Yves Dutrieux
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*Commentationes Mathematicae Universitatis Caroliniae*, Vol. 42 (2001), No. 4, 641--648


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Lipschitz-quotients and the Kunen-Martin Theorem

Yves Dutrieux

Abstract. We show that there is a universal control on the Szlenk index of a Lipschitz-quotient of a Banach space with countable Szlenk index. It is in particular the case when two Banach spaces are Lipschitz-homeomorphic. This provides information on the Cantor index of scattered compact sets $K$ and $L$ such that $C(L)$ is a Lipschitz-quotient of $C(K)$ (that is the case in particular when these two spaces are Lipschitz-homeomorphic). The proof requires tools of descriptive set theory.

Keywords: Lipschitz equivalences, Szenk index

Classification: 03E15, 46B20

In the non-linear classification of Banach spaces, it is an open problem to know whether two separable Lipschitz-homeomorphic Banach spaces are isomorphic. Several partial results appeared recently on the subject. We refer to [10] (especially Chapters 7 and 11) for an up-to-date account of the theory. In Theorem 3.18 of [2], it is shown that the class of Asplund spaces is stable under Lipschitz-quotient (this is false under uniform homeomorphism; see Theorem 1 in [12]). The aim of this paper is to precise this result: we show that there exists a universal control on the Szlenk index of a Lipschitz-quotient of a Banach space $X$, provided $X$ has a countable Szlenk index. For that, we need to estimate the topological complexity of the relation of Lipschitz-quotient and apply the Kunen-Martin theorem.

1. Analyticity of the relation of Lipschitz quotient

The aim of this section is to prove that the relation of Lipschitz-quotient (see Definitions 3.1 and 3.2 in [2]) is analytic in a sense which will be made precise later. First, let us introduce some notation:

Notation. • $E$ will denote the space $C(2^\omega)$ of all continuous functions on the Cantor set. Let us recall that $E$ is universal for all separable Banach spaces.
• $S$ will denote the set of all closed subspaces of $E$. It is shown in Proposition 2.1 of [3] (see also pages 15 and 16) that the restriction of the Effros Borel structure on the closed subsets of $E$ makes it into a standard Borel set.
• If $X$ and $Y$ are two Banach spaces, the fact that $Y$ is a Lipschitz-quotient of $X$ will be written $X \rightarrow_{\ell} Y$. 
When we say that the relation of Lipschitz-quotient is analytic, we mean that the set \( \{(X,Y) \in S^2; X \to \ell Y\} \) is analytic in the standard Borel structure of \( S \) (see Definition 0.4, page 9 in [8]).

We will show the following crucial technical proposition:

**Proposition 1.** \( \to \ell \) is analytic.

Let us introduce some more notation:

**Notation.**

- The sequence of the vectors \( x_n \) will be denoted by \( x \).
- When the sequence \( x \) is dense in \( X \), we write \( X = \overline{x} \).
- \( x \) and \( y \) being two sequences of vectors, we will write \( x \to \ell y \) to mean that there exist two constants \( L \) and \( C \) in \( \omega \) such that 
  \[
  \forall k,l \in \omega, \quad \|y_k - y_l\| \leq L \|x_k - x_l\|.
  \]
  and such that, for any \( n,p \in \omega \) and any \( r \in \mathbb{Q}_+^* \) such that \( \|y_p - y_n\| \leq r/C \), there exists a convergent subsequence \( x_\varphi = (x_\varphi(m))_{m \in \omega} \) verifying:
  \[
  x_\varphi \in B_X(x_n, r)^\omega \quad \text{and} \quad y_\varphi \to y_p.
  \]

The link between \( \to \ell \) for spaces and \( \to \ell \) for sequences is given by the following lemma:

**Lemma 2.** Let \( X \) and \( Y \) be two separable Banach spaces. Then \( X \to \ell Y \) if and only if there exist two sequences \( x \) and \( y \) such that \( X = \overline{x} \), \( Y = \overline{y} \) and \( x \to \ell y \).

**Proof:** If there exists a \( L \)-Lipschitz and \( C \)-co-Lipschitz map \( f \) from \( X \) to \( Y \) then, taking any dense sequence \( x \) and defining \( y \) as the image of \( x \) by \( f \), we clearly have

\[
\forall k,l \in \omega, \quad \|y_k - y_l\| \leq L \|x_k - x_l\|.
\]

Moreover, let \( n,p \in \omega \) and \( r \in \mathbb{Q}_+^* \) be such that \( \|y_p - y_n\| \leq r/C \). Then, \( y_p \in f(B_X(x_n, r)) \). Since there is a preimage \( x \) of \( y_p \) in \( B_X(x_n, r) \), there exists a subsequence \( x_\varphi \) of \( x \) in the open ball such that \( x_\varphi \to x \). Then \( y_\varphi \to f(x) = y_p \).

Conversely, let us suppose that \( X = \overline{x} \), \( Y = \overline{y} \) and \( x \to \ell y \) with constants \( L \) and \( C \). We can define \( f : X \to Y \) by \( f(x_n) = y_n \) for all \( n \in \omega \) and \( f \) is \( L \)-Lipschitz. Moreover \( f \) clearly satisfies:

\[
\forall n,p \in \omega, \forall r \in \mathbb{Q}_+^*, \quad \|y_p - y_n\| \leq \frac{r}{C}, \quad \exists x \in B_X(x_n, r), \ f(x) = y_p.
\]

Let us state and prove some facts:
Fact 1. For every \( x \in X, \ p \in \omega, \ r \in \mathbb{Q}_+^* \) and \( C' > C \) such that the inequality \( \|y_p - f(x)\| \leq r/C' \) holds, there exists \( z \in B_X(x, r) \) such that \( f(z) = y_p \).

Let \( x_\varphi \) be a subsequence of \( x \) converging to \( x \) and verifying, for all \( n \in \omega \), \( \|x - x_\varphi(n)\| \leq k \), \( k > 0 \) being chosen such that \( Lk + r/C' \leq r/C'' \), with \( C'' > C \) and \( C''/C \in \mathbb{Q} \). Then we have \( \|y_p - f(x_\varphi(n))\| \leq r/C'' \). By (1), there exists \( z_n \in B_X(x_\varphi(n), C r/C'') \) such that \( f(z_n) = y_p \). Since \( x_\varphi \to x \), for \( n \) large enough, \( z_n \in B(x, r) \). Taking \( z = z_n \) for such an \( n \) gives the result.

Fact 2. \( f \) is surjective.

Let \( y \in Y \) and let \( y_\varphi \) be a subsequence such that \( \|y - y_\varphi(n)\| \leq 2^{-n-1}/C' \) \((C' > C)\), for all \( n \in \omega \). Applying Fact 1, one can define by induction a sequence \( z \) such that \( z_0 = x_\varphi(0), \|z_{k+1} - z_k\| \leq 2^{-k} \) and \( f(z_k) = y_\varphi(k) \) for all \( k \in \omega \). The limit \( z \) of \( z \) satisfies \( f(z) = y \).

Fact 3. For every \( C' > C \), \( f \) is \( C' \)-co-Lipschitz.

The proof is similar to the proof of Fact 2 and will be omitted.

Finally, \( f \) is a Lipschitz-quotient map from \( X \) to \( Y \) and \( X \to \ell Y \). \( \square \)

We now give a characterization of the condition \( x \to \ell y \) which is useful for our purpose. We denote by \( G \) the set of all infinite subsets of \( \omega \). As a \( G_\delta \) set of a compact set, it is a Polish space. Let us also define

\[ G = G^\omega \times \omega \times \mathbb{Q}_+^*. \]

Lemma 3. Let \( x \) and \( y \) be two sequences of vectors. The condition \( x \to \ell y \) is equivalent to the existence of \( P \in G \) such that the conjunction of the following two conditions holds:

1. There exists \( L \in \omega \) such that

\[ \forall k, l \in \omega, \quad \|y_k - y_l\| \leq L \|x_k - x_l\|. \]

This first condition will be denoted by \( L(x, y) \).

2. There exists \( C \in \omega \) that satisfies: for any \( n, p \in \omega \) and \( r \in \mathbb{Q}_+^* \) such that \( \|y_p - y_n\| \leq r/C \), we have \( \|x_m - x_n\| \leq r \) for all \( m \in P_{n, p, r} \) and

\[ \forall q \in \omega, \exists Q \in 2^{<\omega}; \forall m', m \in P_{n, p, r} \setminus Q, \quad \|x_{m'} - x_m\| + \|y_m - y_p\| \leq 1/q. \]

This second condition will be denoted by \( C(x, y, P) \).
PROOF: It is an easy reformulation of the condition $\mathbf{x} \rightarrow_{\ell} \mathbf{y}$: for a given $(n,p,r)$, $P_{n,p,r}$ is the set $\{\varphi(m); m \in \omega\}$ where $x_{\varphi}$ is the subsequence of the definition of $\mathbf{x} \rightarrow_{\ell} \mathbf{y}$.

□

Lemma 4. Let $\mathcal{A}$ be the set
\[ \left\{ (X,Y,\mathbf{x},\mathbf{y},P) \in S^{2} \times (E^{\omega})^{2} \times \mathcal{G}; \, X = \overline{\mathbf{x}}, \, Y = \overline{\mathbf{y}}, \, L(\mathbf{x},\mathbf{y}), \, C(\mathbf{x},\mathbf{y},P) \right\}. \]

Then $\mathcal{A}$ is a Borel set.

PROOF: It is enough to see that the sets
\[ \mathcal{B} = \{(X,\mathbf{x}) \in E \times E^{\omega}; \, X = \overline{\mathbf{x}}\}, \quad \mathcal{C} = \{(\mathbf{x},\mathbf{y}) \in (E^{\omega})^{2}; \, L(\mathbf{x},\mathbf{y})\} \]
\[ \text{and} \quad \mathcal{D} = \{(\mathbf{x},\mathbf{y},P) \in (E^{\omega})^{2} \times \mathcal{G}; \, C(\mathbf{x},\mathbf{y},P)\} \]
are Borel sets.

It is easy to check that $\mathcal{C}$ is an $F_{\sigma}$.

Let us define $\mathcal{O}$ a countable basis of the topology of $E$. Recall that the Effros Borel structure on the closed subsets of $E$ is generated by the basis:
\[ \left( \{F \subseteq E; \, O \cap F \neq \emptyset\} \right)_{O \in \mathcal{O}}. \]

$X = \overline{\mathbf{x}}$ is equivalent to the two conditions:

(i) $x_{n} \in O$ implies $O \cap X \neq \emptyset$, for all $n \in \omega$ and all $O \in \mathcal{O}$.

(ii) For all $O \in \mathcal{O}$, $O \cap X \neq \emptyset$ implies that there exists $n \in \omega$ such that $x_{n} \in O$.

Then, it is easy to see that $\mathcal{B}$ is a Borel set.

$\mathcal{D}$ is the union over $\mathcal{C}$ of the intersection over $n,p,r$ of:
\[ \left\{ \|y_{n} - y_{p}\| > r/C \right\} \cup \left( \bigcap_{m \in \omega} \left( \{m \notin P_{n,p,r}\} \cup \left\{ \|x_{m} - x_{n}\| \leq r\right\} \right) \cap \left( \bigcap_{q \in \omega} \bigcup_{Q \in 2^{\omega \setminus m,m' \in \omega}} \left[ \{m \notin P_{n,p,r} \text{ or } m' \notin P_{n,p,r}\} \cup \left\{ \|x_{m'} - x_{m}\| + \|y_{m} - y_{p}\| \leq 1/q\right\} \right] \right). \]

Therefore, $\mathcal{D}$ is a Borel set.

The set $\{(X,Y); \mathbf{x} \rightarrow_{\ell} \mathbf{y}\}$ being the projection on the first two coordinates of the set $\mathcal{A}$, it is analytic. This concludes the proof of our technical proposition.
Before investigating the consequences of Proposition 1, let us add some more details on the Lipschitz-homeomorphisms between Banach spaces. In Theorem 2.4 of [3], Benoît Bossard proved that the linear isomorphism relation is analytic and non Borel. It is therefore natural to ask whether the Lipschitz-homeomorphism relation is also non Borel.

**Notation.** Let $X$ and $Y$ be two subspaces of $E$. When $X$ and $Y$ are Lipschitz-homeomorphic, we write $X \sim_\ell Y$.

**Proposition 5.** The relation $\sim_\ell$ is analytic and non Borel.

**Proof:** The proof of the analyticity of $\sim_\ell$ is similar to (and technically simpler than) the proof of the analyticity of $\rightarrow_\ell$. It will thus be omitted.

Let us show that $\sim_\ell$ is non Borel. Let us introduce $C = \omega^{<\omega}$ and the group $G = 2^C$. $G$ is isomorphic to the Cantor group. Let $p$ be a real number greater than 1 and different from 2. It suffices for our purpose to show that the set $L = \{ X \in S; X \sim_\ell L_p(G) \}$ is non Borel.

The dual of $G$ is the group $\hat{G}$ of all finite subsets of $C$ where we identify $b$, a finite subset of $C$, and its Walsh function $w_b$. For any tree $T$ on $\omega$, let us define the set $FB(T)$ of all finite branches of $T$. The space $L^T_p$ is the closed (for the $L_p$ norm) linear span of the set $\{ w_b; b \in FB(T) \}$. Theorem 4.34 in [7] shows that all the spaces $L^T_p$ are complemented subspaces of $L_p(G)$. According to Theorem 4.35 in [7], $L_p(G)$ does not embed in $L^T_p$ if $T$ is well-founded (that we write $T \notin WF$).

Conversely, if $T$ has an infinite branch, then obviously $L_p(G)$ is isomorphic to a complemented subspace of $L^T_p$. Pełczyński’s decomposition method then implies that $L_p(G)$ is isomorphic to $L^T_p$ if and only if $T \notin WF$. Now we need the following fact:

**Fact 4.** The map $\theta$ defined on the set $T$ of all trees on $\omega$ by $\theta(T) = L^T_p$ is Borel.

Let $O$ be an open set of $E$. It is enough to show that the set $\Omega = \{ T \in T; \theta(T) \cap O \neq \emptyset \}$ is Borel. Since $\theta(T) = \overline{\text{span}}\{ w_b; b \in FB(T) \}$, we have, defining $\Lambda = \{ (\lambda_b) \in \mathbb{Q}^{FB(C)}; \sum_b \lambda_b w_b \in O \}$:

$$\Omega = \bigcup_{(\lambda_b) \in \Lambda} \bigcap_{\{ b; \lambda_b \neq 0 \}} \{ T; b \subseteq T \}.$$

It is now clear that $\Omega$ is a Borel set, which ends the proof of Fact 4.

According to Corollary 2.9 in [6], $\mathcal{L} = \{ X \in S; X \text{ isomorphic to } L_p(G) \}$. Thus, $\mathcal{L} = \theta(T \setminus WF)$ is non Borel. Indeed, if it was Borel then, since $T \setminus WF = \theta^{-1}(\mathcal{L})$ and $\theta$ is Borel, $T \setminus WF$ would be Borel which is absurd. \(\square\)

It would come as a very big surprise for us if the relation of Lipschitz-quotient is actually Borel.
2. Control on the Szlenk index of a Lipschitz quotient

Our main result is a consequence of Proposition 1:

**Theorem 6.** There exists a universal function \( \psi_1 : \omega_1 \rightarrow \omega_1 \) such that, if \( X \) is a Banach space with countable Szlenk index and \( Y \) a Lipschitz-quotient of \( X \), then \( \text{Sz}(Y) \leq \psi_1(\text{Sz}(X)) \).

**Proof:** Let us recall that, for separable Banach spaces, having a countable Szlenk index is equivalent to having a separable dual (see Proposition 4.12 of [3] for example). Thus, we will show that the general case boils down to the separable case and then use Theorem 3.18 of [2] concerning Asplund spaces.

According to Corollary 3.17 in [2], if \( f \) is a Lipschitz-quotient from a Banach space \( X \) onto another Banach space \( Y \), then for any separable subspaces \( X_0 \) and \( Y_0 \) in \( X \) and \( Y \) respectively, there exist \( X_1 \) and \( Y_1 \), separable subspaces of \( X \) and \( Y \) respectively such that \( X_0 \subseteq X_1 \), \( Y_0 \subseteq Y_1 \) and the restriction of \( f \) to \( X_1 \) is a Lipschitz quotient mapping from \( X_1 \) onto \( Y_1 \). Moreover, the Szlenk index of a Banach space, when countable, is the supremum of the Szlenk indices of its separable subspaces (Proposition 3.1 in [4]). Thus, it is enough to deal with separable Banach spaces in our proof. Since the Szlenk index is invariant under linear isomorphism and since \( E \) is universal for separable Banach spaces, we can restrict our study to subspaces of \( E \). It is shown in Lemma 3.5 and Theorem 4.13 of [3] that the set of all separable Asplund subspaces of \( E \) is a \( \Pi^1_1 \)-rank on it (see page 140 of [5] for a definition of \( \Pi^1_1 \)-rank). For any ordinal \( \xi \), let us call \( S_\xi \) the set of all subspaces of \( E \) whose Szlenk index is less than or equal to \( \xi \) and \( P_\xi \) the set of all subspaces of \( E \) Lipschitz homeomorphic to some element of \( S_\xi \). With this notation, \( S_{\omega_1} \) is the \( \Pi^1_1 \)-analytic set of all Asplund subspaces of \( E \). Let \( \xi \) be a countable ordinal. The set \( S_\xi \) is Borel. According to Proposition 1, the set \( H = \{(X,Y); X \in S_\xi \text{ and } X \rightarrow Y \} \) is analytic. Since \( P_\xi \) is the projection of \( H \) on the second coordinate, it is also analytic. Theorem 3.18 in [2] shows that \( P_\xi \) is included in \( S_{\omega_1} \). Kunen-Martin’s theorem (see Theorem 7 p. 148 in [5] for instance) then proves that \( P_\xi \) is included in \( S_\xi \) for some countable ordinal \( \zeta \). We can define \( \psi_1 \) by \( \psi_1(\xi) = \zeta \).

In the special case of Lipschitz-homeomorphisms, we obtain the following result:

**Corollary 7.** There exists a universal function \( \psi_2 : \omega_1 \rightarrow \omega_1 \) such that, if \( X \) is a Banach space with a countable Szlenk index and \( Y \) is a Banach space which is Lipschitz-homeomorphic to \( X \), then \( \text{Sz}(Y) \leq \psi_2(\text{Sz}(X)) \).

Theorem 5.5 in [1] proves that, if \( X \) and \( Y \) are uniformly homeomorphic, then \( \text{Sz}(X) \leq \omega \) if and only if \( \text{Sz}(Y) \leq \omega \). Thus, if we consider the minimal choices for \( \psi_1 \) and \( \psi_2 \), we have \( \psi_2(\omega) = \omega \). It is not clear to us whether \( \psi_2(\omega^2) \) equals \( \omega^2 \). We do not know either the value of \( \psi_1(\omega) \). More generally, it could be possible that, in fact, \( \psi_1 \) and \( \psi_2 \) are simply the identity.
As a corollary of Theorem 6, we get the following theorem about the Cantor index of scattered compact sets:

**Corollary 8.** There exists a universal function \( \lambda : \omega_1 \to \omega_1 \) such that, if \( K \) is a scattered compact with a countable derivative empty and if \( C(L) \) is a Lipschitz-quotient of \( C(K) \), then the Cantor index \( i(L) \) of \( L \) is less than or equal to \( \lambda(i(K)) \).

**Proof:** This corollary is a straightforward consequence of Theorem 6 and of Theorem 5.1 in [4]. \( \square \)

Example 4.9 from [11] shows that there exist two non metrizable scattered compact sets \( K \) and \( L \) with a countable derivative empty such that \( C(K) \) and \( C(L) \) are Lipschitz-isomorphic but not isomorphic. Thus Corollaries 7 and 8 deal with a situation which is known not to be linear. In the linear case, the Bessaga-Pelczyński result (see Theorem 3 in [13]) gives a necessary and sufficient condition for two countable compact sets \( K \) and \( L \) to be such that \( C(K) \) is isomorphic to \( C(L) \) (namely, that \( i(K) < i(L) \cdot \omega \) and conversely). Thus, in the context of countable compact sets, it would be natural to compare \( \lambda \) and the function \( \xi \mapsto \xi \cdot \omega \).

**References**


Université Pierre et Marie Curie, Equipe d'Analyse fonctionnelle, Boîte 186, 4, place Jussieu, 75005 Paris, France

E-mail: dutrieux@ccr.jussieu.fr

(Received January 12, 2001, revised July 6, 2001)