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On $D$-property of strong $\Sigma$ spaces

RAUSHAN Z. BUZYAKOVA

Abstract. It is shown that every strong $\Sigma$ space is a $D$-space. In particular, it follows that every paracompact $\Sigma$ space is a $D$-space.

Keywords: strong $\Sigma$ space, $D$-space

Classification: 54D20, 54F99

In this paper we will show that any strong $\Sigma$ space is a $D$-space. This result positively answers Borges and Matveev’s question whether any paracompact $\Sigma$ space is a $D$-space. The notion of $D$-space was introduced by Eric van Douwen [6].

A neighborhood assignment for a space $X$ is a function $\varphi$ from $X$ to the topology of $X$ such that $x \in \varphi(x)$ for any $x \in X$. A space $X$ is a $D$-space, if for any neighborhood assignment $\varphi$ for $X$ there exists a closed discrete subset $D$ of $X$ such that $X = \bigcup_{d \in D} \varphi(d)$.

It is natural to ask which spaces possess the $D$-property. It is known that $\sigma$-compact spaces, metrizable spaces, semi-stratifiable spaces, and paracompact $p$-spaces are all $D$-spaces (see [5], [2]). In [5], DeCaux showed that every finite product of copies of the Sorgenfrey line is a $D$-space. The $D$-property of subspaces of generalized ordered spaces was studied in [8]. In a recent paper [10] of Fleissner and Stanley, the authors give conditions under which a subspace of a product of finitely many ordinals is a $D$-space. Several interesting questions on $D$-spaces were raised by E. van Douwen and W.F. Pfeffer in [7], which was the first published paper that contained results on $D$-spaces. Some other results and questions on $D$-spaces can be found in [5], [2], [3], [4], [8], [10].

The result in this article is obtained in an attempt to answer E.K. van Douwen’s question whether each Lindelöf space is a $D$-space. However, this question remains unanswered. And, one of approaches to solve this problem could be to consider continuous images of Lindelöf $D$-spaces.

Question (A. V. Arhangel’skii). Is it true that a continuous image of a Lindelöf $D$-space is a $D$-space?

We consider only Tychonoff spaces. In notation and terminology, we will follow [9].
A space $X$ is a strong $\Sigma$ space if there exist a $\sigma$-locally-finite family $\gamma$ of closed sets in $X$ and a cover $\mathcal{K}$ of $X$ by compact subsets, such that for any open set $U$ containing an element $K$ of $\mathcal{K}$, $K \subseteq \Gamma \subseteq U$ for some $\Gamma \in \gamma$.

The class of strong $\Sigma$ spaces is wide and it contains all metrizable spaces, $\sigma$-compact spaces, Lindelöf $\Sigma$ spaces, paracompact $\Sigma$ spaces, paracompact $p$-spaces, Moore spaces, spaces with countable network, as well as spaces with $\sigma$-discrete network ($\sigma$ spaces). Thus, our result implies that the mentioned spaces are all D-spaces. In addition, the product of a Lindelöf $\Sigma$ space with a Moore space is still a D-space. However, as shown in [4], in general case the product of two D-spaces need not be a D-space.

**Theorem.** Every strong $\Sigma$ space $X$ is a D-space.

**Proof:** Let $\mathcal{K}$ and $\gamma$ be the families from the definition of a strong $\Sigma$ space. Represent $\gamma$ as $\bigcup \{\gamma_n\}$, where each $\gamma_n$ is a locally-finite family of closed sets in $X$ and $\gamma_n \subseteq \gamma_{n+1}$. Enumerate each $\gamma_n = \{\Gamma_n^\alpha\}$, where $\alpha$ ranges through some ordinal number.

Let $\varphi$ be an arbitrary neighborhood assignment. We need to find a discrete closed subset $D$ in $X$ such that $X = \bigcup_{d \in D} \varphi(d)$. Recursively, we will define closed discrete sets $D_n$ such that $D = \bigcup D_n$.

**Step 0.** Set $D_0 = \emptyset$. Assume $D_m$ is defined for all $0 < m < n$.

**Step n.** Recursively, we will define finite sets $D_n^\alpha$ such that $D_n = (\bigcup D_n^\alpha) \cup D_{n-1}$.

**Sub-step 0.** Set $D_0^\alpha = \emptyset$. Assume $D_\beta^\alpha$ is defined for all $0 < \beta < \alpha$.

**Sub-step $\alpha$.** Let $U = \bigcup \{\varphi(d) : d \in (\bigcup_{\beta < \alpha} D_\beta^\alpha) \cup D_{n-1}\}$. Take the first $\Gamma$ in $\gamma_n$ that satisfies the following requirement.

**Requirement $R_n^\alpha$:** there exists $K \in \mathcal{K}$ which is not fully covered by $U$. And there exist $x_1, \ldots, x_k \in K \setminus U$ such that $K \setminus U \subseteq \Gamma \setminus U \subseteq \varphi(x_1) \cup \cdots \cup \varphi(x_k)$.

If no such $\Gamma$ exists, sub-recursion stops. Put $D_n^\alpha = \{x_1, \ldots, x_k\}$.

Let $D_n = (\bigcup D_n^\alpha) \cup D_{n-1}$. We need to show that $D_n$ is closed and discrete in $X$. Take an arbitrary $x \in X$. We need to separate $x$ from $D_n \setminus \{x\}$ by a neighborhood. Consider the family

$$\gamma' = \{\Gamma_\beta : \Gamma_\beta \text{ is the first in } \gamma_n \text{ satisfying Requirement } R_n^\alpha \text{ for some } \alpha\}.$$

Since $\gamma'_n \subseteq \gamma_n$, $\gamma'_n$ is locally-finite too. Therefore, there exists a neighborhood of $x$ that intersects only a finite number of elements in $\gamma'_n$, and therefore, only
finite number of sets $D^n_\alpha$'s. Since the $D^n_\alpha$'s are finite, $x$ is not in the closure of $(\bigcup D^n_\alpha) \setminus \{x\}$. And $x$ can be separated from $D_{n-1} \setminus \{x\}$ since the latter is closed and discrete by assumption.

The construction is complete. Put $D = \bigcup D_n$.

Let us show that $X = \bigcup_{d \in D} \varphi(d)$. Assume the contrary. Then there exists a $K$ in $\mathcal{K}$ such that $K' = K \setminus \bigcup_{d \in D} \varphi(d) \neq \emptyset$. Since $K'$ is compact we can find $x_1, \ldots, x_k \in K'$ such that $K' \subseteq \varphi(x_1) \cup \cdots \cup \varphi(x_k)$. Consider a compactum $K'' = K \setminus (\varphi(x_1) \cup \cdots \cup \varphi(x_k))$. Find the smallest $n$ such that $K'' \subseteq \bigcup_{d \in D_n} \varphi(d)$.

Now take the first $\gamma_l$ containing such a $\Gamma$ that

$$K \subseteq \Gamma \subseteq \varphi(x_1) \cup \cdots \cup \varphi(x_k) \cup \left( \bigcup_{d \in D_n} \varphi(d) \right).$$

Let $m = \max\{n, l\}$. Then $\gamma_l \subseteq \gamma_{m+1}$, and therefore, $\Gamma \in \gamma_{m+1}$. By the choice of $n$ and $l$, $\Gamma$ satisfies the Requirement starting not later than from Sub-step 1 of Step $m+1$. And eventually, $\Gamma$ will be the first in $\gamma_{m+1}$ satisfying the Requirement. Therefore, $\Gamma$ must be covered by $\bigcup_{d \in D} \varphi(d)$, and so must $K$.

Let us show now that $D$ is closed and discrete. Take an arbitrary $x \in X$. We need to show that $x$ can be separated from $D \setminus \{x\}$ by a neighborhood of $x$. There exists an $n$ such that $x \in \bigcup_{d \in D_n} \varphi(d)$. This means that $x$ is separated from $D_n \setminus \{x\}$ by $\bigcup_{d \in D_n} \varphi(d)$ (follows from the construction of $D_n$'s). And $x$ can be separated from $D_n \setminus \{x\}$, since $D_n$ is closed and discrete. □

References


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