

Yong Ge Tian; George P. H. Styan

A new rank formula for idempotent matrices with applications

Commentationes Mathematicae Universitatis Carolinae, Vol. 43 (2002), No. 2, 379--384

Persistent URL: <http://dml.cz/dmlcz/119327>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2002

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

A new rank formula for idempotent matrices with applications

YONGGE TIAN, GEORGE P.H. STYAN

Abstract. It is shown that

$$\text{rank}(P^*AQ) = \text{rank}(P^*A) + \text{rank}(AQ) - \text{rank}(A),$$

where A is idempotent, $[P, Q]$ has full row rank and $P^*Q = 0$. Some applications of the rank formula to generalized inverses of matrices are also presented.

Keywords: Drazin inverse, group inverse, idempotent matrix, inner inverse, rank, tripotent matrix

Classification: 15A03, 15A09

A complex square matrix A is said to be idempotent, or a projector, whenever $A^2 = A$; when A is idempotent, and Hermitian (or real symmetric), it is often called an orthogonal projector, otherwise an oblique projector. Projectors are closely linked to generalized inverses of matrices. For example, for a given matrix A the product $P_A = AA^+$ is well known as the orthogonal projector on the range (column space) of A , where A^+ is the Moore-Penrose inverse of A ; which is the unique solution of the following four Penrose equations

$$(i) AA^+A = A, \quad (ii) A^+AA^+ = A^+, \quad (iii) (AA^+)^* = AA^+, \quad (iv) (A^+A)^* = A^+A.$$

In addition, the products $AA^\#, AA^D$ and AA^- are also idempotent matrices, where $A^\#, A^D$ and A^- are the group inverse, the Drazin inverse, and an inner inverse of A , respectively. In a recent paper by Drury, Liu, Lu, Puntanen and Styan [1], a rank formula for the orthogonal projector P_A is established as follows

$$(1) \quad \text{rank}(P^*AA^+Q) = \text{rank}(AP) + \text{rank}(AQ) - \text{rank}(A),$$

where $A \in \mathbb{C}^{n \times n}$ is Hermitian nonnegative definite, $P \in \mathbb{C}^{n \times p}$ and $Q \in \mathbb{C}^{n \times q}$ such that $[P, Q]$ has full row rank and $P^*Q = 0$. This formula can help to establish several useful rank equalities for block matrices and orthogonal projectors when X and Y are properly chosen, see Drury *et al.* [1] and Tian [2]. This work motivates us to consider in general the rank of P^*AQ and various related topics, where A is idempotent, $[P, Q]$ has full row rank and $P^*Q = 0$. To do so, we need the following result.

Lemma 1. *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times k}$ and $C \in \mathbb{C}^{k \times l}$ be given. Then*

$$(2) \quad \text{rank}(ABC) = \text{rank}(AB) + \text{rank}(BC) - \text{rank}(B)$$

holds if and only if there are matrices X and Y such that $BCX + YAB = B$.

In fact it is well known that the equation $AX + YB = C$ is consistent if and only if

$$\text{rank} \begin{bmatrix} C & A \\ B & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

Applying this result to the equation $BCX + YAB = B$, we obtain Lemma 1.

Our main results are given below.

Theorem 2. *Let $A \in \mathbb{C}^{m \times m}$ be an idempotent matrix, and let $P \in \mathbb{C}^{m \times p}$ and $Q \in \mathbb{C}^{m \times q}$ be any two matrices such that $[P, Q]$ has full row rank and $P^*Q = 0$. Then*

$$(3) \quad \text{rank}(P^*AQ) = \text{rank}(P^*A) + \text{rank}(AQ) - \text{rank}(A).$$

PROOF: Since $[P, Q]$ has full row rank and $P^*Q = 0$, it follows that

$$[P, Q]^+ = \begin{bmatrix} P^+ \\ Q^+ \end{bmatrix} \quad \text{and} \quad [P, Q][P, Q]^+ = PP^+ + QQ^+ = I_m.$$

Let $X = Q^+A$ and $Y = A(P^+)^*$. Then we have

$$AQX + YP^*A = AQQ^+A + A(P^+)^*P^*A = A(I_m - PP^+)A + APP^+A = A.$$

Applying Lemma 1 to this equality yields (3). □

Now let $P = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$. Then $[P, Q]$ is of full row rank and $P^*Q = 0$. We derive from (3) the following result.

Corollary 3. *Suppose that*

$$(4) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m \times m}, \quad A_{12} \in \mathbb{C}^{m \times k}, \quad A_{21} \in \mathbb{C}^{k \times m}, \quad A_{22} \in \mathbb{C}^{k \times k}$$

is an idempotent matrix. Then the rank of A satisfies the following two rank equalities

$$(5) \quad \text{rank}(A) = \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} + \text{rank}[A_{11}, A_{12}] - \text{rank}(A_{12}),$$

and

$$(6) \quad \text{rank}(A) = \text{rank} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + \text{rank}[A_{21}, A_{22}] - \text{rank}(A_{21}).$$

Moreover, if

$$(7) \quad A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad A_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \leq i, j \leq p$$

is idempotent, then the rank of A satisfies the rank equalities

$$(8) \quad \text{rank}(A) = \text{rank}(Q_{1i}) + \text{rank}(Q_{i+1,p}) - \text{rank}(Q_{i+1,i}), \quad i = 1, 2, \dots, p-1,$$

where

$$Q_{ij} = \begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pj} \end{bmatrix}, \quad 1 \leq i, j \leq p.$$

The rank formulas in (8) are derived from (6). If the matrix A in (4) is an orthogonal projector, then (5) becomes

$$\text{rank}(A) = \text{rank}(A_{11}) + \text{rank}(A_{22}) - \text{rank}(A_{12}).$$

If we replace the idempotent matrix A in (5) by the idempotent matrix $I_{m+k} - A$, and note that $\text{rank}(I_{m+k} - A) = m + k - \text{rank}(A)$, then (5) becomes

$$\text{rank}(A) = m + k + \text{rank}(A_{12}) - \text{rank}[I_m - A_{11}, A_{12}] - \text{rank} \begin{bmatrix} A_{12} \\ I_k - A_{22} \end{bmatrix}.$$

The Drazin inverse A^D of a square matrix A with $\text{index}(A) = k$ is defined to be the unique solution of the three matrix equations

$$(i) \quad A^k X A = A^k, \quad (ii) \quad X A X = X, \quad (iii) \quad A X = X A.$$

When $\text{index}(A) = 1$, i.e., $\text{rank}(A^2) = \text{rank}(A)$, A^D is called the group inverse of A and denoted by $A^\#$. From $A^D A A^D = A^D$ we see that $A A^D A A^D = A A^D$. Thus $A A^D$ is idempotent. In addition, $\text{rank}(A^D) = \text{rank}(A A^D) = \text{rank}(A^k)$. In this case, applying Theorem 2 to $P^* A A^D Q$ and $P^* A A^\# Q$, we get the following corollary.

Corollary 4. *Let $A \in \mathbb{C}^{m \times m}$ be given with $\text{index}(A) = k$, let $P \in \mathbb{C}^{m \times p}$ and $Q \in \mathbb{C}^{m \times q}$ be any two matrices such that $[P, Q]$ has full row rank and $P^*Q = 0$. Then*

$$(9) \quad \text{rank}(P^*AA^DQ) = \text{rank}(P^*A^k) + \text{rank}(A^kQ) - \text{rank}(A^k).$$

In particular,

$$(10) \quad \text{rank}(P^*AA^\#Q) = \text{rank}(P^*A) + \text{rank}(AQ) - \text{rank}(A).$$

Let $P = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ in (10). We also have the following corollary.

Corollary 5. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m \times m}, \quad A_{12} \in \mathbb{C}^{m \times k}, \quad A_{21} \in \mathbb{C}^{k \times m}, \quad A_{22} \in \mathbb{C}^{k \times k}$$

with $\text{index}(A) = 1$, and denote by $(AA^\#)_{12}$ the upper-right $m \times k$ block of the projector $AA^\#$. Then the rank of $(AA^\#)_{12}$ is

$$(11) \quad \text{rank}[(AA^\#)_{12}] = \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} + \text{rank}[A_{11}, A_{12}] - \text{rk}(A).$$

A square matrix A is called tripotent if $A^3 = A$. For the tripotent matrix A , its group inverse is $A^\# = A$. Now applying (9) to a tripotent matrix A and noting that

$$(AA^\#)_{12} = (A^2)_{12} = [A_{11}, A_{12}] \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix},$$

we obtain the following result.

Corollary 6. *Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m \times m}, \quad A_{12} \in \mathbb{C}^{m \times k}, \quad A_{21} \in \mathbb{C}^{k \times m}, \quad A_{22} \in \mathbb{C}^{k \times k}$$

is a tripotent matrix. Then the rank of A satisfies the following two rank equalities

$$(12) \quad \text{rank}(A) = \text{rank}[A_{11}, A_{12}] + \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} - \text{rank} \left([A_{11}, A_{12}] \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \right),$$

and

$$(13) \quad \text{rank}(A) = \text{rank}[A_{21}, A_{22}] + \text{rank} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} - \text{rank} \left([A_{21}, A_{22}] \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \right).$$

Finally, we present a result for a triangular inner inverse of an idempotent matrix. We will use the following result due to Tian [3, Corollary 4.3].

Lemma 7. *The block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad \text{where } A_{ij} \in \mathbb{C}^{s_i \times t_j}, \quad 1 \leq i, j \leq p,$$

has an inner inverse with the upper triangular block form

$$A^- = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ & S_{22} & \cdots & S_{2p} \\ & & \ddots & \vdots \\ & & & S_{pp} \end{bmatrix}, \quad S_{ij} \in \mathbb{C}^{t_i \times s_j}, \quad 1 \leq i, j \leq p$$

if and only if

$$\text{rank}(A) = \text{rank}(Q_{1i}) + \text{rank}(Q_{i+1,p}) - \text{rank}(Q_{i+1,i}), \quad i = 1, 2, \dots, p-1,$$

where

$$Q_{ij} = \begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pj} \end{bmatrix}, \quad 1 \leq i, j \leq p.$$

Applying this lemma to the idempotent block matrix A in (7) under (8), we immediately see that

Theorem 8. *If the block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad \text{where } A_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \leq i, j \leq p,$$

is idempotent, then it must have an inner inverse with the upper triangular block form

$$A^- = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ & S_{22} & \cdots & S_{2p} \\ & & \ddots & \vdots \\ & & & S_{pp} \end{bmatrix}, \quad S_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \leq i, j \leq p.$$

In particular, if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \in \mathbb{C}^{m \times m}, A_{12} \in \mathbb{C}^{m \times k}, A_{21} \in \mathbb{C}^{k \times m}, A_{22} \in \mathbb{C}^{k \times k},$$

is idempotent, then it must have an inner inverse with the upper triangular block form

$$A^- = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}, \quad G_{11} \in \mathbb{C}^{m \times m}, \quad G_{12} \in \mathbb{C}^{m \times k}, \quad G_{22} \in \mathbb{C}^{k \times k}.$$

For more rank equalities for idempotent matrices, see the authors' recent paper [4].

Acknowledgment. The research of both authors was supported in part by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

- [1] Drury S.W., Liu S., Lu C.Y., Puntanen S., Styan G.P.H., *Some comments on several matrix inequalities with applications to canonical correlations: historical background and recent developments*, Report A332 (December 2000), Dept. of Mathematics, Statistics and Philosophy, University of Tampere, Tampere, Finland, 63 pp. To be published in the special issue of *Sankhyā: The Indian Journal of Statistics, Series A* associated with "An International Conference in Honor of Professor C.R. Rao on the Occasion of his 80th Birthday, Statistics: Reflections on the Past and Visions for the Future, The University of Texas at San Antonio, March 2000".
- [2] Tian Y., *Two rank equalities associated with blocks of orthogonal projector*. Problem 25-4, Image, The Bulletin of the International Linear Algebra Society **25** (2000), p. 16 [Solutions by J.K. Baksalary & O.M. Baksalary, by H.J. Werner, and by S. Puntanen, G.P.H. Styan & Y. Tian, Image, The Bulletin of the International Linear Algebra Society **26** (2001), 6-9].
- [3] Tian Y., *Completing block matrices with maximal and minimal ranks*, Linear Algebra Appl. **321** (2000), 327-345.
- [4] Tian Y., Styan, G.P.H., *Some rank equalities for idempotent and involutory matrices*, Linear Algebra Appl. **335** (2001), 101-117.

DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON,
ONTARIO K7L 3N6, CANADA

E-mail: ytian@mast.queensu.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTRÉAL,
QUÉBEC H3A 2K6, CANADA

E-mail: styan@together.net

(Received May 25, 2001, revised January 28, 2002)