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Natural affinors on \((J^{r,s,q}(\cdot, \mathbb{R}^{1,1})_0)^*\)

\[\text{Włodzimierz M. Mikulski}\]

Abstract. Let \(r, s, q, m, n \in \mathbb{N}\) be such that \(s \geq r \leq q\). Let \(Y\) be a fibered manifold with \(m\)-dimensional basis and \(n\)-dimensional fibers. All natural affinors on \((J^{r,s,q}(Y, \mathbb{R}^{1,1})_0)^*\) are classified. It is deduced that there is no natural generalized connection on \((J^{r,s,q}(Y, \mathbb{R}^{1,1})_0)^*\). Similar problems with \((J^{r,s}(Y, \mathbb{R})_0)^*\) instead of \((J^{r,s,q}(Y, \mathbb{R}^{1,1})_0)^*\) are solved.

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0. Let us recall the following definitions (see e.g. [3]).

Let \(F : \mathcal{FM}_{m,n} \to \mathcal{FM}\) be a functor from the category \(\mathcal{FM}_{m,n}\) of all fibered manifolds with \(m\)-dimensional bases and \(n\)-dimensional fibers and their local fibered diffeomorphisms into the category \(\mathcal{FM}\) of fibered manifolds and fibered maps. Let \(B : \mathcal{FM} \to \mathcal{Mf}\) be the base functor from \(\mathcal{FM}\) into the category \(\mathcal{Mf}\) of manifolds. Let \(T : \mathcal{FM} \to \mathcal{Mf}\) be the total space functor.

A bundle functor over \(\mathcal{FM}_{m,n}\) is a (covariant) functor \(F\) satisfying \(B \circ F = T\mathcal{FM}_{m,n}\) and the localization condition: for every inclusion of an open subset \(i_U : U \to Y\), \(FU\) is the restriction \(p_Y^{-1}(U)\) of \(p_Y : FY \to Y\) over \(U\) and \(Fi_U\) is the inclusion \(p_Y^{-1}(U) \to FY\).

An affinor \(D\) on a manifold \(M\) is a tensor type (1, 1), i.e. a linear morphism \(D : TM \to TM\) over \(\text{id}_M\).

A natural affinor on a bundle functor \(F\) is a system of affinors \(D : TFY \to TFY\) on \(FY\) for every \(\mathcal{FM}_{m,n}\)-object \(Y\) satisfying \(TFf \circ D = D \circ TFf\) for every local \(\mathcal{FM}_{m,n}\)-diffeomorphism \(f : Y \to \overline{Y}\).

A connection on a fibre bundle \(Z\) is an affinor \(\Gamma : TZ \to TZ\) on \(Z\) such that \(\Gamma \circ \Gamma = \Gamma\) and \(\text{im}(\Gamma) = VZ\), the vertical bundle of \(Z\).

A natural connection on a bundle functor \(F\) is a system of connections \(\Gamma : TFY \to TFY\) on \(FY\) for every \(\mathcal{FM}_{m,n}\)-object \(Y\) which is (additionally) a natural affinor on \(F\).

In [5] it was shown how natural affinors \(Q\) on some bundle functor \(FY\) can be used to study the torsion \(\tau = [\Gamma, Q]\) of connections \(\Gamma\) on \(FY\). That is why, natural affinors have been classified in many papers, [1], [2], [7]–[11]. For example, in [2] natural affinors on the \(r\)-th order vector tangent bundle \((J^r(M, \mathbb{R})_0)^*\) over \(m\)-manifolds \(M \in \text{obj}(\mathcal{FM}_{m,0})\) were classified.
In this paper we fix numbers \( r, s, q, m, n \in \mathbb{N} \) such that \( s \geq r \leq q \) and consider the bundle functor \( F = T^{(r,s,q)}_{\mathcal{FM}_{m,n}} \), where \( T^{(r,s,q)}_{\mathcal{FM}} = (J^{r,s,q}(. \times \mathbb{R}^{1,1})_0)^* : \mathcal{FM} \to \mathcal{FM} \) is the (introduced in [4]) bundle functor associating to every fibered manifold \( Y \) the vector bundle \( (J^{r,s,q}(Y, \mathbb{R}^{1,1})_0)^* \) over \( Y \). We prove that the set of all natural affinors on \( T^{(r,s,q)}_{\mathcal{FM}_{m,n}} \) is a 3-dimensional vector space over \( \mathbb{R} \) and we construct explicitly the basis of this vector space.

We also solve the similar problem with \( T^{(r,s)} = (J^{r,s}(. \times \mathbb{R}^{1,1})_0)^* : \mathcal{FM} \to \mathcal{FM} \) instead of \( T^{(r,s,q)} \).

As an application of the obtained results we deduce that there are no natural connections on \( T^{(r,s,q)} \) and \( T^{(r,s)} \).

The above results extend [2].

Throughout this paper \( r, s, q, m, n \in \mathbb{N} \) are numbers with \( s \geq r \leq q \).

The usual fiber coordinates on \( \mathbb{R}^{m,n} \), the trivial bundle \( \mathbb{R}^m \times \mathbb{R}^n \) over \( \mathbb{R}^m \), are denoted by \( x^1, \ldots, x^m, y^1, \ldots, y^n \).

All manifolds and maps are assumed to be of class \( C^\infty \).

1. The concept of classical \( r \)-jets can be generalized as follows. Let \( Y \to M \) and \( Z \to N \) be fibered manifolds. We recall that two \( \mathcal{FM} \)-morphisms \( f, g : Y \to Z \) with base maps \( \bar{f}, \bar{g} : M \to N \) determine the same \( (r, s, q) \)-jet \( j^r,s,q_y f = j^r,s,q_y g \) at \( y \in Y_x, \ x \in M \), if \( j^r_y f = j^r_y g, \ j^s_y(f|_{Y_x}) = j^s_y(g|_{Y_x}) \) and \( j^q_x f = j^q_x g \). The space of all \( (r, s, q) \)-jets of \( Y \) into \( Z \) is denoted by \( J^{r,s,q}(Y, Z) \). The composition of \( \mathcal{FM} \)-morphisms induces the composition of \( (r, s, q) \)-jets ([3, p. 126]).

The space \( T^{r,s,q,*}_y Y = J^{r,s,q}(Y, \mathbb{R}^{1,1})_0, \ 0 \in \mathbb{R}^2 \), has an induced structure of a vector bundle over \( Y \). Every \( \mathcal{FM} \)-morphism \( f : Y \to Z, \ f(y) = z \), induces a linear map \( \lambda(j^r_y,s,q^y f) : T^{r,s,q,*}_y Z \to T^{r,s,q,*}_y Y \) by means of the jet composition. If we denote by \( T^{(r,s,q)}_y Y \) the dual vector bundle of \( T^{r,s,q,*}Y \) and define \( T^{(r,s,q)}_y Y \to T^{(r,s,q)}_y Z \) by using the dual maps to \( \lambda(j^r_y,s,q^y f) \), we obtain (similarly as in [3, p. 123]) a vector bundle functor \( T^{(r,s,q)}_y \) on \( \mathcal{FM} \), see [4].

2. In this section all natural transformations \( T^{(r,s,q)} \to T^{(r,s,q)} \) over \( \mathcal{FM}_{m,n} \) will be classified. This extends [6].

A natural transformation \( T^{(r,s,q)} \to T^{(r,s,q)} \) over \( \mathcal{FM}_{m,n} \) is a system of fibered maps \( A : T^{(r,s,q)}_y Y \to T^{(r,s,q)}_y Y \) covering the identity \( \text{id}_Y \) for every \( \mathcal{FM}_{m,n} \)-object \( Y \) satisfying \( T^{(r,s,q)} f \circ A = A \circ T^{(r,s,q)} f \) for every local \( \mathcal{FM}_{m,n} \)-map \( f : Y \to Y \).

Example 1. Let \( Y \) be an \( \mathcal{FM}_{m,n} \)-object. For a fibered map \( \gamma = (\gamma^1, \gamma^2) : Y \to \mathbb{R}^{1,1} \) we have fibered maps \( \gamma^{(1)} = (\gamma^1, 0), \ \gamma^{(2)} = (0, \gamma^2), \ \gamma^{(3)} = (0, \gamma^1) : Y \to \mathbb{R}^{1,1} \). Clearly, \( j^r_y,s,q^{(1)}_y, j^r_y,s,q^{(2)}_y, j^r_y,s,q^{(3)}_y \) depend linearly on \( j^r_y,s,q^y \gamma \) for \( y \in Y \). Define
fibered maps $P_{r}^{(1)}, P_{r}^{(2)}, P_{r}^{(3)} : T(r,s,q)Y \to T(r,s,q)Y$ over id$_{Y}$ by

$$\langle P_{r}^{(1)}(\omega), j_{r,s,q}^{r,s,q,\gamma}\rangle = \langle \omega, j_{r,s,q}^{r,s,q,\gamma}^{(1)}\rangle,$$

$$\langle P_{r}^{(2)}(\omega), j_{r,s,q}^{r,s,q,\gamma}\rangle = \langle \omega, j_{r,s,q}^{r,s,q,\gamma}^{(2)}\rangle,$$

$$\langle P_{r}^{(3)}(\omega), j_{r,s,q}^{r,s,q,\gamma}\rangle = \langle \omega, j_{r,s,q}^{r,s,q,\gamma}^{(3)}\rangle,$$

$\omega \in T_{y}(r,s,q)Y$, $y \in Y$, $\gamma = (\gamma_{1,}, \gamma_{2}) : Y \to \mathbb{R}^{1,1}$ is fibered, $\gamma(y) = 0$. The families $P_{r}^{(1)}, P_{r}^{(2)}, P_{r}^{(3)} : T(r,s,q) \to T(r,s,q)$ are natural transformations over $\mathcal{FM}_{m,n}$.

**Proposition 1.** Every natural transformation $A : T(r,s,q) \to T(r,s,q)$ over $\mathcal{FM}_{m,n}$ is a linear combination of $P_{r}^{(1)}, P_{r}^{(2)}$ and $P_{r}^{(3)}$.

**Proof:** The elements $j_{0}^{r,s,q}(x^{\alpha}, 0)$ and $j_{0}^{r,s,q}(0, x^{\beta}y^{\delta})$ for multiindices $\alpha$ and $(\beta, \delta)$ from obvious sets form the basis of $J_{0}^{r,s,q}(\mathbb{R}^{m,n}, \mathbb{R}^{1,1})$. Then (by the naturality) $A$ is uniquely determined by the contractions $\langle A(\omega), j_{0}^{r,s,q}(x^{1}, y^{1})\rangle$ for all $\omega \in T_{0}^{(r,s,q)}\mathbb{R}^{m,n}$. So, it suffices to deduce that $\langle A(\cdot), j_{0}^{r,s,q}(x^{1}, y^{1})\rangle : T_{0}^{(r,s,q)}\mathbb{R}^{m,n} \to \mathbb{R}$ is a linear combination of $j_{0}^{r,s,q}(x^{1}, 0), j_{0}^{r,s,q}(0, x^{1}), j_{0}^{r,s,q}(0, y^{1}) : T_{0}^{(r,s,q)}\mathbb{R}^{m,n} \to \mathbb{R}$, i.e. that the vector space of all $A$ as above has dimension $\leq 3$.

By the naturality of $A$ with respect to the homotheties $a_{t} = t \text{id}_{\mathbb{R}^{m,n}} : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ for $t \neq 0$ and the homogeneous function theorem (see [3]), we deduce that $\langle A(\cdot), j_{0}^{r,s,q}(x^{1}, y^{1})\rangle$ is a linear combination of $j_{0}^{r,s,q}(x^{i}, 0)$, $j_{0}^{r,s,q}(0, x^{i})$ and $j_{0}^{r,s,q}(0, y^{j})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Next, using the naturality of $A$ with respect to the fibered maps $b_{t} = (x^{1}, tx^{2}, \ldots, tx^{n}, y^{1}, ty^{2}, \ldots, ty^{n}) : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ for $t \neq 0$ we finish the proof. \qed

**3.** In this section all linear natural transformations $TT(r,s,q) \to T(r,s,q)$ over $\mathcal{FM}_{m,n}$ will be classified.

A natural transformation $TT(r,s,q) \to T(r,s,q)$ over $\mathcal{FM}_{m,n}$ is a system of fibered maps $B : TT(r,s,q)Y \to T(r,s,q)Y$ covering the identity id$_{Y}$ for every $\mathcal{FM}_{m,n}$-object $Y$ satisfying $T(r,s,q)f \circ B = B \circ TT(r,s,q)f$ for every local $\mathcal{FM}_{m,n}$-diffeomorphism $f : Y \to Y$. The linearity of $B : TT(r,s,q) \to T(r,s,q)$ means that the restriction and corestriction $B_{\omega} : T_{\omega}T(r,s,q)Y \to Y_{Y}$ of $B : TT(r,s,q)Y \to T(r,s,q)Y$ is linear for any $\omega \in T_{y}(r,s,q)Y$, $y \in Y$ and $Y \in \text{obj}(\mathcal{FM}_{m,n})$.

**Example 2.** Given an $\mathcal{FM}_{m,n}$-object $Y$ let $B^{(1)}, B^{(2)} : TT(r,s,q)Y \to T(r,s,q)Y$ be fibered maps over id$_{Y}$ such that

$$\langle B^{(1)}(v), j_{0}^{r,s,q,\gamma}\rangle = d_{y}\gamma^{1}(T\pi(v)),$$

$$\langle B^{(2)}(v), j_{0}^{r,s,q,\gamma}\rangle = d_{y}\gamma^{2}(T\pi(v)),$$
\( v \in (TT^{(r,s,q)})_y Y, \ y \in Y, \ \gamma = (\gamma_1, \gamma_2) : Y \to \mathbb{R}^{1,1} \) is fibered, \( \gamma(y) = 0 \), where \( \pi : T^{(r,s,q)} Y \to Y \) is the bundle projection, \( T\pi : TT^{(r,s,q)} Y \to TY \) is its tangent map and \( d_y \gamma_1 : T Y \to \mathbb{R} \) is the differential of \( \gamma_1 \) at \( y \). Then \( B^{(1)}, B^{(2)} : TT^{(r,s,q)} \to T^{(r,s,q)} \) are linear natural transformations over \( \mathcal{F}M_{m,n} \).

**Proposition 2.** Every linear natural transformation \( B : TT^{(r,s,q)} \to T^{(r,s,q)} \) over \( \mathcal{F}M_{m,n} \) is a linear combination of \( B^{(1)} \) and \( B^{(2)} \).

**Proof:** We use the notations from the proof of Proposition 1. Let \( (j_0^{r,s,q}(x^\alpha,0))^*, (j_0^{r,s,q}(0,x^\beta y^\delta))^* \in T_0^{(r,s,q)} \mathbb{R}^{m,n} \) be the basis dual to the one of \( j_0^{r,s,q}(\mathbb{R}^{m,n}, \mathbb{R}^{1,1})_0 \).

Let

\[
\begin{align*}
pr_1 & : \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \to \mathbb{R}^m \times \mathbb{R}^n, \\
pr_2 & : \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \to T_0^{(r,s,q)} \mathbb{R}^{m,n}, \\
pr_3 & : \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \to T_0^{(r,s,q)} \mathbb{R}^{m,n}
\end{align*}
\]

be the projections.

Similarly as in the proof of Proposition 1, \( B \) is uniquely determined by the contractions \( \langle B(v), j_0^{r,s,q}(x^1, y^1) \rangle \) for all \( v \in (TT^{(r,s,q)})_0 \mathbb{R}^{m,n} \overset{\cong}{=} \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \), where \( \cong \) is the standard identification. So, it remains to deduce that

\[
\langle B(\cdot), j_0^{r,s,q}(x^1, y^1) \rangle : \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \times T_0^{(r,s,q)} \mathbb{R}^{m,n} \to \mathbb{R}
\]

is a linear combination of \( x^1 \circ pr_1 \) and \( y^1 \circ pr_1 \).

Using similar arguments as in the proof of Proposition 1 (the naturality of \( B \) with respect to \( a_t \) and \( b_t \) and the homogeneous function theorem), we deduce that \( \langle B(\cdot), j_0^{r,s,q}(x^1, y^1) \rangle \) is a linear combination of \( x^1 \circ pr_1, y^1 \circ pr_1, j_0^{r,s,q}(x^1, 0) \circ pr_2, j_0^{r,s,q}(0, x^1) \circ pr_2, j_0^{r,s,q}(y^1, 0) \circ pr_3 \) and \( j_0^{r,s,q}(0, y^1) \circ pr_3 \). Since \( B \) is linear, \( \langle B(\cdot), j_0^{r,s,q}(x^1, y^1) \rangle \) is a linear combination of \( x^1 \circ pr_1, y^1 \circ pr_1, j_0^{r,s,q}(x^1, 0) \circ pr_3, j_0^{r,s,q}(0, x^1) \circ pr_3 \) and \( j_0^{r,s,q}(0, y^1) \circ pr_3 \). Replacing \( B \) by \( B - \lambda_1 B^{(1)} - \lambda_2 B^{(2)} \) we can assume that \( \langle B(\cdot), j_0^{r,s,q}(x^1, y^1) \rangle \) is a linear combination of \( j_0^{r,s,q}(x^1, 0) \circ pr_3, j_0^{r,s,q}(0, x^1) \circ pr_3 \) and \( j_0^{r,s,q}(0, y^1) \circ pr_3 \).

(Then \( \langle B(\partial_1^C \mid \omega), j_0^{r,s,q}(x^1, y^1) \rangle = 0 \) and \( \langle B(\partial_1^C \mid \omega), j_0^{r,s,q}(x^1, y^1) \rangle = 0 \) for any \( \omega \in T_0^{(r,s,q)} \mathbb{R}^{m,n} \), where \( \partial_1 = \frac{\partial}{\partial x^1} \), \( \overline{\partial}_1 = \frac{\partial}{\partial y^1} \) and \( (\partial_1^C) \) is the flow lift of projectable vector fields to \( T^{(r,s,q)} \).) It remains to show

\[
(1) \quad \langle B(0, 0, \bar{\omega}), j_0^{r,s,q}(x^1, y^1) \rangle = 0
\]

for \( \bar{\omega} \in \{ j_0^{r,s,q}(x^1, 0)^*, (j_0^{r,s,q}(0, x^1))^*, (j_0^{r,s,q}(0, y^1))^* \} \). We consider 3 cases.
(1) Assume \( \tilde{\omega} = (j_{0}^{r.s.q}(x^1,0))^* \). For showing (1), we prove

\[
0 = \langle A((\partial_1 + (x^1)^q(\partial_1))^C), j_{0}^{r.s.q}(x^1, y^1) \rangle \\
= \langle A(((x^1)^q(\partial_1))^C), j_{0}^{r.s.q}(x^1, y^1) \rangle \\
= \langle A(0, \omega, \tilde{\omega} + \ldots), j_{0}^{r.s.q}(x^1, y^1) \rangle \\
= \langle A(0, 0, \tilde{\omega}), j_{0}^{r.s.q}(x^1, y^1) \rangle,
\]

where \( \omega = (j_{0}^{r.s.q}((x^1)^q, 0))^* \) and the dots is the linear combination of the elements \( \tilde{\omega} \) from the dual basis of \( T_{0}^{(r,s,q)} \mathbb{R}^{m,n} \) with \( \tilde{\omega} \not\in \{(j_{0}^{r.s.q}(x^1, 0))^*, (j_{0}^{r.s.q}(0, x^1))^*, (j_{0}^{r.s.q}(0, y^1))^* \} \).

The second equality of (2) is clear as \( \langle B(\partial_1^C|\omega), j_{0}^{r.s.q}(x^1, y^1) \rangle = 0 \) and \( A \) is an affinor. The fourth equality of (2) is clear as \( \langle B(\cdot), j_{0}^{r.s.q}(x^1, y^1) \rangle \) is a linear combination of \( j_{0}^{r.s.q}(x^1, 0) \circ \text{pr}_3, j_{0}^{r.s.q}(0, x^1) \circ \text{pr}_3 \) and \( j_{0}^{r.s.q}(0, y^1) \circ \text{pr}_3 \).

We can prove the first equality of (2) as follows. We consider for a moment \( \partial_1 \) and \( \partial_1 + (x^1)^q(\partial_1) \) as the vector fields on \( \mathbb{R} \). They have the same \((q - 1)\)-jets at 0 \( \in \mathbb{R} \). Then there exists a diffeomorphism \( \psi : \mathbb{R} \to \mathbb{R} \) such that \( j_{0}^{q} \psi = \text{id} \) and \( \psi(\partial_1) = \partial_1 + (x^1)^q(\partial_1) \) near 0 \( \in \mathbb{R} \), see Lemma 42.4 in [3] (or [12]). Let \( \varphi = \psi \times \text{id}_{\mathbb{R}^{m-1}} \times \text{id}_{\mathbb{R}^n} \). Then \( \varphi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n} \) is an \( \mathcal{F}\mathcal{M}_{m,n} \)-morphism such that \( j_{0}^{r.s.q} \varphi = \text{id} \) and \( \varphi(\partial_1) = \partial_1 + (x^1)^q(\partial_1) \) near 0. Clearly, \( \varphi \) preserves \( j_{0}^{r.s.q}(x^1, y^1) \) because of the jet argument. Then, using the naturality of \( A \) with respect to \( \varphi \), from \( \langle B(\partial_1^C|\omega), j_{0}^{r.s.q}(x^1, y^1) \rangle = 0 \) for any \( \omega \in T_{0}^{(r,s,q)} \mathbb{R}^{m,n} \) it follows the first equality for any \( \omega \in T_{0}^{(r,s,q)} \mathbb{R}^{m,n} \).

It remains to show the third equality of (2). Let \( \varphi_t \) be the flow of \( (x^1)^q(\partial_1) \). Then

\[
\langle ((x^1)^q(\partial_1))^C, j_{0}^{r.s.q}(x^1, 0) \rangle = \langle \frac{d}{dt}|_{t=0} T^{(r.s,q)}(\varphi_t)(\omega), j_{0}^{r.s.q}(x^1, 0) \rangle \\
= \langle \omega, j_{0}^{r.s.q}(\frac{d}{dt}|_{t=0}(x^1, 0) \circ \varphi_t) \rangle \\
= \langle \omega, j_{0}^{r.s.q}((x^1)^q, 0) \rangle \\
= 1
\]

because of the definition of \( \omega \). Similarly \( \langle ((x^1)^q(\partial_1))^C, j_{0}^{r.s.q}(0, x^1) \rangle = 0 \) and \( \langle ((x^1)^q(\partial_1))^C, j_{0}^{r.s.q}(0, y^1) \rangle = 0 \). Then \( ((x^1)^q(\partial_1))^C \omega = (j_{0}^{r.s.q}(x^1, 0))^* + \ldots \) under the isomorphism \( V_{\omega} T^{(r,s,q)} \mathbb{R}^{m,n} \cong T_{0}^{(r,s,q)} \mathbb{R}^{m,n} \), where the dots stand for a linear combination of the elements \( \tilde{\omega} \) from the dual basis of \( T_{0}^{(r,s,q)} \mathbb{R}^{m,n} \) with \( \tilde{\omega} \not\in \{(j_{0}^{r.s.q}(x^1, 0))^*, (j_{0}^{r.s.q}(0, x^1))^*, (j_{0}^{r.s.q}(0, y^1))^* \} \). It implies the third equality of (2).
(II) Assume \( \tilde{\omega} = (j_0^{r,s,q}(0, x^1))^* \). For showing (1), we prove (2), where \( \omega = (j_0^{r,s,q}(0, (x^1)^q))^* \) and the dots stand for a linear combination of the elements \( \overline{\omega} \) from the dual basis of \( T_0^{r,s,q}(\mathbb{R}^{m,n}) \) with \( \overline{\omega} \notin \{(j_0^{r,s,q}(x^1,0))^*, (j_0^{r,s,q}(0, x^1))^*, (j_0^{r,s,q}(0, y^1))^*\} \).

The proof of the third equality of (2) is almost the same as in case (I).

In this section we classify all natural transformation \( T \). We leave the details to the reader.

(III) Assume \( \tilde{\omega} = (j_0^{r,s,q}(0, y^1))^* \). For showing (1), it suffices to prove

\[
0 = \langle A((\overline{\omega} + (y^1)^s\overline{1}_{(\omega)}), j_0^{r,s,q}(x^1, y^1))
= \langle A((y^1)^s\overline{1}_{(\omega)}), j_0^{r,s,q}(x^1, y^1))
= \langle A(0, \omega, \tilde{\omega} + \ldots), j_0^{r,s,q}(x^1, y^1))
= \langle A(0, 0, \tilde{\omega}), j_0^{r,s,q}(x^1, y^1))
\]

where \( \omega = (j_0^{r,s,q}(0, (y^1)^s))^* \) and the dots stand for a linear combination of the elements \( \overline{\omega} \) from the dual basis of \( T_0^{r,s,q}(\mathbb{R}^{m,n}) \) with \( \overline{\omega} \notin \{(j_0^{r,s,q}(x^1,0))^*, (j_0^{r,s,q}(0, x^1))^*, (j_0^{r,s,q}(0, y^1))^*\} \). The proof of (2)' is similar to that of (2) in case (II). We leave the details to the reader. \( \square \)

4. In this section we classify all natural transformation \( TT^{(r,s,q)} \to T \) over \( \mathcal{FM}_{m,n} \). (The definition is similar to the one from Section 2.)

**Example 3.** Given an \( \mathcal{FM}_{m,n} \)-object \( Y \), let \( \tau : TT^{(r,s,q)}Y \to TY \) be as in Section 3. Then \( \tau : TT^{(r,s,q)} \to T \) is a linear natural transformation over \( \mathcal{FM}_{m,n} \).

**Proposition 3.** Every linear natural transformation \( C : TT^{(r,s,q)} \to T \) over \( \mathcal{FM}_{m,n} \) is a constant multiple of \( \tau \).

**Proof:** Using \( C \), we construct a linear natural transformation \( \tilde{C} : TT^{(r,s,q)} \to T^{(r,s,q)} \) over \( \mathcal{FM}_{m,n} \) as follows. For any \( Y \in \text{obj}(\mathcal{FM}_{m,n}) \) we define a fibered map \( \tilde{C} : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y \) over \( \text{id}_Y \) by

\[
\langle \tilde{C}(v), j_y^{r,s,q}\gamma \rangle = d_y\gamma_1(C(v)),
\]

\( v \in (TT^{(r,s,q)})_y Y, \gamma = (\gamma_1, \gamma_2) : Y \to \mathbb{R}^{1,1} \) is fibered, \( \gamma(y) = 0 \).
Now, by Proposition 2, there exist numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\langle \tilde{C}(v), j_y^{r,s,q}\gamma \rangle = \lambda_1 \cdot d_y\gamma_1(T\pi(v)) + \lambda_2 \cdot d_y\gamma_2(T\pi(v))$$

for any $v \in (TT^{(r,s,q)})_yY$, $y \in Y$, $Y \in \text{obj}(\mathcal{FM}_{m,n})$ and any fibered map $\gamma = (\gamma_1, \gamma_2) : Y \to \mathbb{R}^{1,1}$ with $\gamma(y) = 0$. Then $\lambda_2 = 0$ and $C = \lambda_1 \cdot T\pi$. \qed

5. In this section we prove the main result of this paper.

**Example 4.** For every $\mathcal{FM}_{m,n}$-object $Y$ let $\text{Id} : TT^{(r,s,q)}Y \to TT^{(r,s,q)}Y$ be the identity map and let $\tilde{B}^{(1)}, \tilde{B}^{(2)} : TT^{(r,s,q)}Y \to TT^{(r,s,q)}Y$ be affinors on $T^{(r,s,q)}Y$ such that

$$\tilde{B}^{(1)}(v) = (\omega, B^{(1)}(v)) \in T^{(r,s,q)}Y \times_T T^{(r,s,q)}Y = VT^{(r,s,q)}Y \subset TT^{(r,s,q)}Y,$$

$$\tilde{B}^{(2)}(v) = (\omega, B^{(2)}(v)) \in TT^{(r,s,q)}Y, \ v \in T\omega T^{(r,s,q)}Y, \ \omega \in T^{(r,s,q)}Y,$$

where $B^{(1)}, B^{(2)} : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ are as in Section 3. Then $\text{Id}$, $\tilde{B}^{(1)}$ and $\tilde{B}^{(2)}$ are natural affinors on $T^{(r,s,q)}_{\mathcal{FM}_{m,n}}$.

**Theorem 1.** Every natural affinor $D$ on $T^{(r,s,q)}_{\mathcal{FM}_{m,n}}$ is a linear combination of $\text{Id}$, $\tilde{B}^{(1)}$ and $\tilde{B}^{(2)}$.

**Proof:** The family $T\pi \circ D : TT^{(r,s,q)}Y \to TY$ for $Y \in \text{obj}(\mathcal{FM}_{m,n})$ is a linear natural transformation $TT^{(r,s,q)} \to T$ over $\mathcal{FM}_{m,n}$. Then, by Proposition 3, there exists the real number $\lambda$ such that $T\pi \circ D = \lambda \cdot T\pi$. Then $D - \lambda \cdot \text{Id} : TT^{(r,s,q)}Y \to VT^{(r,s,q)}Y$ for any $\mathcal{FM}_{m,n}$-object $Y$. Let $\text{pr} : VT^{(r,s,q)}Y \cong T^{(r,s,q)}Y \times_T T^{(r,s,q)}Y \to T^{(r,s,q)}Y$ be the projection onto second factor for any $Y$ as above. Then the family $\text{pr} \circ (D - \lambda \cdot \text{Id}) : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ for any $Y$ as above is a linear natural transformation over $\mathcal{FM}_{m,n}$. Now, by Proposition 2, there exist the numbers $\mu_1, \mu_2 \in \mathbb{R}$ such that $\text{pr} \circ (D - \lambda \cdot \text{Id}) = \mu_1 \cdot \tilde{B}^{(1)} + \mu_2 \cdot \tilde{B}^{(2)}$. Then $D = \lambda \cdot \text{Id} + \mu_1 \cdot \tilde{B}^{(1)} + \mu_2 \cdot \tilde{B}^{(2)}$. \qed

6. We have the following corollary of Theorem 1.

**Corollary 1.** There is no natural generalized connection on $T^{(r,s,q)}_{\mathcal{FM}_{m,n}}$.

**Proof:** Suppose that $\Gamma$ is such a connection. By Theorem 1, there are numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that $\Gamma = \lambda_1 \cdot \text{Id} + \lambda_2 \cdot \tilde{B}^{(1)} + \lambda_3 \cdot \tilde{B}^{(2)}$. Let $Y$ be an $\mathcal{FM}_{m,n}$-object. Since $\text{im}(\Gamma) = VT^{(r,s,q)}Y$ and $\text{im}(\tilde{B}^{(1)}) \subset VT^{(r,s,q)}Y$ and $\text{im}(\tilde{B}^{(2)}) \subset VT^{(r,s,q)}Y$, we get $\lambda_1 = 0$. It is easy to see that $VT^{(r,s,q)}Y \subset \ker(\tilde{B}^{(1)})$ and $VT^{(r,s,q)}Y \subset \ker(\tilde{B}^{(2)})$. Then $\Gamma \circ \Gamma = 0 \neq \Gamma$, a contradiction. \qed
7. We can solve similar problems with \( T^{(r,s)} = (J^{r,s}(\cdot, \mathbb{R})_0)^* : \mathcal{FM} \to \mathcal{FM} \) instead of \( T^{(r,s,q)} \) as follows.

(i) Let \( Y \to M \) be a fibered manifold and \( Q \) a manifold. Two maps \( f, g : Y \to Q \) determine the same \((r,s)\)-jet \( j^r_y j^s_z f = j^r_y j^s_z g \) at \( y \in Y_x, x \in M \), if \( j^r_y f = j^r_y g \), and \( j^r_y (f|_{Y_x}) = j^s_y (g|_{Y_x}) \). The space of all \((r,s)\)-jets of \( Y \) into \( Q \) is denoted by \( J^{r,s}(Y, Q) \), see \([3, p. 126]\).

The space \( T^{r,s,y} = J^{r,s}(Y, \mathbb{R})_0 \) has an induced structure of a vector bundle over \( Y \). Every \( \mathcal{FM}\)-morphism \( h : Z \to Y \), \( h(z) = y \), induces a linear map \( \lambda(h)y, z : T_z^{r,s,z} Y \to T_z^{r,s,z} Z \), \( j_z^{r,s} f \to j_y^{r,s}(f \circ h) \). If we denote by \( T^{(r,s)} Y \) the dual vector bundle of \( T^{r,s,y} \) and define \( T^{(r,s)} h : T^{(r,s,q)} Z \to T^{(r,s)} Y \) by using the dual maps to \( \lambda(h)y, z \), we obtain a vector bundle functor \( T^{(r,s)} \) on \( \mathcal{FM} \).

(ii) The family \( \text{id} : T^{(r,s)} Y \to T^{(r,s)} Y \) for any \( \mathcal{FM}_{m,n} \)-object \( Y \) is a natural transformation \( T^{(r,s)} \to T^{(r,s)} \) over \( \mathcal{FM}_{m,n} \).

**Proposition 1’.** Every natural transformation \( A : T^{(r,s)} \to T^{(r,s)} \) over \( \mathcal{FM}_{m,n} \) is a constant multiple of the identity natural transformation.

**Proof:** The proof is quite similar to the proof of Proposition 1. \( \square \)

(iii) For every \( \mathcal{FM}_{m,n} \)-object \( Y \) let \( B^{(l)} : T^{(r,s)} Y \to T^{(r,s)} Y \) be a fibered map over \( \text{id}_Y \) such that \( \langle B^{(l)}(v), j_y^{r,s} \gamma \rangle = d_y \gamma(T \pi(v)) \), \( v \in (T^{(r,s)} Y)_y, y \in Y \), \( \gamma : Y \to \mathbb{R}, \gamma(y) = 0 \), where \( \pi : T^{(r,s)} Y \to Y \) is the bundle projection and \( T \pi : T^{(r,s)} Y \to TY \) is its tangent map. Then \( B^{(l)} : T^{(r,s)} \to T^{(r,s)} \) is a linear natural transformation over \( \mathcal{FM}_{m,n} \).

**Proposition 2’.** Every linear natural transformation \( B : T^{(r,s)} \to T^{(r,s)} \) over \( \mathcal{FM}_{m,n} \) is a constant multiple of \( B^{(l)} \).

**Proof:** The proof is quite similar to the proof of Proposition 2. \( \square \)

(iv) Given an \( \mathcal{FM}_{m,n} \)-object \( Y \) let \( T \pi : T^{(r,s)} Y \to TY \) be as in (iii). Then \( T \pi : T^{(r,s)} \to T \) is a linear natural transformation over \( \mathcal{FM}_{m,n} \).

**Proposition 3’.** Every linear natural transformation \( C : T^{(r,s)} \to T \) over \( \mathcal{FM}_{m,n} \) is a constant multiple of \( T \pi \).

**Proof:** The proof is quite similar to the proof of Proposition 3. \( \square \)

(v) For every \( \mathcal{FM}_{m,n} \)-object \( Y \), let \( \text{Id} : T^{(r,s)} Y \to T^{(r,s)} Y \) be the identity map and let \( \tilde{B}^{(l)} : T^{(r,s)} Y \to T^{(r,s)} Y \) be an affinor on \( T^{(r,s)} Y \) such that \( \tilde{B}^{(l)}(v) = (\omega, B^{(l)}(v)) \in T^{(r,s)} Y \times Y, v \in T \omega T^{(r,s)} Y \), \( T \omega T^{(r,s)} Y \), \( \omega \in T^{(r,s)} Y \), where \( B^{(l)} : T^{(r,s)} Y \to T^{(r,s)} Y \) is as in Proposition 1’. Then \( \text{Id} \) and \( \tilde{B}^{(l)} \) are natural affinors on \( T^{(r,s)} \).
Theorem 1’. Every natural affinor $D$ on $T^{(r,s)}_{\mathcal{FM},m,n}$ is a linear combination of $\text{Id}$ and $\tilde{B}^{(l)}$.

Proof: The proof is quite similar to the proof of Theorem 1. 

(vi) We have the following corollary of Theorem 1’.

Corollary 1’. There is no natural generalized connection on $T^{(r,s)}_{\mathcal{FM},m,n}$.

References


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