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# MAD families and the rationals

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*Abstract.* Rational numbers are used to classify maximal almost disjoint (MAD) families of subsets of the integers. Combinatorial characterization of indestructibility of MAD families by the likes of Cohen, Miller and Sacks forcings are presented. Using these it is shown that Sacks indestructible MAD family exists in ZFC and that $\mathfrak{b} = \mathfrak{c}$ implies that there is a Cohen indestructible MAD family. It follows that a Cohen indestructible MAD family is in fact indestructible by Sacks and Miller forcings. A connection with Roitman's problem of whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$ is also discussed.

*Keywords:* maximal almost disjoint family; Cohen, Miller, Sacks forcing; cardinal invariants of the continuum

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Recall that an infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint* (AD) family if every two distinct elements of $\mathcal{A}$ have finite intersection and it is *maximal* (MAD) if it is maximal with that property. A standard construction of a MAD family uses the structure of the real line. For every real number $r$ pick an increasing sequence $A_r$ of rational numbers converging to $r$. Then $\{A_r : r \in \mathbb{R}\}$ is an AD family of subsets of the rationals which can be, by a routine application of Zorn-Kuratowski Lemma, extended to a maximal one. P. Simon, in a private conversation, raised a question whether there is an "essentially different" construction of a MAD family in ZFC. This question is one of the motivations for the work presented here.

The other motivating factor is the general question of when is a particular MAD family destroyed by a given forcing $\mathbb{P}$. K. Kunen in [Ku] constructed a Cohen-indestructible MAD family assuming CH. Later J. Steprāns (see [St]) asked whether there is a Cohen-indestructible MAD in ZFC. We provide a combinatorial characterization of Cohen-indestructible MAD families and give partial answers to Steprāns' question and analyze the situation for other standard forcing notions.

The used set theoretic notation is mostly standard and follows [Ku]. Familiarity with the method of forcing is assumed. If $\mathcal{A}$ is a MAD family then $\mathcal{I}(\mathcal{A})$ denotes the ideal of all subsets of $\omega$ which can be almost covered by finitely many elements of $\mathcal{A}$, the dual filter is denoted by $\mathcal{I}^*(\mathcal{A})$. Given a forcing notion $\mathbb{P}$ a MAD family $\mathcal{A}$ is $\mathbb{P}$-*indestructible* if $\mathcal{A}$ remains MAD after forcing with $\mathbb{P}$. This is obviously

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equivalent to $\mathbb{P}$ not diagonalizing (not adding a pseudo-intersection to) $\mathcal{I}^*(\mathcal{A})$. If a MAD family is not $\mathbb{P}$-indestructible we say that it is $\mathbb{P}$-*destructible*. The definitions are extended to proper ideals $\mathcal{I}$ on $\omega$ containing all finite sets. All ideals considered in this paper are proper, contain all finite subsets of $\omega$ and are *tall*, i.e. $\mathcal{I}^*$ does not have a pseudo-intersection. Note that for an AD family $\mathcal{A}$, $\mathcal{I}(\mathcal{A})$ is tall if and only if $\mathcal{A}$ is MAD.

Recall that the *Sacks forcing* $\mathbb{S}$ consists of perfect subtrees of $2^{<\omega}$ ordered by inclusion. A $p \subseteq 2^{<\omega}$ is a *perfect tree* provided that $\forall s \in p\, \forall n \in \omega\ s \restriction n \in p$ and $\forall s \in p\, \exists n \in \omega\, \exists t \neq t' \in 2^n \cap p$ such that $s \subseteq t, t'$. For $p$ a perfect tree we let $[p] = \{f \in 2^\omega : \forall n \in \omega\ f \restriction n \in p\}$. If $p \in \mathbb{S}$ and $s \in p$ then $p_s = \{t \in p : t \subseteq s$ or $s \subseteq t\}$. Given a $p \in \mathbb{S}$ let $Br(p) = \{t \in p : t^\frown 0 \in p$ and $t^\frown 1 \in p\}$ and $Br_n(p) = \{t \in Br(p) : |\{s \in Br(p) : s \subseteq t\}| = n\}$.

$\mathbf{0}$ denotes the constant zero function with domain $\omega$. $\mathbb{Q}$ denotes the set of rational numbers, identified with $\{f \in 2^\omega : \forall^\infty n\ f(n) = 0\} \setminus \{\mathbf{0}\}$. Similarly the reals are identified with the Cantor set $2^\omega$. If $q \in \mathbb{Q}$ then let $s_q = q \restriction n$ where $n = \max\{k : q(k) = 1\} + 1$ and if $s \in 2^{<\omega}$ then $q_s = s^\frown \mathbf{0}$. Recall that a set $P$ subset of $2^\omega$ is perfect (non-empty without isolated points) if and only if $P = [p]$ for some $p \in \mathbb{S}$. For each $p \in \mathbb{S}$ let $\mathbb{Q}_p = \{q_{s^\frown 1} : s \in Br(p)\}$. We let $\mathbb{Q}_P = \mathbb{Q}_p$ if $P = [p]$. Note that $\mathbb{Q}_p$ is order isomorphic to $\mathbb{Q}$ and $p \leq q$ if and only if $\mathbb{Q}_p \subseteq \mathbb{Q}_q$. Note that $\overline{\mathbb{Q}_p} \setminus \mathbb{Q}_p \subseteq [p]$, where the closure is taken in $2^\omega$.

**Theorem 1.** *Let $\mathcal{I}$ be an ideal on $\omega$. Then the following are equivalent:*

(1) *$\mathcal{I}$ is $\mathbb{S}$-indestructible,*
(2) *$\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing notion $\mathbb{P}$ adding a real,*
(3) $\forall f : \mathbb{Q} \longrightarrow \omega \quad \exists I \in \mathcal{I} \quad \overline{f^{-1}[I]}$ *is uncountable.*

Proof: (1) implies (2) trivially. For (2) implies (3) consider the contrapositive, i.e there is an $f : \mathbb{Q} \longrightarrow \omega$ such that $\overline{f^{-1}[I]}$ is countable for every $I \in \mathcal{I}$. Let $r \in 2^\omega$ be a new real. Then $r \in 2^\omega \setminus \bigcup\{\overline{f^{-1}[A]} : A \in \mathcal{A}\}$ as no new real is contained in a countable closed set coded in the ground model. Let $A = f[\{q_{r\restriction n} : n \in \omega\}]$. First note that $A$ is infinite, as $r \in \overline{f^{-1}[A]}$. To see that $A$, indeed, "destroys" $\mathcal{I}$ assume that there is an $I \in \mathcal{I}$ such that $A \cap I$ is infinite. This, however implies that $q_{r\restriction n} \in f^{-1}[I]$ for infinitely many $n \in \omega$, hence, $r \in \overline{f^{-1}[I]}$ which is a contradiction.

To see that (3) implies (1) let $\mathcal{I}$ be an ideal satisfying (3) and assume that there is a $p \in \mathbb{S}$ and an $\mathbb{S}$-name $\dot{x}$ for an infinite subset of $\omega$ such that $p \Vdash$ "$\forall I \in \mathcal{I} \quad |\dot{x} \cap I| < \aleph_0$". Using a standard fusion argument find a $p' \leq p$ such that for every $s \in Br_n(p')$ $p'_s$ decides $\dot{x} \cap n$. Identify $\mathbb{Q}$ with $\mathbb{Q}_{p'}$.

Note that, even though, in general $\mathbb{Q}_{p'}$ is not homeomorphic to $\mathbb{Q}$ (typically, $\mathbb{Q}_{p'}$ is discrete, as a subspace of $\mathbb{Q}$), it is order-isomorphic to $\mathbb{Q}$. Note also that a subset of the rationals has an uncountable closure if and only if it contains a subset order-isomorphic to $\mathbb{Q}$.

Define a function $f : \mathbb{Q}_{p'} \longrightarrow \omega$ by

$$f(q) = \max\{k : p'_{s_q} \Vdash \text{“}k \in \dot{x}\text{”}\}.$$

Note that $f$ is well-defined as $\mathcal{I}$ is a tall ideal. By (3) there is an $I \in \mathcal{I}$ such that $\overline{f^{-1}[I]}$ is uncountable. In particular, there is a $p'' \in \mathbb{S}$ such that $\mathbb{Q}_{p''} \subseteq \mathbb{Q}_{p'}$, hence $p'' \leq p'$, and $[p''] \subseteq \overline{f^{-1}[I]}$. This, however, means that $p'' \Vdash$ “$|\dot{x} \cap I| = \aleph_0$” which is a contradiction. $\square$

Now we return to Simon's question. Note that the standard construction of a MAD family as outlined in the introduction produces an $\mathcal{A}$ which is Sacks-destructible (any new real diagonalizes $\mathcal{I}^*(\mathcal{A})$). The following proposition provides an answer to the question. Recall that $\mathfrak{a}$ denotes the minimal cardinality of a MAD family.

**Proposition 2.** *There is an $\mathbb{S}$-indestructible MAD family in ZFC.*

Proof: First note that if $\mathcal{A}$ is a MAD family of size less that $\mathfrak{c}$ then $\mathcal{A}$ is $\mathbb{S}$-indestructible. If not then by Theorem 1 there is a function $f : \mathbb{Q} \longrightarrow \omega$ such that $\overline{f^{-1}[A]}$ is countable for every $A \in \mathcal{I}(\mathcal{A})$. It is enough to pick an $r \in 2^\omega \setminus \bigcup\{\overline{f^{-1}[A]} : A \in \mathcal{I}(\mathcal{A})\}$. Then the set $B = f[\{q_{r \restriction n} : n \in \omega\}]$ is infinite and $B$ is almost disjoint from all $A \in \mathcal{A}$ contradicting the maximality of $\mathcal{A}$.

So without loss of generality we can assume that $\mathfrak{a} = \mathfrak{c}$. Enumerate all functions from $\mathbb{Q}$ to $\omega$ as $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. Construct recursively $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ a sequence of infinite subsets of $\omega$ so that:

a) $\{A_i : i < \omega\}$ is a partition of $\omega$ into infinite sets

and for every $\omega \leq \alpha < \mathfrak{c}$

b) $\forall \beta < \alpha \ |A_\alpha \cap A_\beta| < \aleph_0$ and

c) $\exists \beta \leq \alpha \ \overline{f_\alpha^{-1}[A_\beta]}$ is uncountable.

Assume first that the induction can be carried through. Then $\mathcal{A}$ is a MAD family (if not then letting $f$ be a bijection between $\mathbb{Q}$ and a set almost disjoint from all elements of $\mathcal{A}$ provides a contradiction as $f$ is listed as some $f_\alpha$). Similarly, $\mathcal{A}$ is $\mathbb{S}$-indestructible by clause (3) of Theorem 1.

To see that we can proceed with the induction let $\alpha$ be an infinite ordinal less than $\mathfrak{c}$ and assume that $A_\beta$ has been defined for every $\beta < \alpha$. Consider $f_\alpha$. If there is a $\gamma < \alpha$ such that $\overline{f_\alpha^{-1}[A_\gamma]}$ is uncountable, let $A_\alpha$ be any infinite subset of $\omega$ almost disjoint from all $A_\beta$, $\beta < \alpha$. We can do this as $\mathfrak{a} = \mathfrak{c}$.

If not, then let $X = \bigcup\{\overline{f_\alpha^{-1}[A_\beta]} : \beta < \alpha\}$. Note that $|X| < \mathfrak{c}$ as $\overline{f_\alpha^{-1}[A_\beta]}$ is countable for every $\beta < \alpha$. Note also that $\overline{f_\alpha^{-1}(n)}$ is countable for every integer $n$ as otherwise $\overline{f_\alpha^{-1}[A_i]}$ would be uncountable for some $i < \omega \leq \alpha$. Hence $f_\alpha[\mathbb{Q}_P]$

is infinite for every perfect subset of $2^\omega$. Now, let $P$ be a perfect subset of $2^\omega$ disjoint from $X$. Such a set exists as $2^\omega$ can be partitioned into $\mathfrak{c}$-many perfect sets and $X$ can not intersect them all. Let $A_\alpha = f_\alpha[\mathbb{Q}_P]$. Then $A_\alpha$ is an infinite subset of $\omega$ and $\overline{f_\alpha^{-1}[A_\alpha]}$ is uncountable. All that is left to prove is that $A_\alpha \cap A_\beta$ is finite for every $\beta < \alpha$. To that end assume the contrary and pick $s_i \in 2^{<\omega}$ for each $i \in \omega$ so that $f(q_{s_i{}^\frown 1}) \in A_\alpha \cap A_\beta$ and $f(q_{s_i{}^\frown 1}) \neq f(q_{s_j{}^\frown 1})$ for distinct $i$ and $j$. By König's lemma there is a $g \in \overline{\{q_{s_i{}^\frown 1} : i \in \omega\}} \setminus \{q_{s_i{}^\frown 1} : i \in \omega\}$ that, however, contradicts the fact that $P \cap \overline{f_\alpha^{-1}[A_\beta]} = \emptyset$. □

How does the above answer Simon's question? Well, every $\mathbb{S}$-destructible MAD family is (at least locally) essentially the same as the "standard" MAD by clause (3) of Theorem 1. So in some sense Proposition 2 gives an affirmative answer to the question. The reason why the answer may not be quite satisfactory is that we do not know whether there is an $\mathbb{S}$-indestructible MAD family of size $\mathfrak{c}$ in ZFC.

**Question 3.** *Is it consistent that no MAD family of size $\mathfrak{c}$ is $\mathbb{S}$-indestructible?*

We suspect that there is no such MAD family in the Sacks model (i.e. a model obtained from a model of CH by countable support iteration of Sacks forcing of length $\omega_2$). For a similar observation about Cohen-indestructible MAD families see Proposition 7.

Let us turn our attention towards other standard forcing notions. *Miller forcing* $\mathbb{M}$ consists of perfect subtrees $p$ of $\omega^{<\omega}$ such that for every $t \in p$ $|\{n : t^\frown n \in p\}| \in \{1, \omega\}$. For $p \in \mathbb{M}$ define $Br(p)$ and $Br_n(p)$ as for the Sacks forcing. An alternative description views Miller forcing as the set of those $p \in \mathbb{S}$ such that $\mathbb{Q} \cap [p]$ is dense in $[p]$. The order isomorphism is induced by a map $\Phi : \omega^{<\omega} \longrightarrow 2^{<\omega}$ defined by $\Phi(\emptyset) = \emptyset$ and $\Phi(s^\frown n) = \Phi(s)^\frown 0^\frown \ldots {}^\frown 0^\frown 1$, the sequence of 0's being of length $n$. Note that the above considerations show that $\mathbb{M}$ is isomorphic to $\{A \subseteq \mathbb{Q} : A \simeq \mathbb{Q}\}$, where $A \simeq B$ means that $A$ and $B$ are homeomorphic as opposed to $\mathbb{S}$, which is isomorphic to $\{A \subseteq \mathbb{Q} : A \simeq \mathbb{Q}\}$, where $A \simeq B$ here means that $A$ and $B$ are order isomorphic. For $p \in \mathbb{M}$ denote by $\mathbb{Q}^p$ the set $[\Phi``p] \cap \mathbb{Q}$. Miller forcing is often referred to as *rational perfect set* forcing. Recall that a perfect set $P \subseteq 2^\omega$ is a *rational perfect* set if $P \cap \mathbb{Q}$ is dense in $P$.

**Theorem 4.** *Let $\mathcal{I}$ be an ideal on $\omega$. Then the following are equivalent:*

(1) *$\mathcal{I}$ is $\mathbb{M}$-indestructible,*

(2) $\forall f : \mathbb{Q} \longrightarrow \omega \quad \exists I \in \mathcal{I} \quad \overline{f^{-1}[I]}$ *contains a rational perfect set.*

Proof: The proof follows closely the proof of Theorem 1. For (1) implies (2) consider the contrapositive and note that a Miller real is not included in any ground model closed set which has a scattered intersection with the rationals.

To see that (2) implies (1) let $\mathcal{I}$ be an ideal satisfying (2) and assume that there is a $p \in \mathbb{M}$ and an $\mathbb{M}$-name $\dot{x}$ for an infinite subset of $\omega$ such that $p \Vdash$ "$\forall I \in$

$\mathcal{I} \quad |\dot{x} \cap I| < \aleph_0$". Use fusion to find a $p' \leq p$ such that $p'_s$ decides $\dot{x} \cap n$ for every $s \in Br_n(p')$. Identify $\mathbb{Q}$ with $\mathbb{Q}^{p'}$. Define a function $f : \mathbb{Q}^{p'} \longrightarrow \omega$ by

$$f(q) = \max\{k : p'_{s_q} \Vdash \text{“}k \in \dot{x}\text{”}\}.$$

Again, $f$ is well-defined. By (2) there is an $I \in \mathcal{I}$ such that $\overline{f^{-1}[I]}$ contains a rational perfect set. In particular, there is a $p'' \in \mathbb{M}$ such that $\mathbb{Q}^{p''} \subseteq \mathbb{Q}^{p'}$ ($p'' \leq p'$) and $\mathbb{Q}^{p''} \subseteq f^{-1}[I]$. Then $p'' \Vdash$ "$|\dot{x} \cap I| = \aleph_0$" which is absurd. □

Next we prove an analogous result for Cohen forcing. *Cohen forcing* $\mathbb{C}$ is construed here as $2^{<\omega}$ ordered by extension.

**Theorem 5.** *Let $\mathcal{I}$ be an ideal on $\omega$. Then the following are equivalent:*

(1) *$\mathcal{I}$ is $\mathbb{C}$-indestructible,*
(2) $\forall f : \mathbb{Q} \longrightarrow \omega \quad \exists I \in \mathcal{I} \quad \overline{f^{-1}[I]}$ *has non-empty interior,*
(3) $\forall f : \mathbb{Q} \longrightarrow \omega$ *one-to-one* $\exists I \in \mathcal{I} \quad \overline{f^{-1}[I]}$ *has non-empty interior.*

Proof: (1) implies (2) as a Cohen real is not contained in any closed nowhere dense set coded in the ground model.

To see that (2) implies (1) let $\mathcal{I}$ be an ideal satisfying (2) and assume that there is an $s \in \mathbb{C}$ and a $\mathbb{C}$-name $\dot{x}$ for an infinite subset of $\omega$ such that $s \Vdash$ "$\forall I \in \mathcal{I} \quad |\dot{x} \cap I| < \aleph_0$". Define a function $f : \mathbb{Q} \longrightarrow \omega$ by

$$f(q) = \max\{k : s^\frown s_q \Vdash \text{“}k \in \dot{x}\text{”}\}.$$

$f$ is well-defined as $\mathcal{I}$ is a tall ideal. By (2) there is an $I \in \mathcal{I}$ such that $f^{-1}[I]$ is somewhere dense, that is there is a $t \in \mathbb{C}$ such that for every $r \supseteq t$ there is a $q \in f^{-1}[I]$ such that $r \subset q$. This, however, means that $s^\frown t \Vdash$ "$|\dot{x} \cap I| = \aleph_0$" which is a contradiction.

(2) obviously implies (3). To see that (3) implies (2) assume that there is an $f : \mathbb{Q} \longrightarrow \omega$ such that $\overline{f^{-1}[I]}$ is nowhere dense for every $I \in \mathcal{I}$. It is easy to see that there is a one-to-one function $h$ such that $\mathrm{dom}(h)$ is a dense subset of $\mathbb{Q}$, $\mathrm{rng}(h) = \mathrm{rng}(f)$ and $h(q) = f(q)$ for every $q \in \mathrm{dom}(h)$. By identifying $\mathbb{Q}$ with the domain of $h$, $h$ is a one-to-one function from $\mathbb{Q}$ to $\omega$ such that for every $I \in \mathcal{I}$ $\overline{h^{-1}[I]}$ is nowhere dense. This finishes the proof. □

Now we return to the question of existence of Cohen-indestructible MAD families. J. Steprāns in [St] observed that there is one in any model obtained by adding $\aleph_1$-many Cohen reals and asked whether there is one in ZFC. The following two propositions provide a partial answer to his question. Recall that $\mathfrak{b}$ denotes the minimal cardinality of an unbounded (undominated) subset of $\omega^\omega$ ordered by eventual domination. $\mathrm{cov}(\mathcal{M})$ is the minimal cardinality of a family of nowhere dense subsets of $2^\omega$ covering $2^\omega$. Recall that $\mathfrak{b} \leq \mathfrak{a}$.

**Proposition 6.** *Each of the following implies that there is a $\mathbb{C}$-indestructible MAD family:*

(1) $\mathfrak{a} < cov(\mathcal{M})$,
(2) $\mathfrak{b} = \mathfrak{c}$.

PROOF: For (1) note that if $\mathcal{A}$ is a MAD family of size $\mathfrak{a} < \mathrm{cov}(\mathcal{M})$ then $\mathcal{A}$ is $\mathbb{C}$-indestructible. To see this let $M$ be an elementary submodel of $H(\mathfrak{c}^+)$ of size $\mathfrak{a}$ such that $\mathcal{A} \subseteq M$. As $|M| < \mathrm{cov}(\mathcal{M})$ there is a real $c$ Cohen over $M$ and $\mathcal{A}$ remains maximal in $M[c]$ hence is $\mathbb{C}$-indestructible.

In order to prove (2) enumerate all one-to-one functions from $\mathbb{Q}$ to $\omega$ as $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. Let $\{A_i : i \in \omega\}$ be a partition of $\omega$ into infinite sets. Inductively construct sets $A_\alpha$ so that:

a) $\forall \beta < \alpha \quad |A_\alpha \cap A_\beta| < \aleph_0$ and
b) $\exists \beta \leq \alpha \quad f_\alpha^{-1}[A_\beta]$ is somewhere dense.

It is obvious that if we can fulfill these requirements then the family $\{A_\alpha : \alpha < \mathfrak{c}\}$ is MAD and $\mathbb{C}$-indestructible.

At stage $\alpha$ consider the function $f_\alpha$. If there is a $\beta < \alpha$ such that $f_\alpha^{-1}[A_\beta]$ is somewhere dense let $A_\alpha$ be any infinite subset of $\omega$ satisfying a). If not, enumerate a basis for the topology on $\mathbb{Q}$ as $\{U_i : i \in \omega\}$. Recursively choose $\beta_i < \alpha$ distinct such that $|f_\alpha^{-1}[A_{\beta_i}] \cap U_i| = \aleph_0$. Note that you can always do this as by our assumption $f_\alpha^{-1}[A_\beta]$ is nowhere dense for every $\beta < \alpha$. Now for every $\beta \in \alpha \setminus \{\beta_i : i \in \omega\}$ let

$$g_\beta(i) = \max(A_\beta \cap A_{\beta_i}).$$

As $\alpha < \mathfrak{c} = \mathfrak{b}$ There is a $g : \omega \longrightarrow \omega$ which dominates all $g_\beta$. Pick for every $i \in \omega$

$$k_i \in \{m \in A_{\beta_i} : f_\alpha^{-1}(m) \in U_i \text{ and } m > g(i)\} \setminus \bigcup_{j<i} A_{\beta_j}$$

and let $A_\alpha = \{k_i : i \in \omega\}$. Then $A_\alpha$ is almost disjoint from all $A_\beta$, $\beta < \alpha$, and $f_\alpha^{-1}[A_\alpha]$ is dense. □

**Proposition 7.** *It is relatively consistent with ZFC that no MAD family of size $\mathfrak{c}$ is $\mathbb{C}$-indestructible.*

PROOF: Let $V$ be a model of CH and let $G$ be $\mathbb{C}_{\omega_2}$-generic over $V$, where $\mathbb{C}_{\omega_2}$ denotes the standard poset for adding $\aleph_2$-many Cohen reals. The resulting model is often referred to as the *Cohen model*. Let $G_\alpha$ denote the restriction of $G$ to $\mathbb{C}_\alpha$.

Let $\dot{\mathcal{A}}$ be a $\mathbb{C}_{\omega_2}$-name for a $\mathbb{C}$-indestructible MAD family. Then there is an $\alpha < \omega_2$ such that, in $V[G_\alpha]$, the family $\dot{\mathcal{A}}[G_\alpha]$ is $\mathbb{C}$-indestructible. To see this use clause (3) of Theorem 5. Now, as every real in $V[G]$ is contained in $V[G_\alpha][H]$ ,where $H$ is $\mathbb{C}$-generic over $V[G_\alpha]$, $\dot{\mathcal{A}}[G_\alpha] = \dot{\mathcal{A}}[G]$ and hence $|\dot{\mathcal{A}}[G]| = \omega_1 < \mathfrak{c} = \omega_2$. □

The main question, however, still remains unanswered.

**Question 8** (Steprāns). *Is there a Cohen-indestructible MAD family in ZFC?*

Note that Theorems 1, 4 and 5 show that every Cohen-indestructible MAD family is Miller-indestructible and hence also Sacks-indestructible.

One of the most interesting open problems about the cardinal invariants of the continuum is the problem as to whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$. This problem is sometimes attributed to J. Roitman. Recall that $\mathfrak{d}$ denotes the minimal cardinality of a dominating subset of $\omega^\omega$. The framework for the anticipated negative solution is set up by the preservation theorems of S. Shelah (see [Sh]). Hence, the question reduces to the following: *Assume* CH *and let $\mathcal{A}$ be a MAD family. Is there a proper $\omega^\omega$-bounding forcing $\mathbb{P}$ destroying $\mathcal{A}$*? For some more on the difficulties connected with this problem see [Hr]. C. Laflamme (in [La]) made progress towards the solution by showing that every $F_\sigma$ ideal can be diagonalized by an $\omega^\omega$-bounding forcing. The following questions seem natural and the answers to them necessary for the solution of Roitman's question. It would be sufficient to answer them in the context of CH.

**Question 9.** *Let $\mathcal{A}$ be $\mathbb{C}$-indestructible MAD family. Is there an $\omega^\omega$-bounding forcing $\mathbb{P}$ destroying $\mathcal{A}$?*
*Is there a characterization of Random-indestructibility in the spirit of this paper?*

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