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A new proof of weighted weak-type inequalities for fractional integrals

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Abstract. We give a new and simpler proof of a two-weight, weak \((p, p)\) inequality for fractional integrals first proved by Cruz-Uribe and Pérez [4].

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1. Introduction

For \(0 < \alpha < n\), the fractional integral operator \(I_\alpha\) is defined by

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.
\]

In [4], Cruz-Uribe and Pérez proved a two-weight, weak-type norm inequality which answered a question posed by Sawyer and Wheeden [9].

Theorem. Given a pair of weights \((u, v)\), \(p, 1 < p < \infty\), and \(\alpha, 0 < \alpha < n\), suppose that for some \(r > 1\) and for all cubes \(Q\),

\[
|Q|^{\alpha/n} \left( \frac{1}{|Q|} \int_{Q} u^r \, dx \right)^{1/rp} \left( \frac{1}{|Q|} \int_{Q} v^{-p'/p} \, dx \right)^{1/p'} \leq C < \infty.
\]

Then the fractional integral operator \(I_\alpha\) satisfies the weak \((p, p)\) inequality

\[
u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \leq \frac{C}{tp} \int_{\mathbb{R}^n} |f|^p v \, dx.
\]

Their proof of Theorem 1.1 was fairly complex, and depended on a technical lemma resembling a good-\(\lambda\) inequality. The purpose of this paper is to give another, more elementary proof. It is based on three weighted norm inequalities

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for the fractional integral operator and the closely related fractional maximal operator,

\[ M_\alpha f(x) = \sup_{Q \ni x} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q |f(y)| \, dy. \]

These are stated in Section 2 and the proof of Theorem 1.1 is in Section 3.

Finally, we remark that we believe that condition (1.1) in Theorem 1.1 is not the best possible. In [4], Cruz-Uribe and Pérez proved an analogue of Theorem 1.1 for singular integrals, and in [3] they sharpened this result by replacing the local \( L^r \) norm on the left-hand side of (1.1) by the smaller Orlicz space norm \( \| \cdot \|_{L(\log L)^{p-1+\delta}} \), \( \delta > 0 \). (A similar condition is sufficient for a strong \((p,p)\) inequality for fractional integrals. See Pérez [7].) We conjecture that the corresponding weak \((p,p)\) result holds for fractional integrals. For a partial result, see Cruz-Uribe and Fiorenza [2].

2. Preliminary results

The first result we need is due to Muckenhoupt and Wheeden [6, p. 262].

**Theorem 2.1.** Given \( 0 < \alpha < n \), and a weight \( w \in A_\infty \), there exists a constant \( C \), depending only on \( \alpha \), \( n \) and the \( A_\infty \) constant of \( w \), such that for all functions \( f \),

\[ \sup_{t>0} t w(\{ x \in \mathbb{R}^n : |I_\alpha f(x)| > t \}) \leq C \sup_{t>0} t w(\{ x \in \mathbb{R}^n : M_\alpha f(x) > t \}). \]

The second result is due to Sawyer [8, p. 285]; for a simple proof see [1].

**Theorem 2.2.** Given \( 0 < \alpha < n \), and a weight \( w \), there exists a constant \( C \), depending only on \( \alpha \) and \( n \), such that for all functions \( f \),

\[ w(\{ x \in \mathbb{R}^n : M_\alpha f(x) > t \}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f| M_\alpha w \, dx. \]

The third result is a special case of a theorem due to Pérez [7, p. 668].

**Theorem 2.3.** Given \( 0 < \alpha < n \), \( r > 1 \), and a pair of weights \((u, v)\) such that (1.1) holds, then there exists a constant \( s \), \( 1 < s < \min(n/\alpha, r) \), and a constant \( C \) such that for all functions \( f \),

\[ \int_{\mathbb{R}^n} (M_\alpha s f)^{p'/s} v^{-p'/p} \, dx \leq C \int_{\mathbb{R}^n} |f|^{p'/s} u^{-p'/p} \, dx. \]

**Proof:** Given \( s \), \( 1 < s < \min(n/\alpha, r) \), condition (1.1) is equivalent to

\[ |Q|^{s/\alpha} \left( \frac{1}{|Q|} \int_Q |v^{-s/p}|^{p'/s} \, dx \right)^{s/p'} \left( \frac{1}{|Q|} \int_Q |u^{s/p}|^{rp/s} \, dx \right)^{s/rp} \leq C. \]

Pérez showed that this implies (2.1), provided that \((rp/s)' < p'/s\). This is true for \( s = 1 \), so by continuity it is true for \( s > 1 \) sufficiently small. \( \square \)
3. Proof of Theorem 1.1

The proof requires one lemma.

Lemma 3.1. Given \( \alpha, 0 < \alpha < n, \) and \( s, 1 < s < n/\alpha, \) then for all non-negative, locally integrable functions \( g, M_\alpha(M(g^s)^{1/s})(x) \leq CM_\alpha s(g^s)(x)^{1/s}, \) where \( M \) is the Hardy-Littlewood maximal operator.

Proof: Our proof is modeled on the proof of a similar result in García-Cuerva and Rubio de Francia [5, p.158]. It will suffice to show that there exists a constant \( C \) such that for any \( x \) and any cube \( Q \) containing \( x, \)

\[
\frac{|Q|^\alpha/n}{|Q|} \int_Q M(g^s)(y)^{1/s} \, dy \leq CM_\alpha s(g^s)(x)^{1/s}.
\]

Let \( g = g_1 + g_2, \) where \( g_1 = g \chi_{3Q}. \) Then \( M(g^s)(x)^{1/s} \leq M(g_1^s)(x)^{1/s} + M(g_2^s)(x)^{1/s}. \) Since \( M \) is weak \((1, 1), \) by Kolmogorov’s inequality (cf. [5, p.485]),

\[
\frac{|Q|^\alpha/n}{|Q|} \int_Q M(g_1^s)(y)^{1/s} \, dy \leq C|Q|^\alpha/n \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} g_1^s \, dy \right)^{1/s},
\]

\[
\leq C|Q|^\alpha/n \left( \frac{1}{|3Q|} \int_{3Q} g^s \, dy \right)^{1/s},
\]

\[
\leq CM_\alpha s(g^s)(x)^{1/s}.
\]

Further, \( M(g_2^s)^{1/s} \in A_1 \) with a constant independent of \( g, \) (see [5, p.158]), so

\[
\frac{|Q|^\alpha/n}{|Q|} \int_Q M(g_2^s)(y)^{1/s} \, dy \leq C|Q|^\alpha/n M(g_2^s)(x)^{1/s}.
\]

There exists a cube \( P \) containing \( x \) such that

\[
M(g_2^s)(x) \leq \frac{2}{|P|} \int_P g_2^s \, dy.
\]

Since \( P \) must intersect \( \mathbb{R}^n \setminus 3Q, Q \subset 3P. \) Therefore,

\[
|Q|^\alpha/n M(g_2^s)(x)^{1/s} \leq C|3P|^\alpha/n \left( \frac{1}{|3P|} \int_{3P} g^s \, dy \right)^{1/s} \leq CM_\alpha s(g^s)(x)^{1/s}.
\]

\( \square \)

We can now prove Theorem 1.1. Fix \( p, 1 < p < \infty, \) and a function \( f \in L^p(v); \) by a standard argument we may assume that \( f \) is non-negative, bounded and has
compact support. For each \( t > 0 \), let \( E_t = \{ x \in \mathbb{R}^n : I_{\alpha f}(x) > t \} \). By duality there exists a function \( G_t \in L^{p'} \), \( \|G_t\|_{p'} = 1 \), such that

\[
u(E_t)^{1/p} = \left\|u^{1/p} \chi_{E_t}\right\|_p = \int_{E_t} u^{1/p} G_t \, dx.
\]

Fix \( s > 1 \) as in Theorem 2.3, and let \( w_t = M(u^{s/p} G_t^s)^{1/s} \). Then \( w_t \in A_1 \subset A_\infty \), and the \( A_\infty \) constant of \( w_t \) depends only on \( s \). Hence, by Theorems 2.1 and 2.2, and by Lemma 3.1,

\[
sup_{t>0} t \nu(E_t)^{1/p} = \sup_{t>0} t \int_{E_t} u^{1/p} G_t \, dx \\
\leq \sup_{t>0} t w_t(E_t) \\
\leq C \sup_{t>0} t w_t(\{ x \in \mathbb{R}^n : M_{\alpha f}(x) > t \}) \\
\leq C \sup_{t>0} \int_{\mathbb{R}^n} f M_{\alpha}(w_t) \, dx \\
\leq C \sup_{t>0} \int_{\mathbb{R}^n} f M_{\alpha s}(u^{s/p} G_t^s)^{1/s} \, dx.
\]

Then, by Hölder’s inequality and Theorem 2.3,

\[
sup_{t>0} t \nu(E_t)^{1/p} \leq C \sup_{t>0} \left( \int_{\mathbb{R}^n} f^p v \, dx \right)^{1/p} \left( \int_{\mathbb{R}^n} M_{\alpha s}(u^{s/p} G_t^s)^{p'/s} v^{p'-p/p} \, dx \right)^{1/p'} \\
\leq C \sup_{t>0} \left( \int_{\mathbb{R}^n} f^p v \, dx \right)^{1/p} \left( \int_{\mathbb{R}^n} (u^{s/p} G_t^s)^{p'/s} u^{p'-p/p} \, dx \right)^{1/p'} \\
= C \sup_{t>0} \left( \int_{\mathbb{R}^n} f^p v \, dx \right)^{1/p} .
\]

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