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Countable compactness and p-limits

S. Garcia-Ferreira, A.H. Tomita

Abstract. For $\emptyset \neq M \subseteq \omega^*$, we say that $X$ is quasi $M$-compact, if for every $f : \omega \to X$ there is $p \in M$ such that $\overline{f}(p) \in X$, where $\overline{f}$ is the Stone-Čech extension of $f$. In this context, a space $X$ is countably compact iff $X$ is quasi $\omega^*$-compact. If $X$ is quasi $M$-compact and $M$ is either finite or countable discrete in $\omega^*$, then all powers of $X$ are countably compact. Assuming CH, we give an example of a countable subset $M \subseteq \omega^*$ and a quasi $M$-compact space $X$ whose square is not countably compact, and show that in a model of A. Blass and S. Shelah every quasi $M$-compact space is $p$-compact ($=\text{quasi} \{p\}$-compact) for some $p \in \omega^*$. We list some open problems.

Keywords: $p$-limit, $p$-compact, almost $p$-compact, quasi $M$-compact, countably compact

Classification: Primary 54A20, 54A35; Secondary 54B99

0. Introduction

All our spaces are Tychonoff. If $f : X \to Y$ is a continuous function, then $\overline{f} : \beta(X) \to \beta(Y)$ denotes the Stone-Čech extension of $f$. $\beta(\omega)$ is identified with the set of all ultrafilters on $\omega$, and $\beta(\omega) \setminus \omega = \omega^*$ is the set of all free ultrafilters on $\omega$. For $A \subseteq \omega$, $\hat{A} = \{p \in \beta(\omega) : A \in p\} = \text{cl}_{\beta(\omega)} A$.

In the context of nonstandard analysis, the point $\overline{f}(p) \in X$, where $f : \omega \to X$ is a function and $p \in \omega^*$, has the following interpretation:

Definition 0.1 ([Be]). Let $p \in \omega^*$ and let $(x_n)_{n<\omega}$ be a sequence in a space $X$. We say that $x$ is the $p$-limit point of $(x_n)_{n<\omega}$, we write $x = p - \lim_{n\to\omega} x_n$, if for every neighborhood $V$ of $x$, $\{n < \omega : x_n \in V\} \in p$.

If $x = p - \lim_{n\to\omega} x_n$, then $x = \overline{f}(p)$, where $f : \omega \to X$ is defined by $f(n) = x_n$ for every $n < \omega$. It is known that, in the category of Tychonoff spaces, a space $X$ is countably compact iff every sequence of points in $X$ has a $p$-limit point for some $p \in \omega^*$: By using functions, $X$ is countably compact iff for every $f : \omega \to X$ there is $p \in \omega^*$ such that $\overline{f}(p) \in X$. This last observation leads us to consider the following class of spaces.
Definition 0.2 ([Be]). Let \( p \in \omega^* \). A space \( X \) is said to be \( p \)-compact if for every sequence \( (x_n)_{n<\omega} \) of points of \( X \) there is \( x \in X \) such that \( x = p - \lim_{n \to \omega} x_n \).

Thus, a space \( X \) is \( p \)-compact, for \( p \in \omega^* \), if \( \overline{f}(p) \in X \) for every \( f : \omega \to X \). It is shown in [GS] that all powers of a space \( X \) are countably compact iff there is \( p \in \omega^* \) such that \( X \) is \( p \)-compact. A.R. Bernstein [Be] proved that \( p \)-compactness is preserved under arbitrary products, for every \( p \in \omega^* \). Since countable compactness is not preserved under products, there are countably compact spaces which are not \( p \)-compact for any \( p \in \omega^* \) (see [GJ]).

The following definition plays the main role in this paper:

Definition 0.3 ([G]). Let \( \emptyset \neq M \subseteq \omega^* \). A space \( X \) is said to be quasi \( M \)-compact if for every \( f : \omega \to X \) there is \( p \in M \) such that \( \overline{f}(p) \in X \).

Thus, a space \( X \) is countably compact iff \( X \) is quasi \( \omega^* \)-compact, and \( p \)-compactness agrees with quasi \( \{p\} \)-compactness. Given a countably compact space \( X \), we may ask about that smallest cardinality of a nonempty subset \( M \subseteq \omega^* \) such that for every \( f : \omega \to X \) there is \( p \in M \) such that \( \overline{f}(p) \in X \). For instance, we mentioned above that if all the powers of a space \( X \) are countably compact, then set \( M \) may consist of just one single point. We show that if \( X \) is a countably compact space and one of its powers is not countably compact, then \( M \) cannot be neither finite and nor discrete. Under the assumption of \( CH \), we give an Example of a countable subset \( M \) of \( \omega^* \), with one non-isolated point, and a countably compact space \( X \) such that \( X \) is quasi \( M \)-compact and fails to be \( p \)-compact for any \( p \in \omega^* \) (see [GJ]).

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1. Quasi \( M \)-compact spaces

Our first result is a particular case of Theorem 1.25 from [G].

Theorem 1.1. Let \( \emptyset \neq M \subseteq \omega^* \). If there is \( f : \omega \to \omega \) and \( p \in \omega^* \) such that \( M \subseteq \overline{f}^{-1}(p) \), then every quasi \( M \)-compact space is \( p \)-compact.

Proof: Let \( X \) be a quasi \( M \)-compact space and let \( g : \omega \to X \) be a function. Consider the composition \( g \circ f \). Since \( X \) is quasi \( M \)-compact, there is \( r \in M \) such that \( \overline{g}(\overline{f}(r)) \in X \) and then \( \overline{g}(p) \in X \), because of \( \overline{f}(r) = p \). Thus, \( X \) is \( p \)-compact.

Theorem 1.2. If \( X \) is quasi \( M \)-compact for some countable discrete subset \( M \subseteq \omega^* \), then \( X \) is \( p \)-compact for some \( p \in M \).

Proof: Let \( M \subseteq \omega^* \) be discrete and let \( X \) be a quasi \( M \)-space. Assume that \( X \) is not \( p \)-compact for any \( p \in M \). Enumerate \( M \) as \( \{p_n : n < \omega \} \) and let
\{A_n : n < \omega\} be a partition of \omega such that \(A_n \in p_n\) for every \(n < \omega\). By assumption, for every \(n < \omega\), there is \(f_n : \omega \to X\) such that \(\overline{f_n}(p_n) \notin X\). Let us define \(f : \omega \to X\) by \(f|_{A_n} = f_n|_{A_n}\) for every \(n < \omega\). Then, there is \(m < \omega\) such that \(\overline{f}(p_m) \in X\). But, by the definition of \(f\), \(\overline{f}(p_m) = \overline{f_m}(p_m)\) which is a contradiction since \(\overline{f_m}(p_m) \notin X\). \(\square\)

In our first Example, we will need the following pre-orderings on \(\omega^*\): For \(p, q \in \omega^*\), we say that \(p \leq_{RK} q\) if there is a function \(f : \omega \to \omega\) such that \(\overline{f}(q) = p\), and \(p \leq_{RF} q\) if there is an embedding \(e : \omega \to \beta(\omega)\) such that \(\overline{f}(p) = q\).

If \(p, q \in \omega^*\), then we say that \(p \approx q\) if \(p \leq_{RK} q\) and \(q \leq_{RK} p\), and \(p <_{RK} q\) (resp., \(p <_{RF} q\)) means that \(p \leq_{RK} q\) (resp., \(p \leq_{RF} q\)) but \(p\) and \(q\) are not equivalent. The type of \(p \in \omega^*\) is the set \(T(p) = \{q \in \omega^* : p \approx q\}\). A RK-minimal ultrafilter on \(\omega\) is usually called selective. We list the basic properties of these two pre-orderings that we shall use (proofs of these facts may be found in [Co], [CN], [Ku] and [vM]):

**Lemma 1.3.** The following properties hold:

1. \(\leq_{RF} \subset \leq_{RK}\);
2. For \(p, q \in \omega^*\), \(p \approx q\) iff there is a bijection \(f : \omega \to \omega\) such that \(\overline{f}(p) = q\);
3. Let \(f : \omega \to \omega\) and \(p \in \omega^*\). Then, \(p \approx \overline{f}(p)\) if and only if there is \(A \in p\) such that \(f|_A\) is one-to-one;
4. Every weak \(P\)-point of \(\omega^*\) is RF-minimal, and there are \(2^\omega\)-many weak \(P\)-points of \(\omega^*\) which are not selective;
5. If \(p \in \omega^*\) is selective and \(f : \omega \to \omega^*\) is a function such that \(\overline{f}(p) \notin f[\omega]\), then \(p <_{RF} \overline{f}(p)\);
6. If \(p \leq_{RF} r\) and \(q \leq_{RF} r\), then \(p\) and \(q\) are RF-comparable;
7. If \(f : \omega \to \omega^*\) is an embedding, then \(p <_{RF} \overline{f}(p)\) for every \(p \in \omega^*\);
8. If \(X, Y \subseteq \omega^*\) are countable, then \(X \cap Y = \emptyset\) iff \(X \cap Y = \emptyset\) and \(X \cap \overline{Y} = \emptyset\). In particular, if \(X\) and \(Y\) are disjoint countable sets of weak \(P\)-points of \(\omega^*\), then \(X \cap \overline{Y} = \emptyset\).

To state our preliminary results, we introduce the following notion: Let \(F \in [\omega^*]^{\omega}\), let \(e : \omega \to F\) be a function and let \(p \in \omega^*\). Then, a function \(f : \omega \to \omega^*\) is called a \((F, e, p)\)-function if \(q <_{RF} \overline{f}(q)\) for every \(q \in F\), and \(p <_{RF} \overline{f}(\overline{e}(p))\).

Notice that if \(F = \{p_n : n < \omega\}\) are RK-incomparable selective ultrafilters on \(\omega\), \(p \in \omega^*\) and \(e : \omega \to \omega^*\) is defined by \(e(n) = p_n\) for all \(n < \omega\), then every \((F, e, p)\)-function satisfies that \(\overline{f}(p_n) \neq \overline{f}(p_m)\) whenever \(n < m < \omega\).

**Lemma 1.4.** Let \(\{p\} \cup \{p_n : n < \omega\}\) be pairwise RK-incomparable selective ultrafilters on \(\omega\), and let \(e : \omega \to \omega^*\) be defined by \(e(n) = p_n\) for every \(n < \omega\). If \(f : \omega \to \omega^*\) satisfies that \(p_n <_{RF} \overline{f}(p_n)\) for every \(n < \omega\), then \(f\) is a \((\{p_n : n < \omega\}, e, p)\)-function.

**Proof:** We have to show that \(p <_{RF} \overline{f}(\overline{e}(p))\). In fact, if \(\overline{f}(\overline{e}(p)) \neq \overline{f}(p_n)\) for
every \( n < \omega \), then \( \overline{f}(\overline{e}(n)) \notin \{ \overline{f}(e(n)) : n < \omega \} \), and hence, by Lemma 1.3, \( p < RF \overline{f}(p) \). Suppose that \( \overline{f}(\overline{e}(p)) = \overline{f}(p_k) \) for some \( k < \omega \). Our assumption implies that \( \overline{f}(\overline{e}(p)) \neq \overline{f}(p_n) \) for every \( n \in \omega - \{ k \} \). Since \( \overline{f}(\overline{e}(p)) \) is an accumulation point of \( \{ \overline{f}(p_n) : n \in \omega \setminus \{ k \} \} \). Then, we may find a pairwise disjoint family \( \{ B_m : m < \omega \} \) of subsets of \( \omega \) such that \( B_m \notin \overline{f}(p_k) \) for all \( m < \omega \) and \( \{ \overline{f}(p_n) : n \in \omega \setminus \{ k \} \} \subseteq \bigcup_{m < \omega} B_m \). Let \( h : \omega \to \omega \) be the function defined by \( h^{-1}(m) = B_m \) for each \( m < \omega \). Then, \( (\overline{h} \circ \overline{f} \circ e)_{\omega \setminus \{ k \}} \subseteq \omega \) and \( \overline{h}(\overline{f}(\overline{e}(p))) \in \omega^* \). Hence, \( h(\overline{f}(\overline{e}(p))) \leq RK p \). Since \( p \) is selective, \( h(\overline{f}(\overline{e}(p))) \approx p \). By applying Lemma 1.3, we may find \( A \in p \) such that \( h \circ \overline{f} \circ e|_A \) is one-to-one. So, by the definition of \( h \), the function \( \overline{f} \circ e|_A \) is an embedding and hence \( p < RF \overline{f}(p) \). But, by Lemma 1.3, this implies that \( p \) and \( p_k \) are RF-equivalent, which is a contradiction. Then, \( \overline{f}(\overline{e}(p)) \neq \overline{f}(p_n) \) for all \( n < \omega \). Therefore, \( f \) is a \( \{ p_n : n < \omega \}, e, p \)-function. \( \square \)

**Lemma 1.5.** Let \( \{ p \} \cup \{ p_n : n < \omega \} \) be a set of pairwise RK-incomparable selective ultrafilters on \( \omega \), and let \( e : \omega \to \omega^* \) be defined by \( e(n) = p_n \) for every \( n < \omega \). For a subspace \( X \) of \( \omega^* \), the following are equivalent:

1. \( X \) is quasi (\( \{ \overline{e}(p) \} \cup \{ p_n : n < \omega \} \))-compact;
2. for every \( \{ p_n : n < \omega \}, e, p \)-function \( f : \omega \to X \) there is \( q \in \{ \overline{e}(p) \} \cup \{ p_n : n < \omega \} \) such that \( \overline{f}(q) \in X \).

**Proof:** The implication \( (1) \Rightarrow (2) \) is evident.

\( (2) \Rightarrow (1) \). Observe that \( e \) is an embedding and hence \( \overline{e}(p) \neq p_n \) for all \( n < \omega \). Put \( M = \{ \overline{e}(p) \} \cup \{ p_n : n < \omega \} \). Let us assume that \( f : \omega \to X \) is a function such that \( \overline{f}(q) \notin X \) for every \( q \in M \). Then, in particular, \( \overline{f}(p_n) \notin f[\omega] \) for every \( n < \omega \). Thus, by Lemma 1.3, \( p_n < RF \overline{f}(p_n) \) for each \( n < \omega \). So, by Lemma 1.4, \( f \) is a \( \{ p_n : n < \omega \}, e, p \)-function. By assumption, there is \( q \in M \) such that \( \overline{f}(q) \in X \), which is a contradiction. \( \square \)

**Example 1.6.** Let \( \{ p \} \cup \{ p_n : n < \omega \} \) be a set of pairwise RK-incomparable selective ultrafilters on \( \omega \), and let \( e : \omega \to \omega^* \) be defined by \( e(n) = p_n \) for every \( n < \omega \). Then, there is a quasi (\( \{ \overline{e}(p) \} \cup \{ p_n : n < \omega \} \))-compact space that is not \( q \)-compact for any \( q \in \omega^* \).

**Proof:** Let \( \{ q_n : n < \omega \} \) be a set of selective ultrafilters on \( \omega \) such that \( \{ p \} \cup \{ p_n : n < \omega \} \cup \{ q_n : n < \omega \} \) are pairwise RK-incomparable. Notice that \( \overline{e}(p) \) is an accumulation point of \( \{ p_n : n < \omega \} \). Put \( F = M_0 = \{ p_n : n < \omega \} \) and \( N_0 = \{ q_n : n < \omega \} \). It follows from Lemma 1.3 that \( M_0 \cap N_0 = \emptyset \). By transfinite induction, for each \( 0 < \nu < \omega_1 \) we may define \( M_\nu, N_\nu \subseteq \omega^* \) as follows:

1. \( M_\nu = \{ \overline{f}(\overline{e}(p)) : f : \omega \to \bigcup_{\mu < \nu}(M_\mu \cup N_\mu) \) is an \( (F, e, p) \)-function and \( \{ n < \omega : f(n) \in \bigcup_{\mu < \nu} M_\mu \} \in \overline{e}(p) \} \).
2. \( N_\nu = \{ \overline{f}(p_k) : f : \omega \to \bigcup_{\mu < \nu}(M_\mu \cup N_\mu) \) is an \( (F, e, p) \)-function and \( \{ n < \omega : f(n) \notin \bigcup_{\mu < \nu} M_\mu \} \in \overline{e}(p) \cap p_k, k < \omega \).
We have that \( M, M \subseteq \overline{M} \) and \( N, N \subseteq \overline{N} \) for every \( \nu < \omega_1 \). Our space is \( X = \bigcup_{\nu < \omega_1} (M, N) \). By definition and Lemma 1.5, \( X \) is quasi \( (\{e(p)\} \cup \{p_n : n < \omega\}) \)-compact. To prove that \( X \) is not \( q \)-compact for any \( q \in \omega^* \) is enough to show that \( X \times X \) is not countably compact (see [GS]). Assume that \( X \times X \) is countably compact and let us consider the function \( h : \omega \to X \) given by \( h(n) = q_n \), for every \( n < \omega \). Let \( \sigma : \omega \to X \times X \) be defined by \( \sigma(n) = (e(n), h(n)) = (p_n, q_n) \), for each \( n < \omega \). It is clear that \( \sigma \) is an embedding. By assumption, there is \( r \in \omega^* \) such that \( \sigma(r) \in X \times X \). Then, \( \sigma(r), h(r) \in X, r <_{RF} e(r) \) and \( r <_{RF} h(r) \). We also have that \( \sigma(r), h(r) \notin M_0 \cup N_0, \sigma(r) \in M_0 \) and \( h(r) \in N_0 \). Let \( \theta = \min\{\mu < \omega_1 : \sigma(r) \in M \} \) and \( \lambda = \min\{\mu < \omega_1 : h(r) \in M \} \). Hence, we must have that \( \sigma(r) = \overline{\sigma}(p_i) \) and \( h(r) = \overline{\sigma}(p_i) \), for some \( i < \omega \), where \( f : \omega \to \bigcup_{\mu < \theta} (M_\mu) \) is an \( (F, e, p) \)-function, \( \{n < \omega : f(n) \notin \bigcup_{\mu < \theta} M_\mu \in \sigma(p), g : \omega \to \bigcup_{\mu < \lambda} (M_\mu) \in \sigma(p) \) is an \( (F, e, p) \)-function and \( \{n < \omega : g(n) \notin \bigcup_{\mu < \lambda} M_\mu \} \in \sigma(p) \). Then, we have that \( r \) and \( p \) are \( RF \)-comparable and \( r \) and \( p_i \) are \( RF \)-comparable as well. Since \( p \) and \( p_i \) are \( RF \)-minimal, \( p \leq_{RF} r \) and \( p_i \leq_{RF} r \), but this implies, by Lemma 1.3, that \( p \) and \( p_i \) are \( RK \)-comparable, which contradicts our hypothesis. Therefore, \( X \times X \) is not countably compact.

We remark that in Example 1.6 the set \( \{p_n : n < \omega\} \) is discrete and has \( \sigma(p) \) as an accumulation point. A. Blass [BL] proved, in ZFC, that if \( \emptyset \neq M \subseteq \omega^* \) has cardinality \( < \delta \) and every element of \( M \) is generated by \( < \delta \) sets, then there is a finite-to-one function \( f : \omega \to \omega \) such that \( \overline{f}[M] \) is a free ultrafilter on \( \omega \), and hence, by Theorem 1.1, every quasi \( M \)-compact space is \( p \)-compact for some \( p \in \omega^* \). This shows that Example 1.6 cannot take place in some models of ZFC.

**Theorem 1.7.** There is a model of ZFC in which every quasi \( M \)-compact space is \( p \)-compact for some \( p \in \omega^* \), whenever \( M \in [\omega^*]^{<\tau} \).

**Proof:** The authors of [BL] showed that in the models described in [BS1] and [BS2] the following combinatorial principle holds:

(\*) If \( F \) is any free filter on \( \omega \), then there is a finite-to-one function \( f : \omega \to \omega \) such that \( f[F] \) is either the filter of cofinite sets or an ultrafilter.

Fix \( M \in [\omega^*]^{<\tau} \) and put \( F = \bigcap\{q : q \in M\} \). By (\*), there is a finite-to-one function \( f : \omega \to \omega \) such that either \( f[F] \) is the filter of cofinite sets or \( f[F] \) is an ultrafilter. If \( f[F] \) is the filter of cofinite sets, then \( \overline{f}[M] \) would be dense in \( \omega^* \), which is impossible. So, \( f[F] \) must be an ultrafilter, say \( p \), and then \( M \subseteq \overline{f}^{-1}(p) \). According to Theorem 1.1, every quasi \( M \)-compact space is \( p \)-compact.

It is a consequence of Theorem 1.7 that, under (\*), if a quasi \( M \)-compact space \( X \) is not \( p \)-compact for any \( p \in \omega^* \), then \( |M| \geq \tau \).

We turn out to the second example of this section.
Example 1.8. If $\emptyset \notin \{ T_\xi : \xi < 2^c \} \subseteq [\omega^*]<2^c$, then there is a countably compact space $X$ such that it is not quasi $T_\xi$-compact for any $\xi < 2^c$.

Proof: We will use the following fact:

If $X$ is a countable infinite subset of $\beta(\omega)$, then $|X| = 2^c$.

It is well-known that there are $2^c$-many weak $P$-points in $\omega^*$ (see [vM]). We partition the set of weak $P$-points of $\omega^*$ in countable infinite subsets and enumerate them as $\{ S_\xi : \xi < 2^c \}$. Now, for each $\xi < 2^c$, we fix a bijection $h_\xi : \omega \to S_\xi$. We shall use the standard method of constructing countably compact subspaces of $\omega^*$. We put $Y_0 = S_0 - h_0[T_0]$. Suppose that for each $\xi < \lambda < 2^c$ we have defined $Y_\xi \subseteq \omega^*$ such that

1. $Y_\xi \subseteq \bigcup \{ X : X \in \{ \bigcup_{\zeta \leq \xi} S_\zeta \}^\omega \}$ for each $1 \leq \xi < \lambda$;
2. $Y_\xi \subseteq Y_\zeta$ whenever $\xi < \zeta < \lambda$;
3. every countable discrete infinite subset of $Y_\xi$ has an accumulation point in $Y_{\xi+1}$ for each $\xi < \xi + 1 < \lambda$; and
4. $S_\xi \subseteq Y_\xi \subseteq \omega^* \setminus \left[ \left( \bigcup_{\zeta \leq \xi} h_\xi[T_\zeta] \right) \cup \left( \bigcup_{\xi < \zeta < 2^c} S_\zeta \right) \right]$ for each $\xi < \lambda$.

Define $Y = S_\lambda \cup (\bigcup_{\xi < \lambda} Y_\xi)$. First notice that $Y \cap h_\xi[T_\xi] = \emptyset$ for every $\xi < 2^c$. We enumerate all countable discrete infinite subsets of $Y$ as $\{ D_\theta : \theta < |Y| = \kappa \}$. Without loss of generality, we may assume that either $D_\theta \subseteq \bigcup_{\xi < \lambda} Y_\xi$ or $D_\theta \subseteq S_\lambda$.

By Lemma 1.3, $\overline{D_\theta} \cap (\bigcup_{\lambda < \zeta < 2^c} S_\zeta) = \emptyset$, for each $\theta < \kappa$, and $S_\zeta \cap Y_\xi = \emptyset$ whenever $\xi < \zeta \leq \lambda$. For each $\theta < \kappa$, we choose $p_\theta \in \overline{D_\theta}$ as follows:

Suppose that $D_\theta \subseteq \bigcup_{\xi < \lambda} Y_\xi$. By 1, there is a countable subset $I$ of $\lambda$ such that $D_\theta \subseteq \bigcup_{\xi \in I} S_\xi$. Since $|\bigcup_{\xi \in I} h_\xi[T_\xi]| < 2^c$, we may choose $p_\theta \in \overline{D_\theta} \setminus \bigcup_{\xi \in I} h_\xi[T_\xi]$ (by the fact).

If $D_\theta \subseteq S_\lambda$, then we pick any $p_\theta \in \overline{D_\theta} \setminus h[T_\lambda]$, this is possible by the fact.

Then, we define $Y_\lambda = Y \cup \{ p_\theta : \theta < \kappa \}$. It is clear that $Y_\lambda$ satisfies all the conditions. Finally, we put $X = \bigcup_{\lambda < 2^c} Y_\lambda$. By clauses 2 and 3 and the fact that $cf(2^c) > \omega$, $X$ is countably compact and, by clause 4, $X$ is not quasi $T_\xi$-compact for every $\xi < 2^c$.

In particular, if $2^c = (2^c)^{<2^c}$, then there is a countably compact space $X$ such that $X$ is not quasi $M$-compact for any $M \in [\omega^*]^{<2^c}$: The equality $2^c = (2^c)^{<2^c}$ holds when $2^c$ is a regular cardinal. It should be remark that if $X$ is a countably compact space of size $\tau$, then there is $M \in [\omega^*]^{\leq \tau}$ such that $X$ is quasi $M$-compact.

Question 1.9. For each cardinal $\kappa < 2^c$, is there a countably compact space $X$ such that $X^{\kappa}$ is countably compact, and $X$ is not quasi $M$-compact for any $M \in [\omega^*]^{<2^c}$?

By making some minor changes, for each $1 < n < \omega$, we may construct a space $X$ like in Example 1.8 with the additional property that $X^n$ is countably compact.
Question 1.10. Is there a countably compact space $X$ and $M \in [\omega^*]^{\omega_1}$ such that $X$ is quasi $M$-compact, and $X$ is not $N$-compact for any $N \in [\omega^*]^{\leq \omega}$?

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