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## Characterization of $\omega$ -limit sets of continuous maps of the circle

DAVID POKLUDA

*Abstract.* In this paper we extend results of Blokh, Bruckner, Humke and Smítal [Trans. Amer. Math. Soc. **348** (1996), 1357–1372] about characterization of  $\omega$ -limit sets from the class  $\mathcal{C}(I, I)$  of continuous maps of the interval to the class  $\mathcal{C}(\mathbb{S}, \mathbb{S})$  of continuous maps of the circle. Among others we give geometric characterization of  $\omega$ -limit sets and then we prove that the family of  $\omega$ -limit sets is closed with respect to the Hausdorff metric.

*Keywords:* dynamical system, circle map,  $\omega$ -limit set

*Classification:* Primary 37E10, 37B99, 26A18

### 1. Introduction

Continuous maps of the interval and continuous maps of the circle have many properties in common. Some of them are proved in [6]. In this paper we extend results proved in [3] from the class  $\mathcal{C}(I, I)$  of continuous maps of the interval to the class  $\mathcal{C}(\mathbb{S}, \mathbb{S})$  of continuous maps of the circle by using the same technique used in [6]. Other results concerning continuous maps of the circle can be found in [1] or [5].

Throughout the paper, the symbols  $I$  and  $\mathbb{S}$  denote the unit interval  $[0, 1]$  and the circle  $\{z \in \mathbb{C}; |z| = 1\}$ , respectively, and  $X$  denotes either  $I$  or  $\mathbb{S}$ . Denote by  $\mathbb{S}_b$  the circle cut at a point  $b \in \mathbb{S}$ , i.e.  $\mathbb{S}_b = \mathbb{S} \setminus \{b\}$ . Let  $e : \mathbb{R} \rightarrow \mathbb{S}$  be the natural projection defined by  $e(x) = \exp(2\pi i x)$ . Note that the map  $\tilde{e} : (v, v+1) \rightarrow \mathbb{S}_{e(v)}$  obtained by restricting  $e$  to the interval  $(v, v+1)$ , is a homeomorphism. It is clear that if we define a map  $h_v(x) := e(x+v)$ , where  $v \in \mathbb{R}$ , then  $\tilde{h}_v := h_v|_{(0,1)}$  is a homeomorphism from  $(0, 1)$  onto  $\mathbb{S} \setminus \{e(v)\}$  (see Lemma 3.1.3 in [1]). We say that  $\tilde{h}_v(x) \leq \tilde{h}_v(y)$  whenever  $x \leq y$ . For an interval  $A \subset \mathbb{S}_{e(v)}$  a point  $a$  is called the *left endpoint*, resp. the *right endpoint*, of  $A$  if  $a \leq x$ , resp.  $x \leq a$ , for every  $x \in A$ . Recall that the *trajectory* of a point  $x$  under a map  $f$  is the sequence  $\{f^n(x)\}_{n=0}^\infty$ , where  $f^n$  is the  $n$ -th iteration of  $f$ . The set of limit points of the trajectory of

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$x$  is called  $\omega$ -limit set and we denote the set by  $\omega_f(x)$ . A set  $\{U_0, \dots, U_{n-1}\}$  of mutually disjoint intervals is called a *cycle of intervals* if  $f(U_i) = U_{i+1}$  for  $i = 0, 1, \dots, n - 2$  and  $f(U_{n-1}) = U_0$ . The map  $f$  is *transitive* if for every two non-empty open sets  $V, W$  there is a positive integer  $n$  such, that  $f^n(V) \cap W \neq \emptyset$ . Two maps  $f : Y_1 \rightarrow Y_1$  and  $g : Y_2 \rightarrow Y_2$  are *topologically conjugate* if there exists a homeomorphism  $\varphi : Y_1 \rightarrow Y_2$  such that  $\varphi \circ f(x) = g \circ \varphi(x)$  for any  $x \in Y_1$ . For more terminology see standard books like [1] or [2].

Now we introduce some notions used in [3] and modified for maps from  $\mathcal{C}(\mathbb{S}, \mathbb{S})$ . We say that a set  $A \subset \mathbb{S}$  is  $T$ -side or  $T$ -unilateral neighborhood ( $T$  means either “left” or “right”) of an  $x \in \mathbb{S}$  if the set  $A$  is a closed interval and the point  $x$  is  $T$  endpoint of the set  $A$ .

Let  $U = U_0 \cup \dots \cup U_{N-1}$  be a union of pairwise disjoint non-degenerate closed intervals and  $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$ . For any set  $K \subset U$  let  $f_U(K) = f(K) \cap U$  (this may be empty). Inductively define  $f_U^n(K) = f_U(f_U^{n-1}(K))$ . Define  $\tilde{K} \equiv \tilde{K}(U) = \bigcup_{i=1}^\infty f_U^i(K)$ ; although  $\tilde{K}$  depends on  $U$ , to avoid convoluted notation we use  $\tilde{K}$  whenever the set  $U$  is evident. Let  $A \subset \mathbb{S}$  be a closed set and  $x \in A$ . We say that a side  $T$  of a point  $x$  is  $A$ -covering if for any union of finitely many closed intervals  $U$  such that  $A \subset \text{Int}(U)$  and any closed  $T$ -unilateral neighborhood  $V(x)$  there are finitely many components of  $\tilde{V}(x)$  such that the closure of their union covers  $A$ . If  $T$  is an  $A$ -covering side of  $x$  then any  $T$ -unilateral neighborhood  $V(x)$  is also said to be  $A$ -covering. We call the set  $A$  *locally expanding* according to the map  $f$  if every  $x \in A$  has an  $A$ -covering side.

The main theorems of this paper are the following.

**Theorem 1.1.** *Let  $f$  be a map in  $\mathcal{C}(X, X)$ . A closed set  $A \subset X$  is an  $\omega$ -limit set if and only if it is locally expanding.*

**Theorem 1.2.** *Let  $\{\omega_n\}_{n=1}^\infty = \{\omega_f(x_n)\}_{n=1}^\infty$  be a sequence of  $\omega$ -limit sets of a continuous map  $f \in \mathcal{C}(X, X)$  and let a point  $a$  have a side  $T$ , such that for any  $T$ -unilateral neighborhood  $V$  of  $a$ , there exists a positive integer  $N$  such that for each  $n \geq N$ , the orbit of  $x_n$  enters  $V$  infinitely many times. Then  $\bigcap_{k=1}^\infty \overline{\bigcup_{n=k}^\infty \omega_n}$  is an  $\omega$ -limit set.*

**Theorem 1.3.** *Let  $f$  be a map in  $\mathcal{C}(X, X)$ . Then the family of all  $\omega$ -limit sets of  $f$  endowed with the Hausdorff metric is compact.*

**2. Proof of the main theorems**

Let  $b \in \mathbb{S}$  and  $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$ . We denote by  $e^{-1}(b)$  the point  $x \in [0, 1)$  such that  $e(x) = b$ . In the rest of the paper by  $h$  we denote the map  $\tilde{h}_{e^{-1}(b)}$  whenever the point  $b \in \mathbb{S}$  is evident and by  $A^*$  we mean the preimage of the set  $A \subset \mathbb{S}_b$  under the map  $\tilde{h}_{e^{-1}(b)}$ . Denote by  $S$  the set  $\mathbb{S} \setminus \bigcup_{n=0}^\infty f^{-n}(b)$ . Now we can define a map  $f^* \in \mathcal{C}(S^*, S^*)$  as

$$f^* := h^{-1} \circ f \circ h|_{S^*}.$$

The map  $f^*$  is defined only on the subset  $S^*$  of the interval  $(0, 1)$ , but we overcome this difficulty using Lemma 2.1.

**Lemma 2.1.** *Let  $f \in \mathcal{C}(X, X)$  and  $A \subset X$  be a locally expanding set according to the map  $f$ . Then the set  $A$  is invariant, i.e.  $f(A) \subset A$ .*

PROOF: In the case when  $X = I$  the lemma is proved in [3] (Lemma 2.5). It remains to consider the case  $X = \mathbb{S}$ . The case  $A = \mathbb{S}$  is trivial. Let  $A \subset \mathbb{S}_b$ ,  $x \in A$  and  $f(x) \notin A$ . Then there exists a union of finitely many intervals  $U = U_0 \cup \dots \cup U_{n-1}$ ,  $U \supset A$  such that for any sufficiently small neighborhood  $V$  of  $x$  we have  $f(V) \cap U = \emptyset$ . The definition of  $\tilde{V}$  implies that  $\tilde{V} = \emptyset$  which is a contradiction.  $\square$

**Lemma 2.2.** *A set  $A \subset \mathbb{S}$  is a  $T$ -side of a point  $x \in \mathbb{S}$  if and only if the set  $A^*$  is a  $T$ -side of the point  $x^*$ .*

The proof is omitted.

**Lemma 2.3.** *If the whole circle  $\mathbb{S}$  is locally expanding with respect to a map  $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$  then  $f$  is transitive.*

PROOF: Take two nonempty open sets  $V, W$ . Since a point  $x \in \text{Int}(V)$  has  $\mathbb{S}$  covering side then  $\tilde{V} = \mathbb{S}$  and hence there is a positive integer  $n$  such that  $f^n(V) \cap W \neq \emptyset$ . This proves that the map  $f$  is transitive.  $\square$

**Lemma 2.4.** *Let  $f$  be a map in  $\mathcal{C}(\mathbb{S}, \mathbb{S})$ . A closed set  $A \subset \mathbb{S}_b$  is locally expanding according to the map  $f$  if and only if the set  $A^* \subset (0, 1)$  is locally expanding according to the map  $f^*$ .*

PROOF: First assume that the set  $A^*$  is locally expanding. Hence the sets  $A^*, A$  are closed and by Lemma 2.1 the set  $A^*$  is invariant and  $A^* \subset S^*$ . Take a point  $x \in A$ . Since the set  $A^*$  is locally expanding the point  $x^*$  has an  $A^*$ -covering side  $T^*$ . By Lemma 2.2 the set  $T$  is a side of the point  $x$ . Take a union of finitely many closed intervals  $U \subset \mathbb{S}_b$  such that  $A \subset \text{Int}(U)$  and any closed  $T$ -unilateral neighborhood  $V(x)$ . Using the assumptions there are finitely many components of  $\tilde{W}$  where  $W = V(x)^*$  such that the closure of their union covers  $A^*$  and clearly  $\tilde{W} \subset (0, 1)$ . Hence the set  $\tilde{V}(x)$  has finitely many components such that the closure of their union covers  $A$  as well. Thus the set  $A$  is locally expanding.

The proof of the converse is analogous.  $\square$

**Lemma 2.5.** *A set  $A \subset \mathbb{S}_b$  is an  $\omega$ -limit set of the map  $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$  if and only if the set  $A^*$  is an  $\omega$ -limit set of the map  $f^*$ .*

PROOF: First consider the closed set  $A \subset \mathbb{S}_b$  to be an  $\omega$ -limit set. There is a point  $x_0 \in \mathbb{S}$  such that  $\omega_f(x_0) = A$ . If there are two positive integers  $m_1 < m_2$  such that  $f^{m_1}(x_0) = f^{m_2}(x_0) = b$  then the  $\omega$ -limit set  $A$  is finite and  $b \in A$  which is a contradiction. We may assume that  $f^n(x_0) \neq b$  for every positive integer  $n$  (in

the case when there is just one positive integer  $m$  such that  $f^m(x_0) = b$  we replace  $x_0$  by  $f^{m+1}(x_0)$  and thus  $\{f^n(x_0)\}_{n=0}^\infty \subset S$ . Hence  $(\{f^n(x_0)\}_{n=0}^\infty)^* \subset S^*$  and we have  $\omega_{f^*}(x_0^*) = (\omega_f(x_0))^* = A^*$ .

The proof of the converse is analogous. □

Before stating the next lemma, let us recall one of Blokh's results from [4].

**Proposition 2.6.** *Suppose that  $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$  is a transitive map. Then there is a positive integer  $m$ , such that  $\mathbb{S} = \bigcup_{i=0}^{m-1} K_i$ , where all the  $K_i$  are connected compact sets,  $K_i \cap K_j$  is finite for  $i \neq j$ ,  $f(K_i) = K_{i+1}$ ,  $i = 0, 1, \dots, m - 2$ ,  $f(K_{m-1}) = K_0$  and two cases are possible:*

- (1)  $P(f) \neq \emptyset$ ; then  $f^{mq}|_{K_i}$  is transitive for any  $i = 0, 1, \dots, m - 1$  and any positive integer  $q$ ,
- (2)  $P(f) = \emptyset$ ; then  $m = 1$ ,  $K_0 = \mathbb{S}$  and  $f$  is conjugate to an irrational rotation.

**Lemma 2.7** (Lemma 2.6 in [3] for  $\mathcal{C}(I, I)$ ). *Let  $f$  be a map in  $\mathcal{C}(\mathbb{S}, \mathbb{S})$  and  $A \subset \mathbb{S}$  be a locally expanding set according to the map  $f$  with non-empty interior. Then  $A$  is a cycle of intervals and  $f|_A$  is transitive.*

PROOF: Suppose that  $A = \mathbb{S}$ . By Lemma 2.3 the map  $f$  is transitive and by Proposition 2.6 the set  $A$  must be a cycle of intervals. Suppose that  $A \subset \mathbb{S}_b$ . Since  $A$  is locally expanding then by Lemma 2.1  $A \subset S$  and by Lemma 2.4 the set  $A^* \subset S^*$  is locally expanding. By Lemma 2.6 in [3] the set  $A^*$  is a cycle of intervals  $A_0^*, \dots, A_{n-1}^*$  and  $f^*|_{A^*}$  is transitive. The map  $h$  is a homeomorphism and hence the set  $A = h(A^*) = h(A_0^*) \cup \dots \cup h(A_{n-1}^*)$  and

$$\begin{aligned} f(A_i) &= \left( h \circ f^* \circ h^{-1}|_S \right) (A_i) = \left( h \circ f^* \circ h^{-1}|_S \right) (h(A_i^*)) \\ &= h(f^*(A_i^*)) = h(A_{i+1}^*) = A_{i+1}, \end{aligned}$$

where  $A_j = h(A_j^*)$  and  $j$  is taken modulo  $n$ . This means that  $A$  is a cycle of intervals. It remains to show that  $f|_A$  is transitive when  $A \subset \mathbb{S}_b$ . Take two open sets  $V, W \subset A$ . Then the sets  $V^*, W^* \subset S^*$  are open sets and so there is a positive integer  $n$  such that  $(f^*)^n(V^*) \cap W^* \neq \emptyset$ . Hence

$$f^n(V) \cap W = (h \circ (f^*)^n \circ h^{-1}|_S)(V) \cap W = h((f^*)^n(V^*) \cap W^*) \neq \emptyset.$$

□

**Lemma 2.8** (Lemma 2.7 in [3] for  $\mathcal{C}(I, I)$ ). *Let  $f$  be a map in  $\mathcal{C}(\mathbb{S}, \mathbb{S})$  and  $A \subset \mathbb{S}$  be a locally expanding or an  $\omega$ -limit set. Then  $f(A) = A$ .*

PROOF: The case of an  $\omega$ -limit set is trivial and well known. Let  $A$  be a locally expanding set. When  $A = \mathbb{S}$  then  $f$  is transitive (Lemma 2.3) and the lemma is

proved. It remains to consider the case when  $A \subset \mathbb{S}_b$ . By Lemma 2.1  $A \subset S$ , and by Lemma 2.7 in [3] we have  $f^*(A^*) = A^*$ . Clearly

$$f(A) = \left( h \circ f^* \circ h^{-1} \Big|_S \right) (A) = h(f^*(A^*)) = h(A^*) = A.$$

□

We continue by proving the main theorems.

**PROOF OF THEOREM 1.1:** In the case when  $X = I$  the theorem is proved in [3] (Theorem 2.12). It remains to consider the case when  $X = \mathbb{S}$ . First we show that if  $A = \omega_f(x)$  is an  $\omega$ -limit set then  $A$  is locally expanding. In the case  $A \subset \mathbb{S}_b$ ,  $A^*$  is an  $\omega$ -limit set by Lemma 2.5, hence  $A^*$  is locally expanding (see Theorem 2.12 in [3]) and by Lemma 2.4, the set  $A$  is locally expanding as well. It remains to consider the case when  $A = \mathbb{S}$ . Since  $A$  is an  $\omega$ -limit set and it has a non-empty interior,  $A$  is a cycle of intervals (see Theorem 1.1 in [6]). From this it follows that if  $W \subset A$  is an interval, then  $W$  has a dense orbit in  $A$  and hence there is an  $n \in \mathbb{N}$  such that  $f^n(W) \cap W \neq \emptyset$ . Therefore the union  $\bigcup_{i=1}^{\infty} f^i(W)$  is dense in  $A$  and has finitely many component intervals. As this is true for every such interval  $W$ , it follows that  $A$  is locally expanding.

Assume that  $A$  is locally expanding. In the case  $A \subset \mathbb{S}_b$  we can again prove the theorem by using our Lemmas 2.4 and 2.5, and Theorem 2.12 in [3]. It remains to consider the case when  $A = \mathbb{S}$ . By Lemma 2.7 the set  $A$  is a cycle of intervals and  $f|_A$  is transitive. Thus the set  $A$  is an  $\omega$ -limit set. □

**PROOF OF THEOREM 1.2:** In the case when  $X = I$  the theorem is proved in [3] (Theorem 3.1). It remains to consider the case when  $X = \mathbb{S}$ . We will prove this in several steps.

*Case 1.* Assume that  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \subset \mathbb{S}_b$ . Using our Lemma 2.5 and Theorem 3.1 in [3] we get that the set  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n^*}$  is an  $\omega$ -limit set. By Lemma 2.5 the set  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n}$  is an  $\omega$ -limit set as well.

*Case 2.* Next assume that  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} = \mathbb{S}$ . Then it suffices to show that  $f$  is transitive. Take two non-empty open sets  $V, W \subset \mathbb{S}$ .

*Subcase 2.1.* If there is an  $m$  such that  $\omega_m$  intersects both  $V$  and  $W$  we are done since there are positive integers  $p < q$  such that  $f^p(x_m) \in V$  and  $f^q(x_m) \in W$  and consequently,  $f^{q-p}(V) \cap W \neq \emptyset$ .

*Subcase 2.2.* If there is no such  $m$ , then  $V \cap W = \emptyset$ . Let  $\{\omega_{n_i}\}_{i=1}^{\infty}$  be the subsequence of  $\{\omega_n\}_{n=1}^{\infty}$  consisting of  $\omega$ -limit sets intersecting  $V$ . Then  $\omega_V = \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} \omega_{n_i}} \subset \mathbb{S}_b$  for any  $b \in W$ , hence, according to the first part,  $\omega_V = \omega_f(v)$  is an  $\omega$ -limit set,  $\omega_f(v) \cap W = \emptyset$ , and  $a \in \omega_f(v)$  is its cluster point from the side  $T$ . Similarly, for some  $w$ ,  $\omega_f(w)$  is an  $\omega$ -limit set intersecting  $W$  such that  $\omega_f(w) \cap V = \emptyset$  and  $a$  is its cluster point from the side  $T$ . Let  $A = \omega_f(v) \cup \omega_f(w)$ .

*Subcase 2.2.1.* If  $A \neq \mathbb{S}$  then  $A \subset \mathbb{S}_b$  for some  $b$ . We apply the result by Sharkovsky [7] which is also stated in [3]: If, for a map in  $\mathcal{C}(I, I)$ , two  $\omega$ -limit sets have a common cluster point from the same side then their union is an  $\omega$ -limit set. So, by Lemma 2.5  $A$  is an  $\omega$ -limit set since both  $(\omega_V)^*$  and  $(\omega_W)^*$  are and have a point  $a^*$  as a common cluster point from side  $T$ . We have the situation described in Subcase 2.1.

*Subcase 2.2.2.*  $A = \omega_f(v) \cup \omega_f(w) = \mathbb{S}$ . Since any  $\omega$ -limit set in  $\mathbb{S}$  is either nowhere dense or a finite union of non-degenerate intervals, and since  $\omega_f(v) \cap W = \emptyset = \omega_f(w) \cap V$ , both  $\omega_f(v)$  and  $\omega_f(w)$  are finite unions of intervals. If  $\omega_f(v) \cap \omega_f(w)$  is infinite then the two  $\omega$ -limit sets have an interval in common and the transitivity is easily proven. If the intersection  $\omega_f(v) \cap \omega_f(w)$  would be finite then the condition with the  $T$ -side must be violated since the intersection contains  $a$ .  $\square$

PROOF OF THEOREM 1.3: In the case when  $X = I$  the theorem is proved in [3] (Theorem 3.2). It remains to consider the case when  $X = \mathbb{S}$ . Let  $\{\omega_1, \omega_2, \dots\}$  be a sequence of  $\omega$ -limit sets converging in the Hausdorff metric to a set  $A$ . Choosing a subsequence (if necessary) we may also assume that there exists a point  $a$ , a side  $T$  of  $a$  and points  $a_n \in \omega_n$ ,  $a_n \neq a$  converging to  $a$  from  $T$ . As the original sequence converges to  $A$ , the subsequence does as well. To finish the proof it remains to use Theorem 1.2 and to show that  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} = A$ . Since we consider Hausdorff metric and all the sets  $\omega_n$  are closed then the set  $A$  is closed as well. Hence it is clear that  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \supset A$ . Consider the sequence of open sets  $\{A_{1/n}\}_{n=1}^{\infty}$  where  $A_\varepsilon := \{x \in X; \text{dist}(x, A) < \varepsilon\}$ ,  $\text{dist}(x, A) := \inf\{d(x, a); a \in A\}$  and  $d$  is the metric on  $X$ , and note that for every  $m$  there is a positive integer  $k$  such that  $\overline{\bigcup_{n=k}^{\infty} \omega_n} \subset A_{1/m}$ . Therefore  $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \subset A$ .  $\square$

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