L^3(\Omega)$. In Theorem 3 the condition (9) is replaced by the assumption $u = u_0 + u_1$ and $u_0 \in L^\infty(0,T,L^3(\Omega))$, $u_1 \in L^\alpha(0,T,L^q(\Omega))$, sup$_{0< t < T} \|u_0(t)\|_3 < \varepsilon$ and $2/\alpha + 3/q = 1$ with $q \in (3, \infty]$. In Theorem 4, the space $L^3(\Omega)$ is used instead of the space $L^3_w(\Omega)$. Theorem 5 can be stated without any change.

**Conclusion**

The results on regularity of weak solutions to the Navier-Stokes equations presented in this paper have been proved recently in [3]. It is interesting, however, that an easy proof of these results can be based on a well known classical theorem on compact operators. Further, weak solutions of the Navier-Stokes equations in the space $L^2(0,T,W^{1,3}(\Omega))$ are regular (Theorem 5), which is interesting in connection with the famous Prodi-Serrin’s conditions (see [3]).

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**References**


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