Henryk Hudzik; L. Wang; Ting Fu Wang
Smooth, very smooth and strongly smooth points in Musielak-Orlicz function spaces equipped with the Luxemburg norm


Persistent URL: http://dml.cz/dmlcz/119263

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 2001

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
Smooth, very smooth and strongly smooth points in Musielak-Orlicz function spaces equipped with the Luxemburg norm

H. Hudzik, L. Wang, T. Wang

Abstract. First, we extend the criteria for smooth points of $S(L_M)$ from [22] to the whole class of Musielak-Orlicz spaces. Next, we present criteria for very smooth and strongly smooth points of $S(L_M)$.

Keywords: smooth points, very smooth points, strongly smooth points, Musielak-Orlicz function spaces, Luxemburg norm

Classification: 46E30, 46E40, 46B20

1. Introduction

Let us start with some notations and definitions. In the whole paper $X$ denotes a real Banach space and $X^*$ denotes its dual space. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ stand for the set of natural numbers, the set of reals and positive reals, respectively. By $(T, \Sigma, \mu)$ we denote a measure space with $\mu$ being monotonic and $\sigma$-finite. The letter $M$ stands for a Musielak-Orlicz function, i.e. $M$ is a mapping from $T \times \mathbb{R}$ into $[0, +\infty]$ satisfying the following conditions:

(i) there is a null set $A \in \Sigma$ such that for any $t \in T \setminus A$, $M(t, \cdot)$ is an Orlicz function, i.e. $M(t, 0) = 0$, $M(t, \cdot)$ is continuous at zero and left continuous on $(0, \infty)$, $M(t, \cdot)$ is convex and even on $\mathbb{R}$ and $M(t, u) \to \infty$ as $u \to \infty$,
(ii) for any $u \in \mathbb{R}$, $M(\cdot, u)$ is a $\Sigma$-measurable function on $T$.

Let us denote by $L^0 = L^0(T, \Sigma, \mu)$ the space of all (equivalence classes of) $\Sigma$-measurable functions $x : T \to \mathbb{R}$. Given any Musielak-Orlicz function $M$, we define on $L^0$ a convex modular $\varrho_M$ by

$$\varrho_M(x) = \int_T M(t, x(t)) d\mu$$

and a Musielak-Orlicz space $L_M$ by

$$L_M = \{ x \in L^0 : \varrho_M(\lambda x) < \infty \text{ for some } \lambda > 0 \}.$$
We denote by $N$ the Musielak-Orlicz function complementary to $M$ in the sense of Young, i.e.

$$N(t, v) = \sup_{u \geq 0} \{ u|v| - M(t, u) \}$$

for all $u \in \mathbb{R}$ and $t \in T \setminus A$. We define in $L_M$ two norms; the Luxemburg norm

$$\|x\|_M = \inf \{ \lambda > 0 : \varrho_M(x/\lambda) \leq 1 \}$$

and the Amemiya-Orlicz norm

$$\|x\|_M^o = \inf_{k > 0} \frac{1}{k} (1 + \varrho_M(kx)).$$

For simplicity, we write $L_M$ and $L_M^0$ in place of $(L_M, \| \cdot \|_M)$ and $(L_M, \| \cdot \|_M^0)$, respectively. Let us denote by $K(x)$ the set of all $k > 0$ such that the infimum in the last formula is attained at $k$. $L_M$ is a Banach space under either of these two norms (see [2], [15] and in the case of Orlicz spaces also [12], [13], [14] and [17]).

Let $p_-(t, u)$ and $p(t, u)$ denote the left and right derivative of $M(t, \cdot)$ at $u$, respectively, and let us denote for $t \in T$:

$$e(t) = \sup \{ u > 0 : M(t, u) = 0 \}, \quad b(t) = \sup \{ u > 0 : M(t, u) < \infty \},$$

$$\hat{e}(t) = \sup \{ v > 0 : N(t, v) = 0 \}, \quad \hat{b}(t) = \sup \{ v > 0 : N(t, v) < \infty \},$$

$$S_x = \{ t \in T : x(t) \neq 0 \}, \quad O_x = \{ t \in T : x(t) = 0 \} \quad \text{for} \quad x \in L^0,$$

and

$$\xi_M(x) = \inf \{ c > 0 : \varrho_M(x/c) < \infty \} \quad \text{for} \quad x \in L_M.$$

We say that $M$ satisfies the $\triangle_2$-condition ($M \in \triangle_2$ for short) if there are a null set $B \in \Sigma$, a constant $K \geq 2$ and a nonnegative function $h \in L^0$ such that $\varrho_M(h) < \infty$ and $M(t, 2u) \leq KM(t, u)$ for all $u \geq h(t)$ (see [2] and [15]).

It is well known that between various smoothness properties of $X$ and respective rotundity properties of $X^*$ there is an one-side duality. Namely, if $X^*$ is rotund (weakly locally uniformly rotund) [locally uniformly rotund] then $X$ is smooth (very smooth) [strongly smooth].

Let us recall these six notions. $X$ is said to be rotund if for any $x \in S(X)$ (= the unit sphere of $X$) if $y, z \in S(X)$ and $2x = y + z$, then $y = z = x$. $X$ is said to be weakly locally uniformly rotund (locally uniformly rotund) if for any $x \in S(X)$ and $(x_n)$ in $S(X)$ such that $\|x_n + x\| \to 2$ there holds $x_n \rightharpoonup x$ weakly $\left( x_n \rightharpoonup x \right)$ for short), respectively $x_n \to x$ strongly, i.e. $\|x_n - x\| \to 0$.

$X$ is said to be smooth if for any $x \in S(X)$ there is only one support functional $x^*$ at $x$. Recall that $x^* \in X^*$ is said to be a support functional at $x$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. We denote by $\text{Grad}(x)$ the set of all support functionals at $x$. $X$ is said to be strongly (very) smooth if it is smooth and for any $x \in S(X)$ and $(x_n)$ in $S(X)$ the condition $\|x_n - x\| \to 0$ implies that $x_n^* \rightharpoonup x^*$ strongly (weakly), where $\{x_n^*\} = \text{Grad}(x)$ and $\{x_n^*\} = \text{Grad}(x_n)$ for $n = 1, 2, \ldots$.

Smoothness properties of Orlicz spaces and Musielak-Orlicz spaces were considered in [1], [3]–[5], [7]–[11], [18]–[19] and [22]–[23].
2. Results

We start with a criterion for smooth points of $S(L_M)$. Analogous criterion has been obtained in [22] but only for Musielak-Orlicz functions which are smooth at zero. Note that smoothness of $M$ at zero is equivalent to the fact that $\tilde{c}(t) = 0$ for $\mu$-a.e. $t \in T$.

**Theorem 1.** A point $x \in S(L_M)$ is a smooth point if and only if:

(a) $\xi_M(x) < 1$,
(b) $\mu\{t \in O_x : \tilde{c}(t) > 0\} = 0$,
(c) $\mu\{t \in S_x : p_-(t,|x(t)|) < p(t,|x(t)|)\} = 0$.

**Proof:** Assume without loss of generality that $x(t) \geq 0$ for $\mu$-a.e. $t \in T$.

Necessity. The necessity of (a) can be proved in the same way as in [22]. Since (a) must be true we have that $\text{Grad}(x) = R\text{Grad}(x)$, where $R\text{Grad}(x)$ denotes the set of all regular, i.e. order continuous functionals. Recall that $x^* \in (L_M)^*$ is said to be order continuous if $x^*(x_n) \to 0$ whenever $0 \leq x_n \searrow 0$ and that every such functional $x^*$ is represented by some $y \in L^0_N$ (see [17]). We will prove that if $y \in \text{Grad}(x)$, then $k(y) \neq \emptyset$, i.e. $\|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$ for some $k > 0$. Otherwise

$$1 = \|y\|_N^0 = \lim_{k \to \infty} \frac{1}{k}(1 + \varrho_N(ky)) = \int_{S_y} y(t)b(t) \, d\mu = \int_T x(t)y(t) \, d\mu$$

$$= \int_{S_y} x(t)y(t) \, d\mu.$$

Since $x(t) \leq b(t)$ $\mu$-a.e. in $T$, we have $x(t) = b(t)$ $\mu$-a.e. in $S_y$.

It follows from $\xi_M(x) < 1$ that there exists $\lambda > 1$ such that $\xi_M(\lambda x) < \infty$. Thus

$$\infty > \xi_M(\lambda x) = \int_{S_y} M(t, \lambda x(t)) \, d\mu = \int_{S_y} M(t, \lambda b(t)) \, d\mu = \infty.$$

This is a contradiction, which proves that $k(y) \neq \emptyset$.

Now, we are ready to prove the necessity of (b). Assume that $x$ is a smooth point of $S(L_M)$ and (b) is not true. Then $T_0 = \{t \in O_x : \tilde{c}(t) > 0\}$ is a set in $\Sigma$ with $\mu(T_0) > 0$. Assume that $y \in \text{Grad}(x)$ and $\|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$. Take $z \in L^0$ such that $z(t) = y(t)$ for $t \notin T_0$, $kz(t) \leq \tilde{c}(t)$ and $z(t) \neq y(t)$ for $t \in T_0$. Then

$$\|z\|_N^0 \leq \frac{1}{k}(1 + \varrho_N(kz)) = \frac{1}{k}(1 + \int_{T \setminus T_0} N(t,ky(t)) \, d\mu) \leq \frac{1}{k}(1 + \varrho_N(ky))$$

$$= \|y\|_N^0 = 1$$

and

$$\langle x, z \rangle = \int_T x(t)z(t) \, d\mu = \int_{S_x} x(t)z(t) \, d\mu = \int_{S_x} x(t)y(t) \, d\mu = 1.$$
So, \( \|z\|_{N}^{0} = 1 \) and \( z \in \text{Grad}(x) \). But \( z \neq y \), whence \( x \) is not a smooth point, a contradiction.

Assume that \( x \in S(L_{M}) \) is a smooth point and (c) is not true, then \( T_{1} = \{ t \in S_{x} : p_{-}(t, x(t)) < p(t, x(t)) \} \) has positive measure. We may assume that \( 0 < \mu(T_{1}) < \mu(T) \). Take \( y \in \text{RGrad}(x) \) with \( \|y\|_{N}^{0} = \frac{1}{k}(1 + \varrho_{N}(ky)) \) for some \( k > 0 \). It can be proved in the same way as in [4, Theorem 1.5] for Orlicz spaces that
\[
\int_{T} N(t, p_{-}(t, x(t))) \, d\mu \leq \int_{T} N(t, ky(t)) \, d\mu = k - 1 < \infty.
\]
Let
\[
y_{1}(t) = \begin{cases} 
p_{-}(t, x(t)) & \text{for } t \in S_{x} \\
0 & \text{for } t \in O_{x}
\end{cases}
\]
and \( y_{2} \) be a measurable function with \( y_{2}(t) = p_{-}(t, x(t)) \) for \( t \in S_{x} \setminus T_{0} \) and \( y_{2}(t) \in (p_{-}(t, x(t)), p(t, x(t))) \) for \( t \in T_{0} \) and satisfying \( \varrho_{N}(y_{2}) < \infty \). Then \( y_{1}, y_{2} \in L_{N}^{0} \). Let \( z_{1} = y_{1}/\|y_{1}\|_{N}^{0} \) and \( z_{2} = y_{1}/\|y_{2}\|_{N}^{0} \). Then \( z_{1} \neq z_{2} \) and \( z_{1}, z_{2} \in S(L_{N}^{0}) \). Furthermore
\[
1 \geq \langle x, z_{1} \rangle = \frac{1}{\|y_{1}\|_{N}^{0}} \langle x, y_{1} \rangle = \frac{1}{\|y_{1}\|_{N}^{0}} \int_{T} x(t)p_{-}(t, x(t)) \, d\mu
\]
\[
= \frac{1}{\|y_{1}\|_{N}^{0}} \int_{T} (M(t, x(t)) + N(t, p_{-}(t, x(t)))) \, d\mu
\]
\[
= \frac{1}{\|y_{1}\|_{N}^{0}}(1 + \varrho_{N}(y_{1})) = \frac{1}{\|y_{1}\|_{N}^{0}}(1 + \varrho_{N}(\|y_{1}\|z_{1})) \geq \|z_{1}\| = 1,
\]
whence we conclude that \( \|z_{1}\|_{N}^{0} = 1 = \langle x, z_{1} \rangle \). So, \( z_{1} \in \text{Grad}(x) \). Similarly, \( z_{2} \in \text{Grad}(x) \), which means that \( x \) is not a smooth point, a contradiction.

Sufficiency. Let \( f = y + \phi \in \text{Grad}(x) \), where \( y \) and \( \phi \) denote the regular and the singular part of \( f \), respectively. By condition (a), \( \phi = 0 \) and \( 1 = \|y\|_{N}^{0} = \frac{1}{k}(1 + \varrho_{N}(ky)) \) for some \( k > 0 \) (see the beginning of the proof of the necessity). It can be proved in the same way as in [4, Theorem 1.5] for Orlicz spaces that
\[
(1) \quad p_{-}(t, x(t)) \leq ky(t) \leq p(t, x(t)) \quad \text{for } t \in S_{x}.
\]
Moreover, by \( \|x\|_{M} = 1 \) and \( \xi_{M}(x) < 1 \), we have \( \varrho_{M}(x) = 1 \). Therefore, the equality
\[
\int_{O_{x}} x(t)ky(t) \, d\mu = \int_{O_{x}} (M(t, x(t)) + N(t, ky(t))) \, d\mu
\]
yields that \( N(t, ky(t)) = 0 \) for \( t \in O_{x} \). By condition (b), \( y(t) = 0 \) for \( t \in O_{x} \) and by condition (c), \( ky(t) = p(t, x(t)) \) for \( t \in S_{x} \), i.e. \( ky \) is unique. By
Smooth, very smooth and strongly smooth points in Musielak-Orlicz function spaces...

\[ k = \|ky\|_0^N = \|ky\chi_{S_x}\|_0^N = \|p(\cdot, x(\cdot))\chi_{S_x}\|_0^N, \]

we obtain \( k = \frac{1}{\|p(\cdot, x(\cdot))\|_0^N} \). Therefore

\[ y(t) = \begin{cases} \frac{p(t, x(t))}{\|p(\cdot, x(\cdot))\chi_{S_x}\|_0^N} & \text{for } t \in S_x \\
0 & \text{for } t \in 0_x, \end{cases} \]

which means that \( y \) is unique and so \( x \) is a smooth point, which finishes the proof. \( \square \)

**Corollary 1.** The space \( L_M \) is smooth if and only if:

(a) \( M \in \Delta_2 \),
(b) \( \tilde{e}(t) = 0 \) for \( \mu \)-a.e. \( t \in T \),
(c) \( p(t, \cdot) \) is continuous function on \( \mathbb{R} \) for \( \mu \)-a.e. \( t \in T \).

**Proof:** This result follows from Theorem 1. We need only to show the necessity of condition (b) because the rest can be proved in the same way as in [22].

Assume that condition (b) is not satisfied, that is, the set \( A = \{ t \in T : \tilde{e}(t) > 0 \} \) has positive measure. Then we can easily build \( x \in S(L_M) \) with \( \mu(O_x \cap A) > 0 \). By Theorem 1, \( x \) is not a smooth point, which finishes the proof of the necessity of condition (b). \( \square \)

In the proof of the next theorem the following result will be useful.

**Proposition 1.** Let \( M \) be a Musielak-Orlicz function and \( N \) be its complementary function in the sense of Young. Let \( N \in \Delta_2, x \in S(L_M), y_n \in L_0^N, k(y_n) \not= 0, (n = 1, 2, \ldots), \) and \( \langle x, y_n \rangle \to 1 \) as \( n \to \infty \). Then for every \( \varepsilon > 0 \) there is \( T_\varepsilon \in \Sigma \) with \( \mu T_\varepsilon < \infty \) such that \( \sup_n \varrho_N(y_n\chi_{T \setminus T_\varepsilon}) < \varepsilon \).

**Proof:** Take \( T_1 \subset T_2 \subset \ldots \subset T_i \subset T_{i+1} \subset \ldots \) with \( \mu T_i < \infty \) for each \( i \in \mathbb{N} \) and \( \bigcup_i T_i = T \). We will prove that for any \( \varepsilon > 0 \) there is \( i_\varepsilon \in \mathbb{N} \) such that \( \sup_n \varrho_N(y_n\chi_{T \setminus T_{i_\varepsilon}}) < \varepsilon \). Otherwise, there is \( \varepsilon > 0 \) such that for any \( i \in \mathbb{N} \) there is \( n_i \in \mathbb{N} \) such that \( \varrho_N(y_n\chi_{T \setminus T_{n_i}}) > \varepsilon \). We may assume that \( n_i \to \infty \) as \( i \to \infty \) because \( (n_i) \) is unbounded by the fact that the assumption \( N \in \Delta_2 \) yields that

\[ \sup_{n \in N_0} \varrho(y_n\chi_{T \setminus T_{n_i}}) \to 0 \text{ as } i \to \infty \]

for any finite subset \( N_0 \) of \( \mathbb{N} \). Choose \( k_i \in k(y_{n_i}) \). From \( \xi_M(x) < 1 \) it follows that
there is $\lambda > 1$ satisfying $\varrho_M(\lambda x) < \infty$. This yields that for $i \to \infty$ there holds

$$
1 \leftarrow \langle x, y_{n_i} \rangle = \frac{1}{k_i} \langle x, k_i y_{n_i} \rangle \\
= \frac{1}{k_i} \left( \int_{T_i} x(t) k_i y_{n_i}(t) \, d\mu + \int_{T \setminus T_i} x(t) k_i y_{n_i}(t) \, d\mu \right) \\
\leq \frac{1}{k_i} \left( \varrho_M(x\chi_{T_i}) + \varrho_N(k_i y_{n_i} \chi_{T_i}) + \frac{1}{\lambda} \varrho_M(\lambda x\chi_{T \setminus T_i}) \right) \\
\leq \frac{1}{k_i} \left( \varrho_M(x) + \varrho_N(k_i y_{n_i}) - (1 - \frac{1}{\lambda}) \varrho_N(k_i y_{n_i} \chi_{T \setminus T_i}) + \frac{1}{\lambda} \varrho_M(\lambda x\chi_{T \setminus T_i}) \right) \\
\leq \frac{1}{k_i} \left( 1 + \varrho_N(k_i y_{n_i}) - (1 - \frac{1}{\lambda}) \varrho_N(k_i y_{n_i} \chi_{T \setminus T_i}) + \frac{1}{\lambda} \varrho_M(\lambda x\chi_{T \setminus T_i}) \right) \\
\leq \|y_{n_i}\| - (1 - \frac{1}{\lambda}) \varepsilon + \frac{1}{\lambda} \varrho(\lambda x\chi_{T \setminus T_i}) \to 1 - (1 - \frac{1}{\lambda}) \varepsilon,
$$
a contradiction finishing the proof. \hfill \Box

**Theorem 2.** Let $x \in S_{L_M}$. Then the following assertions are equivalent:

1. $x$ is a strongly smooth point,
2. $x$ is a very smooth point,
3. $x$ is a smooth point and $N \in \triangle_2$.

**Proof:** We still assume without loss of generality that $x \geq 0$. The implication (1) $\Rightarrow$ (2) is obvious. Let us prove that (2) $\Rightarrow$ (3). We need only to prove that (2) $\Rightarrow N \in \triangle_2$. Assume that condition (2) holds and $N \notin \triangle_2$. There is $z \in L^0_N$ with $\varrho_N(z) < \infty$ and $\xi_N(y - \frac{z}{k}) =: A > 0$, where $y$ defines the unique support functional for $x$ and $k > 0$ satisfies $1 = \|y\|_N = \frac{1}{k}(1 + \varrho_N(ky))$. Indeed, if $\xi_N(y) = 0$, we take $z \in L^0_N \setminus E^0_N$; if $\xi_N(y) > 0$, we take $z = 0$. Divide $T$ into $T_1, T'_1$ with $\mu(T_1) = \mu(T'_1) = \frac{\mu(T)}{2}$, $T_1 \cap T'_1 = \emptyset$. Lemma 1.67 from [2] is also true for Musielak-Orlicz spaces (without any change of the proof). Namely, for any partition $\{T_i\}_{i=1}^n$ of $T$ and any $x \in L^0_N$, $\xi_N(x) = \max_i \xi_N(x\chi_{T_i})$. So, we may assume that $\xi_N(y - \frac{z}{k}) = \xi_N((y - \frac{z}{k})\chi_{T_1})$.

Divide $T_1$ into $T_2, T'_2$ with $\mu(T_2) = \mu(T'_2) = \frac{\mu(T_1)}{2}$, $T_2 \cap T'_2 = \emptyset$. We may assume that

$$
\xi_N(y - \frac{z}{k}) = \xi_N((y - \frac{z}{k})\chi_{T_1}) = \xi_N((y - \frac{z}{k})\chi_{T_2}).
$$

Continuing this process by induction one can find a sequence $(T_n)_{n=1}^\infty$ of measurable sets in $T$ such that $T \supset T_1 \supset T_2 \supset \cdots \supset T_n \supset \cdots$, $\mu(T_n) = \frac{1}{2^n} \mu(T)$, and $\xi_N(y - \frac{z}{k}) = \xi_N((y - \frac{z}{k})\chi_{T_n})$ for $n = 1, 2, \ldots$. Let

$$
y_n(t) = \begin{cases} 
\frac{z(t)}{k} & \text{for } t \in T_n \\
y(t) & \text{for } t \in T \setminus T_n 
\end{cases} (n = 1, 2, \ldots).
$$
Then
\[ \|y_n\|_N^0 \leq \frac{1}{k}(1 + \varrho_N(ky_n)) \leq \frac{1}{k}(1 + \varrho_N(ky) + \int_{T_N} N(t, z(t)) \, d\mu) \to \|y\|_N^0 = 1. \]

On the other hand
\[ \langle x, y_n \rangle = \int_{T \setminus T_n} x(t)y(t) \, dt + \int_{T_n} x(t)z(t) \, dt \to \langle x, y \rangle = 1. \]

But
\[ \xi_N(\min_{1 < i \leq n} |y - y_i|) = \xi_N((y - \frac{z}{k})x_{T_n}) = \xi_N(y - \frac{z}{k}) = A. \]

Since Theorem 1.68 from [2] holds also for Musielak-Orlicz spaces, that is if \((x_n)\)

is a sequence in \(L_N^0\), then \(\langle x_n, \varphi \rangle \to 0\) for any singular functional \(\varphi \in (L_N^0)^*\) if and only if \(\lim_{m \to \infty} \xi_N(\min_{i \leq m} |y_i|) = 0\) for each subsequence \((y_i)\) of \((x_n)\), we conclude from the last condition that \(y_n \not\to y\) weakly. This contradicts the fact that \(x\) is a very smooth point.

(3) \Rightarrow (1). Assume that (3) holds. Since \(x\) is a smooth point, by Theorem 1 we conclude that \(\xi_M(x) < 1\) and for \(y \in L_N^0\) determining the unique support functional at \(x\) there is \(k > 0\) such that \(1 = \|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky)).\)

Moreover, \(ky(t) = p(t, x(t))\) for \(t \in S_x\) and \(y(t) = 0\) for \(t \in O_x\).

Assume that \(f_n = y_n + \phi_n \in S(L_M^*), f_n(x) \to 1. \) In order to prove that \(\|f_n - y\| \to 0\), we consider six steps.

I. Assume that \(\xi_M(x) < 1 - \theta < 1.\) Take \(z \in E_M\) such that \(\|x - z\|_M < 1 - \theta.\) Then

\[ 1 \to f_n(x) = \langle x, y_n \rangle + \phi_n(x) \leq \|x\|_M \|y_n\|_N^0 + \|\phi_n\|_M \|x - z\|_M \]
\[ \leq \|y_n\|_N^0 + \|\phi_n\|_M (1 - \theta) = \|f_n\| - \theta \|\phi_n\|. \]

Therefore \(\|\phi_n\| \to 0, \|y_n\|_N^0 \to 1\) and \(\langle x, y_n \rangle \to 1.\) Without loss of generality we assume in the following that \(\|y_n\|_N^0 = 1\) for \(n = 1, 2, \ldots\) and \(\langle x, y_n \rangle \to 1.\)

II. Let us prove that \(k(y_n) \neq \emptyset\) for an infinite number of \(n \in \mathbb{N}, i.e.\) there are \(k_n > 0\) such that
\[ \|y_n\|_N^0 = \frac{1}{k_n}(1 + \varrho_N(k_ny_n)). \]

Otherwise \(\|y_n\|_N^0 = \int_T y_n(t)b(t) \, d\mu\) for infinite number of \(n.\) Since \(\xi_M(x) < 1,\)

there is \(\lambda > 1\) such that \(\varrho_M(\lambda x) < 1.\) Hence
\[ 1 = \|y_n\|_N^0 = \int_T y_n(t)b(t) \, d\mu \geq \int_T y_n(t)\lambda x(t) \, d\mu \to \lambda \text{ as } n \to \infty, \text{ which is a contradiction.} \]

So, we may assume in the following, that \(k(y_n) \neq \emptyset\) for all \(n \in \mathbb{N}.\)
III. We will prove that  
\[ \tilde{k} = \sup_n k_n < \infty. \]

Otherwise, we may assume that  \( k_n \to \infty \), whence for  \( \lambda > 1 \) such that  \( \varrho_M(\lambda x) < \infty \), we get

\[
1 \leftarrow \int_T x(t)y_n(t) \, d\mu = \frac{1}{\lambda} \int_{S_{yn}} \lambda x(t)y_n(t) \, d\mu \\
\leq \frac{1}{\lambda} \int_{S_{yn}} b(t)y_n(t) \, d\mu = \frac{1}{\lambda} \int_{S_{yn}} \lim_{v \to \infty} q(t,v)y_n(t) \, d\mu \\
= \frac{1}{\lambda} \int_{S_{yn}} \lim_{v \to \infty} \frac{N(t,v)}{v} y_n(t) \, d\mu = \frac{1}{\lambda} \lim_{n \to \infty} \frac{1}{k_n} (1 + \varrho_N(k_ny_n)) \\
= \frac{1}{\lambda},
\]
a contradiction. Therefore  \( \tilde{k} < \infty \).

IV. Let us prove that

\[
\lim_{\mu(E) \to 0} \left[ \sup_n \int_E N(t,k_ny_n(t)) \, d\mu \right] = 0. \tag{2}
\]

Otherwise, there is  \( \varepsilon > 0 \) such that

\[
\lim_{\mu E \to 0} \left[ \sup_n \int_E N(t,k_ny_n(t)) \, d\mu \right] > \varepsilon.
\]

Given \( \eta_1 > 0 \) there is  \( E_1 \in \Sigma \) with  \( \mu E_1 < \eta_1 \) and  \( n_1 \in \mathbb{N} \) such that  \( \int_{E_1} N(t,k_{n_1}y_{n_1}(t)) \, d\mu > \varepsilon \). By the absolute continuity of integral there is  \( \Theta_1 \) such that

\[
\int_A N(t,k_{n_1}y_{n_1}(t)) \, d\mu < \varepsilon
\]

for any  \( A \in \Sigma \) with  \( \mu A < \Theta_1 \) and  \( n = 1, 2, \ldots, n_1 \). Take \( \eta_2 = \min(\eta_1/2, \Theta_1) \). Then there is  \( E_2 \in \Sigma \) with  \( \mu E_2 < \eta_2 \) and  \( n_2 \in \mathbb{N} \) such that  \( \int_{E_2} N(t,k_{n_2}y_{n_2}(t)) \, d\mu > \varepsilon \). Obviously,  \( n_2 > n_1 \). Proceeding like that by induction, we can construct a sequence  \( (\eta_i) \) of positive numbers with  \( \eta_1 > 2\eta_2 > 2^2\eta_3 > \ldots > 2^{n-1}\eta_n > \ldots \), a sequence  \( (n_i) \) of natural numbers with  \( n_1 < n_2 < n_3 < \ldots \) and a sequence  \( (E_i) \) in  \( \Sigma \) with  \( \mu E_i < \eta_i \) such that

\[
\int_{E_i} N(t,k_{n_i}y_{n_i}(t)) \, d\mu > \varepsilon \quad (i = 1, 2, \ldots).
\]
Hence
\[ 1 \leftarrow \langle x, y_n \rangle = \frac{1}{k_n} \left( \int_{T \setminus E_i} k_n x(t) y_n(t) \, d\mu + \frac{1}{\lambda} \int_{E_i} \lambda x(t) k_n y_n(t) \, d\mu \right) \]
\[ \leq \frac{1}{k_n} \left( \varrho_M(x) \chi_{T \setminus E_i} + \varrho_N(k_n y_n x(T \setminus E_i)) + \frac{1}{\lambda} \varrho_M(\lambda x E_i) \right) \]
\[ + \frac{1}{\lambda} \varrho_N(k_n y_n x E_i) \]
\[ \leq \frac{1}{k_n} \left( \varrho_M(x) + \varrho_N(k_n y_n) - (1 - \frac{1}{\lambda}) \varrho_N(k_n y_n x E_i) + \frac{1}{\lambda} \varrho_M(\lambda x E_i) \right) \]
\[ \leq \|y_n\| - (1 - \frac{1}{\lambda}) \varepsilon + \varrho_M(\lambda x E_i) \rightarrow 1 - (1 - \frac{1}{\lambda}) \varepsilon. \]

This is a contradiction, so equality (2) holds.

V. Now, we will prove that
\[ \lim_{n \to \infty} k_n y_n(t) = k y(t) = \left\{ \begin{array}{ll} p(t, x(t)) = p_-(t, x(t)) & \text{for } t \in S_x \\ 0 & \text{for } t \in O_x. \end{array} \right. \]

From
\[ 0 \leftarrow \|y_n\|_N - <x, y_n> = \frac{1}{k_n} \left( 1 + \varrho_N(k_n y_n) \right) - \frac{1}{k_n} \langle x, k_n y_n \rangle \]
\[ = \frac{1}{k_n} \left( \varrho_M(x) + \varrho_N(k_n y_n) - \langle x, k_n y_n \rangle \right) \]
\[ \geq \frac{1}{k} \int_T (M(t, x(t)) + N(t, k_n y_n(t)) - x(t) k_n y_n(t)) \, d\mu \]

it follows that
\[ M(t, x(t)) + N(t, k_n y_n(t)) - x(t) k_n y_n(t) \to 0 \quad \mu\text{-a.e. in } T. \]

Notice that \( p_-(t, x(t)) = p(t, x(t)) \) for \( t \in S_x \). Therefore, by the Young inequality, we can easily deduce that \( k_n y_n(t) \to p(t, x(t)) \) \( \mu\text{-a.e. in } S_x \). Using condition (b) in Theorem 1, we conclude that \( y_n \to 0 \) \( \mu\text{-a.e. in } T \).

VI. Finally, we will show that \( \|y_n - y\|_N \to 0 \). By Proposition 1, we can assume that \( \mu T < \infty \). Take an arbitrary \( \varepsilon > 0 \). By \( N \in \triangle_2 \) there exist \( k > 0 \) and a nonnegative function \( \delta_0 \in L^1 \) such that
\[ N(t, \frac{v}{2}) \leq k N(t, v) + \delta_0(t) \]

for \( \mu\text{-a.e. } t \in T \). Take \( \eta > 0 \) such that if \( E \subset T \) and \( \mu(E) < \eta \), then \( \int_E \delta_0(t) \, d\mu < \frac{1}{4} \), \( \int_E N(t, ky(t)) \, d\mu < \frac{1}{4k} \) and \( \int_E N(t, k_n y_n(t)) \, d\mu < \frac{1}{4k} \) for any \( n \in \mathbb{N} \) (the last one is possible by (2)).
Since $k_ny_n \to ky \mu$-a.e. in $T$, there is $T_0 \subset T$ such that $\mu(T \setminus T_0) < \eta$ and $N(t, k_ny_n(t) - ky(t)) \to 0$ uniformly in $T_0$. Hence

$$\int_{T_0} N(t, k_ny_n(t) - ky(t)) \frac{1}{2\varepsilon} \, d\mu < \frac{1}{2}$$

for $n$ large enough. Therefore,

$$\|k_ny_n - ky\|^0_N \leq 2\varepsilon(1 + \int_T N(t, \frac{k_ny_n(t) - ky(t)}{2\varepsilon}) \, d\mu)$$

$$\leq 2\varepsilon(1 + \int_{T_0} N(t, \frac{k_ny_n(t) - ky(t)}{2\varepsilon}) \, d\mu)$$

$$+ \frac{1}{2} \int_{T \setminus T_0} N(t, \frac{k_ny_n(t)}{\varepsilon}) + N(t, \frac{ky(t)}{\varepsilon}) \, d\mu$$

$$\leq 2\varepsilon(1 + \frac{1}{2} + \frac{1}{2} \int_{T \setminus T_0} (kN(t, k_ny_n(t)) \, d\mu)$$

$$\leq 4\varepsilon$$

for $n$ large enough, which means that $\|k_ny_n - ky\|^0_N \to 0$ as $n \to 0$. On the other hand $k_n = \|k_ny_n\|^0_N \to \|ky\|^0_N = k$ as $n \to \infty$. Thus $\|y_n - y\|^0_N \to 0$ as $n \to \infty$, which completes the proof.

\[\square\]

**Corollary 2.** The following are equivalent:

1. $L_M$ is strongly smooth,
2. $L_M$ is very smooth,
3. $L_M$ is smooth and $N \in \Delta_2$.

**Proof:** It is an immediate consequence of Theorem 2. \[\square\]

**References**

Smooth, very smooth and strongly smooth points in Musielak-Orlicz function spaces ... 497


H. Hudzik:
Faculty of Mathematics, Adam Mickiewicz University, Poznań, Poland
and
Institute of Mathematics, Poznań University of Technology, Poznań, Poland

L. Wang, T. Wang:
Department of Mathematics, Harbin University of Science and Technology, Harbin, P.R. China

(Received May 31, 2000)