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Abstract. There is a locally compact Hausdorff space which is linearly Lindelöf and not Lindelöf. This answers a question of Arhangel’skii and Buzyakova.

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This note is devoted to the proof of:

Theorem 1. There is a compact Hausdorff space $X$ and a point $p$ in $X$ such that:

(1) $\chi(p, X) > \omega$;
(2) for all regular $\kappa > \omega$, no $\kappa$-sequence of points distinct from $p$ converges to $p$.

As usual, $\chi(p, X)$, the character of $p$ in $X$, is the least size of a local base at $p$. Regarding (2), if $\vec{q} = \langle q_\alpha : \alpha < \kappa \rangle$ is a $\kappa$-sequence, we say $\vec{q} \to p$ iff whenever $U$ is a neighborhood of $p$, $\exists \alpha \forall \beta > \alpha [q_\beta \in U]$. Then, (2) asserts that $\vec{q} \not\to p$ whenever $\kappa > \omega$ is regular and all the $q_\alpha \neq p$. Observe that if $\chi(p, X) = \omega$, then (2) holds trivially.

Theorem 1 answers Question 1 of Arhangel’skii and Buzyakova [1]. They point out that given such an $X, p$, the space $X \setminus \{p\}$ is linearly Lindelöf (by (2)), not Lindelöf (by (1)), and locally compact.

Note that in any compact Hausdorff space $X$, if the point $x$ is not isolated, then there is a sequence of type $\text{cf}(\chi(x, X))$ converging to $x$. Thus, the $X, p$ in Theorem 1 must satisfy $\text{cf}(\chi(p, X)) = \omega$. In our example, $\chi(p, X)$ will be $\beth_\omega$.

Our $X$ will be constructed as an inverse limit. We begin by reviewing some basic terminology:

Definition 2. An inverse system is a sequence $\langle X_n, \pi_{n+1}^n : n \in \omega \rangle$, where each $X_n$ is a compact Hausdorff space, and each $\pi_{n+1}^n$ is a continuous map from $X_{n+1}$ onto $X_n$.

Such an inverse systems yields a compact Hausdorff space,

$$X_\omega = \lim_n X_n = \{ x \in \prod_n X_n : \forall n [x_n = \pi_{n+1}^n(x_{n+1})]\}.$$ 

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It also yields the obvious maps $\pi_m^n : X_m \to X_n$ for $m < n < \omega$.

**Lemma 3.** Suppose that $\langle X_n, \pi^{n+1}_n : n \in \omega \rangle$, is an inverse system and $p = \langle p_n : n \in \omega \rangle \in X = X_\omega$ satisfies:

(A) each $p_n$ is a weak $P_{\Delta_m}$-point in $X_n$;
(B) each $\chi(p_n, X_n) \leq \Delta_{n+1}$;
(C) each $(\pi^0_n)^{-1}\{p_0\}$ is nowhere dense in $X_n$.

Then $X, p$ satisfies Theorem 1 with $\chi(p, X) = \Delta_\omega$.

As usual, $y \in Y$ is a weak $P_\kappa$-point iff $y$ is not in the closure of any subset of $Y \setminus \{y\}$ of size less than $\kappa$, and $y$ is a $P_\kappa$-point if the intersection of fewer than $\kappa$ neighborhoods of $y$ is always a neighborhood of $y$. In a Hausdorff space, every $P_\kappa$-point is a weak $P_\kappa$-point, but note that in (A), the $p_n$ cannot all be $P_{\Delta_m}$-points, as that would contradict (C). Note that (C) cannot be omitted; it is easy to construct an example satisfying (A) and (B) where each $X_n$ is a LOTS and each $\pi^{n+1}_n$ collapses an interval around $p_{n+1}$ to the point $p_n$; then $\chi(p, X) = \omega$.

**PROOF OF LEMMA 3:** First, note that one local base at any $x \in X$ consists of all the $(\pi^0_n)^{-1}(U)$ such that $n \in \omega$ and $U$ is an open neighborhood of $x_n$ in $X_n$. It follows that:

(α) $\chi(p, X_\omega) \leq \sup_n \chi(p_n, X_n) = \Delta_\omega$;
(β) $(\pi^0_\omega)^{-1}\{p_0\}$ is nowhere dense in $X_\omega$.

Now, to verify (2) of Theorem 1, assume that $\tilde{q} = \langle q_\alpha : \alpha < \kappa \rangle \to p$, where $\kappa > \omega$ is regular and all the $q_\alpha \neq p$. The definition of $\tilde{q} \to p$ implies that $\kappa \leq \chi(p, X)$, so fix $m$ with $\kappa < \Delta_m$. Now, $q_\alpha \neq p$ implies that $\pi^\omega_m(q_\alpha) \neq p_n = \pi^\omega_n(p)$ for some $n$, so we can fix $n \geq m$ and an $S \subseteq \kappa$ with $|S| = \kappa$ and $\pi^\omega_n(q_\alpha) \neq p_n$ for all $\alpha \in S$. But then $p_n \in \cl{\{\pi^\omega_n(q_\alpha) : \alpha \in S\}}$, contradicting (A).

In view of (α), to prove that $\chi(p, X) = \Delta_\omega$, it is sufficient to fix $m < \omega$ and prove that $\chi(p, X) \geq \Delta_m$. Suppose that $B$ were a local base at $p$ in $X$ with $|B| < \Delta_m$. Let $F = (\pi^\omega_m)^{-1}\{p_m\}$. By (β), $F$ is nowhere dense in $X$, so for each $U \in B$, we can choose $y_U \in U \setminus F$. Then $p \in \cl{\{y_U : U \in B\}}$, so $p_m = \pi^\omega_m(p) \in \cl{\{\pi^\omega_m(y_U) : U \in B\}}$, contradicting (A).

We now need to find an inverse system satisfying the hypotheses of Lemma 3. $X_\omega$ will be the Čech compactification of a discrete $\kappa$; equivalently, $\beta_\kappa$ is the space of ultrafilters on $\kappa$; thus, the remainder, $\kappa^* = \beta_\kappa \setminus \kappa$, is the space of non-principal ultrafilters on $\kappa$.

The $p_n$ will be good ultrafilters. Following Keisler [5], an ultrafilter $x$ on $\kappa$ is good (i.e., $\kappa^+$-good) iff given $A_s \in x$ for $s \in [\kappa]^{<\omega}$, there are $B_\alpha \in x$ for $\alpha < \kappa$ such that $\bigcap_{\alpha \in s} B_\alpha \subseteq A_s$ for all non-empty $s \in [\kappa]^{<\omega}$. For every infinite $\kappa$, there is a non-principal $x \in \beta_\kappa$ such that $x$ is a good ultrafilter (Keisler [5] under GCH and Kunen [7] in ZFC; see also Chang and Keisler [3, Theorem 6.1.4]). The following folklore result about such ultrafilters is proved in [2] and [4]:


Lemma 4. If \( x \) is a good ultrafilter on \( \kappa \), then \( x \) is a weak \( P_\kappa \)-point in \( \beta \kappa \).

Thus, fixing \( p_n \in \beta \beth_n \) to be good will handle (A) of Lemma 3, but to get \( p = \langle p_n : n \in \omega \rangle \) to really define a point in \( X = X_\omega \), we need to choose the \( \pi^{n+1}_n : \beta \beth_{n+1} \rightarrow \beta \beth_n \) such that each \( p_n = \pi^{n+1}_n(p_{n+1}) \). In fact, \( \pi^{n+1}_n \) will be \( \beta(\Pi^{n+1}_n) \), where \( \Pi^{n+1}_n : \beth_{n+1} \rightarrow \beth_n \). Here, as usual, if \( f : P \rightarrow Q \), where \( P, Q \) are Tychonov spaces, then \( \beta f : \beta P \rightarrow \beta Q \) denotes its Čech extension. In the special case of discrete \( P, Q \), where \( x \in \beta P \) is an ultrafilter on \( P \), \( (\beta f)(x) \in \beta Q \) is the induced measure, \( \{B \subseteq Q : f^{-1}(B) \in x\} \). Now, showing that appropriate \( \Pi^{n+1}_n : \beth_{n+1} \rightarrow \beth_n \) can be chosen requires a digression:

Definition 5. An ultrafilter \( x \) on \( \kappa \) is regular if there are \( E_\alpha \) \( \in x \) for \( \alpha < \kappa \) such that \( \bigcap_n E_\alpha_n = \emptyset \) whenever the \( \alpha_n \) (for \( n \in \omega \)) are distinct.

Clearly, such \( x \) are countably incomplete. Moreover,

Lemma 6. If \( x \) is a countably incomplete good ultrafilter on \( \kappa \), then \( x \) is regular.

This is Exercise 6.1.3 of [3]; a proof is contained in the proof of Lemma 2.1 of Keisler [6]. The proof of universality of regular ultrapowers ([3, Theorem 4.3.12]) is easily modified to produce:

Lemma 7. Suppose that \( \kappa \geq 2^\lambda \) and \( y \) is any ultrafilter on \( \lambda \). Let \( x \) be a regular ultrafilter on \( \kappa \). Then there is an \( f : \kappa \rightarrow \lambda \) such that \( (\beta f)(x) = y \).

Proof: Since \( \kappa \geq 2^\lambda \), we may list the elements of \( y \) (possibly with repetitions) as \( \{B_\alpha : \alpha < \kappa \} \). Let the \( E_\alpha \subseteq \kappa \) be as in Definition 5. Choose \( g : \kappa \rightarrow \lambda \) such that \( g(\xi) \) is some element of \( \bigcap \{B_\alpha : \xi \in E_\alpha \} \) (observe that this is a finite intersection). Then \( (\beta g)(x) = y \) because each \( g^{-1}(B_\alpha) \supseteq E_\alpha \in x \). This \( g \) may fail to be onto, but we may now fix a set \( A \in x \) with \( |\kappa \setminus A| = \kappa \), and choose \( f : \kappa \rightarrow \lambda \) such that \( f \upharpoonright A = g \upharpoonright A \), so that \( (\beta f)(x) = (\beta g)(x) = y \).

Proof of Theorem 1: We obtain the situation of Lemma 3. Fix \( X_n = \beta \beth_n \), and fix \( p_n \in \beta \beth_n \) to be good and non-principal (and hence countably incomplete). Applying Lemmas 6 and 7, fix \( \Pi^{n+1}_n : \beth_{n+1} \rightarrow \beth_n \) so that setting \( \pi^{n+1}_n = \beta(\Pi^{n+1}_n) \) yields \( p_n = \pi^{n+1}_n(p_{n+1}) \). Then (A) follows by Lemma 4, and (B) is clear, since there is a base for the space \( X_n \) of size \( 2^\beth_n = \beth_{n+1} \). Finally, (C) holds because \( (\pi^0_n)^{-1}\{p_0\} \subseteq (\beth_n)^* \), which is nowhere dense in \( \beta \beth_n \).

References


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