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Tightness of compact spaces is preserved by the $t$-equivalence relation

Oleg Okunev

Abstract. We prove that if there is an open mapping from a subspace of $C_p(X)$ onto $C_p(Y)$, then $Y$ is a countable union of images of closed subspaces of finite powers of $X$ under finite-valued upper semicontinuous mappings. This allows, in particular, to prove that if $X$ and $Y$ are $t$-equivalent compact spaces, then $X$ and $Y$ have the same tightness, and that, assuming $2^t > c$, if $X$ and $Y$ are $t$-equivalent compact spaces and $X$ is sequential, then $Y$ is sequential.

Keywords: function spaces, topology of pointwise convergence, tightness

Classification: 54B10, 54D20, 54A25, 54D55

All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We study the spaces $C_p(X, Z)$ of all continuous functions on a space $X$ with the values in a space $Z$ equipped with the topology of pointwise convergence (see [Arh3] for a thorough presentation of the theory of spaces of functions equipped with this topology). The space $C_p(X, \mathbb{R})$ is denoted by $C_p(X)$, and $C_p^*(X)$ denotes the subspace of $C_p(X)$ consisting of all bounded functions; in all cases we denote by 0 the zero constant function on $X$. We say that $Y$ is a $t$-image of $X$ if $C_p(Y)$ is homeomorphic to a subspace (not necessarily linear) of $C_p(X)$. Every continuous image of a space is its $t$-image by virtue of the dual mapping between the function spaces (see [Arh3]). Two spaces $X$ and $Y$ are called $t$-equivalent if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic, and $l$-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Of course, if two spaces are $t$-equivalent, then each of them is a $t$-image of the other; simple examples show that the converse is not true. Note also that the spaces $C_p(X, [0, 1])$ and $C_p^*(X)$ contain homeomorphic copies of $C_p(X)$, and their homeomorphic copies are contained in $C_p(X)$. It follows that if one of the spaces $C_p(Y), C_p^*(Y), C_p(Y, [-1, 1])$, admits a homeomorphic embedding in $C_p(X), C_p^*(X)$, or $C_p(X, [-1, 1])$, then $Y$ is a $t$-image of $X$.

We denote by $t(X)$ and $l(X)$ the tightness and the Lindelöf number of a space $X$ (see e.g. [Eng]); we put $l^*(X) = \sup\{l(X^n) : n \in \mathbb{N}\}$ and $t^*(X) = \{t(X^n) : n \in \mathbb{N}\}$. All cardinals are assumed to be infinite; $\omega$ is the set of all naturals, and $\mathbb{N} = \omega \setminus \{0\}$. The cardinal $t$ is the minimum cardinality of a tower of infinite subsets in $\omega$ (see [vDo]), and $c = 2^\omega$. 
For a set-valued mapping \( p: X \to Y \) and a set \( A \subseteq X \), we define the image of \( A \), \( p(A) \) as the union \( \bigcup \{ p(x) : x \in A \} \). We say that a set-valued mapping \( p: X \to Y \) is onto if \( p(X) = Y \). A set-valued mapping \( p: X \to Y \) is called compact-valued (finite-valued) if for every \( x \in X \) the set \( p(x) \) is compact (finite), and upper semicontinuous if for every closed set \( F \subseteq Y \), the preimage \( p^{-1}(F) = \{ x \in X : p(x) \cap F \neq \emptyset \} \) is closed. We do not require \( p(x) \neq \emptyset \) for every \( x \in X \); this is slightly different from the common usage of the term, but is more convenient in the context of this article. Note that for every upper-semicontinuous mapping \( p: X \to Y \) the set \( p^{-1}(Y) \) of all points of \( X \) with nonempty images is closed in \( X \), and every closed subspace of \( X \) is an image of \( X \) under a finite-valued upper semicontinuous mapping (the one identical on the subspace, and with empty images of the points of the complement), so “an image of \( X \) under an upper semicontinuous mapping” in this article is the same as “an image of a closed subspace of \( X \) under an upper semicontinuous mapping” in the traditional sense. It is easy to verify that a set-valued mapping from a space \( X \) is compact-valued upper semicontinuous if and only if it is the composition of the inverse of a perfect mapping (onto a closed subspace of \( X \)) and a continuous mapping; in particular, this implies the standard fact that we often use in this article: Upper semicontinuous compact-valued mappings preserve compactness and do not raise the Lindelöf number.

A set-valued mapping \( p: X \to Y \) is called upper semicontinuous at a point \( x_0 \in X \) if for every open neighborhood \( V \) of \( p(x_0) \) in \( Y \), there is a neighborhood \( U \) of \( x_0 \) in \( X \) such that \( p(U) \subseteq V \). It is easy to verify that \( p \) is upper semicontinuous if and only if it is upper semicontinuous at every point of \( X \).

In [Ok1] the author proved that if there is an open mapping of a subspace of \( C_p(X) \) onto \( C_p(Y) \), then \( Y \) is a countable union of continuous images of closed subspaces of products of finite powers of \( X \) and a compact space — in other words, \( Y \) is a countable union of images of finite powers of \( X \) under compact-valued upper semicontinuous mappings. In this article we refine this result by showing that \( Y \) is a countable union of images of finite powers of \( X \) under finite-valued upper semicontinuous mappings; this allows to prove that if \( X \) is compact, then the tightness of every compact subspace of \( Y \) does not exceed the tightness of \( X \). In particular, the tightness in compact spaces is not increased by \( t \)-images, which gives a positive answer to Problem 32 (1057) in [Arh2] (the question first appeared in [Tk1] and was repeated in [Tk2].) We also prove that if \( X \) and \( Y \) are compact, \( X \) is sequential, and \( Y \) is a \( t \)-image of \( X \), then \( Y \) is a countable union of sequential compact subspaces, which consistently implies that \( Y \) is sequential. Note that neither tightness, nor sequentiality are preserved by the relation of \( t \)-equivalence without the assumption of compactness ([Ok2]).

1. Statements

1.1 Theorem. Let \( X \) and \( Y \) be spaces, and assume that there is a continuous
open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$. Then there is a sequence of finite-valued upper semicontinuous mappings $T_k: X^k \to Y$, $k \in \mathbb{N}$, such that $Y = \bigcup \{ T_k(X^k) : k \in \mathbb{N} \}$.

1.2 Proposition. Let $\tau$ be a cardinal, $Z$ a space, $K$ a compact space, and $p: Z \to K$ a compact-valued upper semicontinuous mapping such that $p(Z) = K$. If $l(Z)t(Z) \leq \tau$ and $t(p(z)) \leq \tau$ for every $z \in Z$, then $t(K) \leq \tau$.

1.3 Theorem. If there is a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$ (in particular, if $Y$ is a $t$-image of $X$), then for every compact subspace $K$ of $Y$, $t(K) \leq t^*(X)t^*(X)$. In particular, if $X$ is compact, then $t(K) \leq t(X)$.

1.4 Corollary. Let $Y$ be a $k$-space. If $Y$ is a $t$-image of a compact space $X$, then $t(Y) \leq t(X)$.

Indeed, if every compact subspace of a $k$-space $Y$ has the tightness $\leq \tau$, then $t(Y) \leq \tau$.

1.5 Corollary. If $X$ and $Y$ are $t$-equivalent compact spaces, then $t(X) = t(Y)$.

The last statement is an answer to Problem 32(1057) in [Arh2].

Remark. The preservation of the tightness of compact spaces by the relation of $l$-equivalence was proved by Tkachuk in [Tk1].

1.6 Proposition. Let $Z$ and $K$ be compact spaces, and $p: Z \to K$ a finite-valued upper semicontinuous mapping such that $p(Z) = K$. If $Z$ is sequential, then $K$ is sequential.

1.7 Corollary. If $X$ and $Y$ are compact spaces, $X$ is sequential, and there is a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$ (in particular, if $Y$ is a $t$-image of $X$), then $Y$ is a countable union of sequential compact subspaces. In particular, every countably compact subspace of $Y$ is compact, and if $2^1 > \mathfrak{c}$, then $Y$ is sequential.

2. The proofs

Proof of Theorem 1.1: Let $\Phi_0$ be a continuous open mapping from a subspace $C_0$ of $C_p(X)$ onto $C_p(Y)$. Since $C_p(X)$ and $C_p(Y)$ are homogeneous, we may assume without loss of generality that $0 \in C_0$ and $\Phi_0(0) = 0$.

Denote $I = [-1, 1]$. The space $C_p(Y, I)$ is a subspace of $C_p(Y)$; put $C = \Phi_0^{-1}(C_p(Y, I))$ and let $\Phi: C \to C_p(Y, I)$ be the restriction of $\Phi_0$. Then $\Phi$ is continuous, open, onto $C_p(Y, I)$, and $\Phi(0) = 0$.

Let $\beta Y$ be the Stone-Čech compactification of $Y$. For every $g \in C_p(Y, I)$ we denote by $\tilde{g}$ the continuous extension of $g$ over $\beta Y$. 
For every \( k \in \mathbb{N} \), \( \bar{x} = (x_1, \ldots, x_k) \in X^k \), \( \bar{y} = (y_1, \ldots, y_k) \in (\beta Y)^k \) and \( \varepsilon > 0 \) denote

\[
O_X(\bar{x}, \varepsilon) = \{ f \in C : |f(x_1)| < \varepsilon, \ldots, |f(x_k)| < \varepsilon \};
\]

\[
O_Y(\bar{y}, \varepsilon) = \{ g \in C_p(Y, I) : |\tilde{g}(y_1)| < \varepsilon, \ldots, |\tilde{g}(y_k)| < \varepsilon \},
\]

and

\[
\tilde{O}_Y(\bar{y}, \varepsilon) = \{ g \in C_p(Y, I) : |\tilde{g}(y_1)| \leq \varepsilon, \ldots, |\tilde{g}(y_k)| \leq \varepsilon \}.
\]

The sets \( O_X(\bar{x}, 1/k), k \in \mathbb{N}, \bar{x} \in X^k \) form an open base at 0 of the space \( C \).

Similarly, the sets \( O_Y(\bar{y}, 1/k), k \in \mathbb{N}, \bar{y} \in Y^k \) form an open base at 0 of the space \( C_p(Y, I) \) (see e.g. [Arh3]).

For every \( k \in \mathbb{N} \) put

\[
P_k = \{ y \in \beta Y : \text{there is a point } \bar{x} \in X^k \text{ such that } \Phi(O_X(\bar{x}, 1/k)) \subset \tilde{O}_Y(y, 1/2) \}.
\]

From the continuity of \( \Phi \) it follows that \( Y \subset \bigcup \{ P_k : k \in \mathbb{N} \} \).

For every \( \bar{x} \in X^k \) put

\[
T_k(\bar{x}) = \{ y \in \beta Y : \Phi(O_X(\bar{x}, 1/k)) \subset \tilde{O}_Y(y, 1/2) \}.
\]

Obviously, \( T_k(X^k) = P_k \), so \( Y \subset \bigcup \{ T_k(X^k) : k \in \mathbb{N} \} \).

Claim 1. For every \( \bar{x} \in X^k \), \( T_k(\bar{x}) \) is a finite subset of \( Y \).

Since \( \Phi \) is open, the set \( \Phi(O_X(\bar{x}, 1/k)) \) is a neighborhood of 0 in \( C_p(Y, I) \). Hence there are points \( y_1, \ldots, y_m \in Y \) and \( \delta > 0 \) such that \( O_Y(y_1, \ldots, y_m, \delta) \subset \Phi(O_X(\bar{x}, 1/k)) \). Then \( T_k(\bar{x}) \subset \{ y_1, \ldots, y_m \} \). Indeed, if \( y \) is a point of \( \beta Y \) distinct from \( y_1, \ldots, y_m \), then there is a function \( g \in C_p(Y, I) \) such that \( g(y_i) = 0 \), \( i = 1, \ldots, m \), and \( \tilde{g}(y) = 1 \). Then \( g \in O_Y(y_1, \ldots, y_m, \delta) \), and therefore \( g \in \Phi(O_X(\bar{x}, 1/k)) \). Then there is an \( f \in O_X(\bar{x}, 1/k) \) such that \( \Phi(f) = g \); then \( g = \Phi(f) \notin O_Y(y, 1/2) \), so \( y \notin T_k(\bar{x}) \).

Thus, we have defined finite-valued mappings \( T_k : X^k \to Y \) so that \( \bigcup \{ T_k(X^k) : k \in \mathbb{N} \} = Y \).

Claim 2. For every \( k \in \mathbb{N} \), the mapping \( T_k \) is upper semicontinuous.

Obviously, it is sufficient to verify that \( T_k \) is upper semicontinuous as a mapping to \( \beta Y \).

Let \( \bar{x}_0 \) be a point of \( X^k \), and let \( V \) be an open neighborhood of \( T_k(\bar{x}_0) \) in \( \beta Y \). For every \( y \in \beta Y \backslash V \) choose a function \( f_y \in O(\bar{x}_0, 1/k) \) so that \( \tilde{g}_y(y) > 1/2 \) where
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$g_y = \Phi(f_y)$, and put $F_y = g_y^{-1}([−1/2, 1/2])$. Then $F_y$ is closed in $\beta Y$ and $y \notin F_y$, so

$$\bigcap \{F_y : y \in \beta Y \setminus V\} \subset V.$$ 

By the compactness of $\beta Y$, there is a finite set $y_1, \ldots, y_m$ in $\beta Y \setminus V$ such that

$$F_{y_1} \cap \cdots \cap F_{y_m} \subset V.$$ 

Put

$$U = \{ (x_1, \ldots, x_k) \in X^k : |f_{y_i}(x_j)| < 1/k, \; i \leq m, \; j \leq k \}.$$ 

Then $U$ is a neighborhood of $\bar{x}_0$ in $X^k$, and $T_k(U) \subset V$. Indeed, if $\bar{x} \in U$ and $y \notin V$, then $y \notin F_{y_i}$ for some $i \leq m$, so $f_{y_i} \in O(\bar{x}, 1/k)$ and $g_{y_i} = \Phi(f_{y_i}) \notin \bar{O}_Y(y, 1/2)$, so $y \notin T_k(\bar{x})$.

This concludes the proof of Theorem 1.1. \hfill \square

Remark. The above proof may be easily (almost literally) modified to prove the following:

2.1 Theorem. Let $X$ and $Y$ be spaces such that $\text{ind } Y = 0$, and assume that there is a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y, 2)$. Then there is a sequence of finite-valued upper semicontinuous mappings $T_k : X^k \to Y$, $k \in \mathbb{N}$, such that $Y = \bigcup \{ T_k(X^k) : k \in \mathbb{N} \}$.

Proof of Proposition 1.2: Let

$$\Gamma = \{ (z, y) \in Z \times K : y \in p(z) \}.$$ 

Then $\Gamma$ is closed in $Z \times K$. Indeed, if $(z_0, y_0) \notin \Gamma$, then $y_0$ and $p(z_0)$ have disjoint neighborhoods $V$ and $W$ in $K$; put $U = \{ z \in Z : p(z) \subset W \}$. Then $U \times V$ is a neighborhood of $(z_0, y_0)$ disjoint from $\Gamma$.

Let $\pi_Z : Z \times K \to Z$, $\pi_K : Z \times K \to K$ be the projections. Since $K$ is compact, the projection $\pi_Z$ is perfect, so its restriction $h = \pi_Z|\Gamma$ is perfect. In particular, this implies $l(\Gamma) \leq \tau$. Obviously, for every $z \in Z$, $\pi_K$ maps $h^{-1}(z)$ homeomorphically onto $p(z)$, so $h : \Gamma \to Z$ is a closed mapping whose all fibers have the tightness $\leq \tau$. By Theorem 4.5 in [Arh1], $t(\Gamma) \leq \tau$. The statement of the proposition now follows from the next well-known fact (apparently, first discovered by Tkachenko; see also Theorem 1 in [Ra]):

2.2 Proposition. Let $K$ be a compact space, and suppose there is a continuous mapping $p$ from a space $\Gamma$ onto $K$. Then $t(K) \leq l(\Gamma)t(\Gamma)$.

Proof of Theorem 1.3: Let $\Phi$ be a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$, and let $r : C_p(Y) \to C_p(K)$ be the restriction mapping; since
Let \( T_k : X^k \to K \), \( k \in \mathbb{N} \), be as in Theorem 1.1. Put \( M = \bigoplus_{k \in \mathbb{N}} X^k \), and define a mapping \( T : M \to K \) by the rule: \( T(\vec{x}) = T_k(\vec{x}) \) if \( \vec{x} \in X^k \). Obviously, \( T \) is finite-valued and upper semicontinuous. By Proposition 1.2, \( t(K) \leq l(M)t(M) = l^*(X)t^*(X) \).

If \( X \) is compact, then \( l^*(X)t^*(X) = t(X) \) [Mal], so \( t(K) \leq t(X) \). \( \Box \)

**Proof of Proposition 1.6:** Let \( \Gamma, \pi_Z, \pi_K \) and \( h = \pi_Z|\Gamma \) be as in the proof of Proposition 1.2. Since \( Z \) is compact, \( \pi_K \) is perfect, and its restriction \( h \) to the closed set \( \Gamma \) is closed. Thus, it is sufficient to verify that \( \Gamma \) is sequential.

Let \( A \) be a non-closed set in \( \Gamma \); we will prove that \( A \) is not sequentially closed. Let \( a_0 \in \Gamma \setminus A \) be a limit point of \( A \) and \( b_0 = h(a_0) \). Fix a closed neighborhood \( W \) of \( a_0 \) in \( \Gamma \) so that \( \{a_0\} = W \cap h^{-1}(b_0) \), and put \( A_0 = W \cap A \). Then \( h_0 = h|W \) is closed and has finite fibers, and \( a_0 \) is a limit point of \( A_0 \). The point \( b_0 \) is a limit point of \( B = h(A_0) \) and is not in \( B \), so \( B \) is not closed in \( Z \). Since \( Z \) is sequential, there is a sequence \( \{z_n : n \in \omega\} \in B \) that converges to a point \( b_1 \in Z \setminus B \). The set \( M = h_0^{-1}(\{z_n : n \in \omega\})\cup h_0^{-1}(b_1) \) is a countable compact subspace of \( W \), and \( h(M \cap A) = \{z_n : n \in \omega\} \) is not compact. It follows that \( M \cap A \) is not compact, and hence \( A \) is not sequentially closed. \( \Box \)

**Proof of Corollary 1.7:** The first statement follows immediately from Theorem 1.1 and Proposition 1.6. Let \( Y = \bigcup\{Y_n : n \in \mathbb{N}\} \) where each \( Y_n \) is compact and sequential. If \( A \) is a countably compact subspace of \( Y \), then for each \( n \in \mathbb{N} \), \( A \cap Y_n \) is countably compact, and therefore is closed in \( Y_n \). It follows that \( A \) is \( \sigma \)-compact, so it is compact. This proves the second statement. The last statement follows from the fact that \( 2^t > \mathfrak{c} \) implies that a compact space is sequential if and only if every its countably compact subspace is closed (Corollary 6.4 in [vDo]). \( \Box \)

*Remark.* The sequentiality of a compact space that is a countable union of sequential compact subspaces was proved under the assumption of Martin’s Axiom or \( \mathfrak{c} < 2^{\omega_1} \) in [Ra]. Both assumptions are stronger that \( 2^t > \mathfrak{c} \).

### 3. Some open problems

It is shown in [Ok2] that there are \( t \)-equivalent spaces \( X \) and \( Y \) such that \( X \) is bisequivalent and the tightness of \( Y \) is uncountable. The example, however, relies heavily on the non-normality of the space \( X \), so the following questions appear very interesting.

**3.1 Problem.** Let \( X \) and \( Y \) be \( t \)-equivalent normal spaces. Is it true that \( t(X) = t(Y) \)?
3.2 Problem. Let $X$ and $Y$ be $l$-equivalent normal spaces. Is it true that $t(X) = t(Y)$?

From Theorem 2.2 follows that if $X$ is $\sigma$-compact and all finite powers of $X$ have tightness $\leq \tau$, then every compact subspace in $Y$ has the tightness $\leq \tau$. The following version of Problem 1.1 remains open; it also appears more natural, because compactness is not preserved by $t$-equivalence [GH], while $\sigma$-compactness is [Ok1].

3.3 Problem. Let $X$ and $Y$ be $t$-equivalent $\sigma$-compact spaces. Is it true that $t(X) = t(Y)$?

3.4 Problem. Let $X$ and $Y$ be $l$-equivalent $\sigma$-compact spaces. Is it true that $t(X) = t(Y)$?

Note that the tightness is not preserved by $t$-images in the class of $\sigma$-compact spaces. Indeed, there are $\sigma$-compact spaces of uncountable tightness in which all compact subspaces are Fréchet — for example, consider the subspace $X$ of $I^{\omega_1}$ consisting of the $\sigma$-product with the center at 0 and the point whose all coordinates are equal to 1. This space is obviously a continuous image (and hence a $t$-image) of a countable direct sum of Eberlein compact spaces. Furthermore, using the construction as in Theorem III.1.11 in [Arh3] one can show that $X$ is a $t$-image of an Eberlein (hence, Fréchet) compact space.

A positive answer to the next question, suggested by Reznichenko, would be a big improvement of Corollary 1.5.

3.5 Problem. Let $X$ be a compact space. Is it true that $t(K) \leq t(X)$ for every compact subspace $K$ of $C_p(C_p(X))$?

The proof of the preservation of the tightness of compact spaces by the relation of $l$-equivalence given in [Tk1] in fact shows that if $X$ is compact, then $t(K) \leq t(X)$ for every compact set $K$ in the subspace $L_p(X)$ of $C_p(C_p(X))$ consisting of all linear continuous functions on $C_p(X)$.

Corollary 1.7 leaves open the next question:

3.6 Problem. Let $X$ and $Y$ be $t$-equivalent (or $l$-equivalent) compact spaces. Is it true in ZFC that the sequentiality of $X$ implies the sequentiality of $Y$?

Clearly, the answer is positive if it is true in ZFC that every compact space, which is a union of a countable family of sequential closed subspaces, is sequential.

The following interesting question was suggested by the referee:

3.7 Problem. Let $X$ and $Y$ be $t$-equivalent (or $l$-equivalent) compact spaces. Is it true that the orders of sequentiality of $X$ and $Y$ coincide?

In particular, it is unknown whether the Fréchet property is preserved by $l$-equivalence within the class of compact spaces (Problem 33 (1058) in [Arh2]).
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