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Commentationes Mathematicae Universitatis Carolinae, Vol. 43 (2002), No. 4, 675--685

Persistent URL: <http://dml.cz/dmlcz/119356>

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Topological games and product spaces

S. GARCÍA-FERREIRA, R.A. GONZÁLEZ-SILVA, A.H. TOMITA

Abstract. In this paper, we deal with the product of spaces which are either \mathcal{G} -spaces or \mathcal{G}_p -spaces, for some $p \in \omega^*$. These spaces are defined in terms of a two-person infinite game over a topological space. All countably compact spaces are \mathcal{G} -spaces, and every \mathcal{G}_p -space is a \mathcal{G} -space, for every $p \in \omega^*$. We prove that if $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu < \omega_1} X_\mu$ is a \mathcal{G} -space, then there is $A \in [\omega_1]^{\leq \omega}$ such that X_μ is countably compact for every $\mu \in \omega_1 \setminus A$. As a consequence, X^{ω_1} is a \mathcal{G} -space iff X^{ω_1} is countably compact, and if X^{2^c} is a \mathcal{G} -space, then all powers of X are countably compact. It is easy to prove that the product of a countable family of \mathcal{G}_p spaces is a \mathcal{G}_p -space, for every $p \in \omega^*$. For every $1 \leq n < \omega$, we construct a space X such that X^n is countably compact and X^{n+1} is not a \mathcal{G} -space. If $p, q \in \omega^*$ are *RK*-incomparable, then we construct a \mathcal{G}_p -space X and a \mathcal{G}_q -space Y such that $X \times Y$ is not a \mathcal{G} -space. We give an example of two free ultrafilters p and q on ω such that $p <_{RK} q$, p and q are *RF*-incomparable, $p \approx_C q$ (\leq_C is the *Comfort* order on ω^*) and there are a \mathcal{G}_p -space X and a \mathcal{G}_q -space Y whose product $X \times Y$ is not a \mathcal{G} -space.

Keywords: *RF*-order, *RK*-order, *Comfort*-order, *p*-limit, *p*-compact, \mathcal{G} -space, \mathcal{G}_p -space, countably compact

Classification: Primary 54A35, 03E35; Secondary 54A25

1. Introduction

All the spaces are assumed to be Tychonoff. The Stone-Čech compactification $\beta\omega$ of the countable discrete space ω is identified with the set of all ultrafilters on ω and its remainder $\omega^* = \beta\omega \setminus \omega$ is identified with the set of all free ultrafilters on ω .

Let us define the basic common rules of our games:

Let X be a space and $x \in X$. We have two players, *I* and *II* who are going to play around the point x . Player *I* makes the first move by choosing an open neighborhood $U_0 \in \mathcal{N}(x)$. Then, player *II* responds by choosing $x_0 \in U_0$. Player *I* then chooses another open neighborhood $U_1 \in \mathcal{N}(x)$, and then player *II* responds by choosing $x_1 \in U_1$ and so on. Both players repeat this procedure infinitely many times. At the end of the game we have a sequence $(x_n)_{n < \omega}$ of points in X , and a sequence $(U_n)_{n < \omega}$ of neighborhoods of x such that $x_n \in U_n$, for every $n < \omega$. The games differ from each other in the winning condition. Following

Research supported by CONACYT grant no. 32171-E and DGAPA grant no. IN103399.

A. Bouziad [Bo], we say that player I wins in the $\mathcal{G}(x, X)$ -game if $\{x_n : n < \omega\}$ has an adherent point in X . Otherwise, player II is declared to be the winner in the $\mathcal{G}(x, X)$ -game. To define the main infinite games of this paper, we shall recall the definition of the p -limit point of a sequence of points of a space, for an ultrafilter $p \in \omega^*$.

Definition 1.1 (R.A. Bernstein [Be]). *Let $p \in \omega^*$. A point x of a space X is said to be the p -limit point of a sequence $(x_n)_{n < \omega}$ in X , in symbols $x = p\text{-}\lim_{n \rightarrow \omega} x_n$, if for every neighborhood U of x , $\{n < \omega : x_n \in U\} \in p$.*

Bernstein's notion characterizes the points lying in the closure of countable subsets of a space: A point $x \in X$ is an adherent point of a countable subset A of X iff there are a sequence $(x_n)_{n < \omega}$ in A and $p \in \omega^*$ such that $x = p\text{-}\lim_{n \rightarrow \omega} x_n$.

We are ready to state the winning condition of our games. Fix $p \in \omega^*$. As in the paper [GG], we say that player I wins in the $\mathcal{G}_p(x, X)$ -game if the sequence $(x_n)_{n < \omega}$ has a p -limit point in the space X . Otherwise, the second player wins the $\mathcal{G}_p(x, X)$ -game. All these games are natural generalizations of the $W(x, X)$ -game introduced by G. Gruenhage in [Gru].

Definition 1.2. *Let X be a space and $p \in \omega^*$. A strategy for player I is a sequence $\sigma = \{\sigma_n : n < \omega\}$ of functions, where $\sigma_n : X^{n+1} \rightarrow \mathcal{N}(x)$ for every $n < \omega$. Given a strategy σ we say that a sequence $(x_n)_{n < \omega}$ in X is a σ -sequence if $x_{n+1} \in \sigma_n((x_0, x_1, \dots, x_n))$, for each $n < \omega$. For $x \in X$, a strategy $\sigma = \{\sigma_n : n < \omega\}$ for player I in the $\mathcal{G}(x, X)$ -game (respectively, $\mathcal{G}_p(x, X)$ -game) is said to be a winning strategy, if each σ -sequence has an adherent point (respectively, a p -limit point) in X . A space X is called a \mathcal{G} -space (respectively, \mathcal{G}_p -space) if the first player I has a winning strategy in the $\mathcal{G}(x, X)$ -game (respectively, $\mathcal{G}_p(x, X)$ -game), for every $x \in X$.*

Every countably compact space is a \mathcal{G} -space, every \mathcal{G}_p -space is a \mathcal{G} -space and every p -compact space is a \mathcal{G}_p -space, for $p \in \omega^*$ (a space X is called p -compact provided that every sequence in X has a p -limit point in X).

In this paper, we mainly study the product of \mathcal{G} -spaces. In the second section, it is shown that if $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $\prod_{\mu < \omega_1} X_\mu$ is a \mathcal{G} -space, then the X_μ 's are countably compact except for countably many. It follows that if X^{ω_1} is a \mathcal{G} -space, then X^{ω_1} is countably compact. At the end of the second section, we shall prove that the product of a countable family of \mathcal{G}_p -spaces is a \mathcal{G}_p -space, for every $p \in \omega^*$. In the last section, we study the finite products of some \mathcal{G} -spaces.

2. Infinite products

In the first theorem of this section, we will give a necessary condition for a product of ω_1 -many \mathcal{G} -spaces to be a \mathcal{G} -space.

A subbasic open set of the product space $X = \prod_{i \in I} X_i$ is denoted by $[i, V] = \{x \in X : x(i) \in V\}$, where $i \in I$ and V is a nonempty open subset of X_i .

Theorem 2.1. *If $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu < \omega_1} X_\mu$ is a \mathcal{G} -space, then there is $A \in [\omega_1]^{<\omega}$ such that X_μ is countably compact, for every $\mu \in \omega_1 \setminus A$.*

PROOF: Suppose that there is a set $A \in [\omega_1]^{\omega_1}$ such that X_μ is not countably compact for any $\mu \in A$. Then, for every $\mu \in A$, there is a closed discrete countable subset $\{y_n^\mu : n < \omega\}$ of X_μ . Fix $x \in X$ and let us play the $\mathcal{G}(x, X)$ -game. We are going to define a winning strategy for player II. Indeed, player I starts the game by picking $V_0 = \bigcap_{\mu \in F_0} [\mu, W_\mu^0]$, where $F_0 \in [\omega_1]^{<\omega}$ and $W_\mu^0 \in \mathcal{N}(x(\mu))$, for every $\mu \in F_0$. Then, player II chooses $x_0 \in X$ which is defined by

$$x_0(\mu) = \begin{cases} x(\mu) & \text{if } \mu \in (\omega_1 \setminus A) \cup F_0, \\ y_0^\mu & \text{if } \mu \in A \setminus F_0. \end{cases}$$

Now, player I choose $V_1 = \bigcap_{\mu \in F_1} [\mu, W_\mu^1]$, where $F_1 \in [\omega_1]^{<\omega}$ and $W_\mu^0 \in \mathcal{N}(x(\mu))$ for every $\mu \in F_1$. Then, player II responds by picking the point $x_1 \in X$ defined by

$$x_1(\mu) = \begin{cases} x(\mu) & \text{if } \mu \in (\omega_1 \setminus A) \cup F_1, \\ y_1^\mu & \text{if } \mu \in A \setminus F_1, \end{cases}$$

and so on. Clearly, $x_n \in V_n = \bigcap_{\mu \in F_n} [\mu, W_\mu^n]$, for every $n < \omega$. Since the set $\bigcup_{n < \omega} F_n$ is countable there is $\nu \in A \setminus (\bigcup_{n < \omega} F_n)$. Then, the set $\{x_n(\nu) : n < \omega\} = \{y_n^\nu : n < \omega\}$ does not have an accumulation point in X_ν ; hence, the set $\{x_n : n < \omega\}$ cannot have an accumulation point in X , which contradicts the hypothesis. Therefore, A is countable. \square

Corollary 2.2. *Let X be a space. Then, X^{ω_1} is a \mathcal{G} -space iff X^{ω_1} is countably compact.*

A countably compact space X whose square is not countably compact (see for instance [Va, Example 4.8]) is an example of a \mathcal{G} -space such that X^{ω_1} cannot be a \mathcal{G} -space. The proof of the following theorem is analogous to the proof of Theorem 2.1.

Theorem 2.3. *Let $p \in \omega^*$. If $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu < \omega_1} X_\mu$ is a \mathcal{G}_p -space, then there is $A \in [\omega_1]^{<\omega}$ such that X_μ is p -compact for every $\mu \in \omega_1 \setminus A$.*

J. Ginsburg and V. Saks [GS] (see also [Va, Theorem 4.11]) proved that all powers of a space are countably compact the 2^c power of the space is countably compact iff it is p -compact, for some $p \in \omega^*$. This characterization and Theorem 2.3 imply the following two corollaries:

Corollary 2.4. *For $p \in \omega^*$, the following are equivalent.*

- (1) All powers of X are \mathcal{G}_p -spaces.
- (2) X^{ω_1} is a \mathcal{G}_p -space.
- (3) X is p -compact.

Corollary 2.5. *For a space X , the following are equivalent.*

- (1) All powers of X are \mathcal{G} -spaces.
- (2) X^{2^c} is a \mathcal{G} -space.
- (3) All powers of X are countably compact.
- (4) X is p -compact for some $p \in \omega^*$.

PROOF: The equivalence $3 \Leftrightarrow 4$ is taken from [GS]. It suffices to show the implication $2 \Rightarrow 3$. Consider the space $Y = (X^{2^c})^{\omega_1}$ which is homeomorphic to X^{2^c} . By assumption, Y is a \mathcal{G} -space. So, by Corollary 2.2, X^{2^c} is countably compact. By the characterization of J. Ginsburg and V. Saks [GS] quoted above, we conclude that all powers of X are countably compact. \square

Theorem 2.6. *For $p \in \omega^*$, the product of countably many \mathcal{G}_p -spaces is a \mathcal{G}_p -space.*

PROOF: Let $\{X_i : i < \omega\}$ be a family of \mathcal{G}_p -spaces. Put $X = \prod_{i < \omega} X_i$ and fix $x \in X$. For every $i < \omega$, let $\sigma^i = \{\sigma_n^i : X_i^{n+1} \rightarrow \mathcal{N}(x(i)) : n < \omega\}$ be a winning strategy for the $\mathcal{G}_p(x(i), X_i)$ -game. For every $n < \omega$, we define $\sigma_n : X^{n+1} \rightarrow \mathcal{N}(x)$ by $\sigma_n(y_0, \dots, y_n) = \bigcap_{i \leq n} [i, \sigma_n^i(y_0(i), \dots, y_n(i))]$, for every $(y_0, \dots, y_n) \in X^{n+1}$. We claim that $\sigma = \{\sigma_n : n < \omega\}$ is a winning strategy for the $\mathcal{G}_p(x, X)$ -game. In fact, let $(y_n)_{n < \omega}$ be a σ -sequence. By definition, we have that $y_{n+1} \in \sigma_n(y_0, \dots, y_n)$, for every $n < \omega$. Hence, $y_{n+1}(i) \in \sigma_n^i(y_0(i), \dots, y_n(i))$ for every $i, n < \omega$. That is, $(y_n(i))_{n < \omega}$ is a σ^i -sequence, for every $i < \omega$. By assumption, for every $i < \omega$, we have that $y(i) = p - \lim_{n \rightarrow \omega} y_n(i)$ exists. Thus, we obtain that $y = p - \lim_{n \rightarrow \omega} y_n$. This proves the claim and the theorem as well. \square

For $p \in \omega^*$, $X_p = \beta_p(\omega) \setminus \{p\}$ is a \mathcal{G}_p -space that is not p -compact, where $\beta_p(\omega)$ is the p -compactification of ω (see [G]). It follows from Corollary 2.6 that X_p^ω is a \mathcal{G}_p -space and $X_p^{\omega_1}$ is not a \mathcal{G}_p -space, for every $p \in \omega^*$.

3. Finite products

We have shown in Theorem 2.6 that the product of countably many \mathcal{G}_p -spaces is again a \mathcal{G}_p -space, for each $p \in \omega^*$. However, for \mathcal{G} -spaces this is not true as we will see in the first example of this section. For this task, we will slightly modify Frolík’s Example given in [Fro]. We shall present most of the details of Frolík’s construction since his notation, in the original paper [Fro], is not standard:

If $p, q \in \omega^*$, then $p \approx q$ means that there is a bijection $f : \omega \rightarrow \omega$ such that $\hat{f}(p) = q$, where $\hat{f} : \beta(\omega) \rightarrow \beta(\omega)$ denotes the Stone-Čech extension of f . It is

clear that \approx is an equivalence relation on ω^* and the equivalence class of a point $p \in \omega^*$ is called the *type* of p and it is denoted by $T(p) = \{q \in \omega^* : q \approx p\}$. We know that $p \approx q$ iff there is a function $f : \omega \rightarrow \omega$ and $A \in p$ such that $\hat{f}(p) = q$ and $f|_A$ is one-to-one (for a proof see [CN, Theorem 9.2(b)]). The *RK-ordering* and the *RF-ordering* on ω^* are defined as follows:

For $p, q \in \omega^*$, we say that $p \leq_{RK} q$ if there is a function $f : \omega \rightarrow \omega$ such that $\hat{f}(q) = p$, and we say that $p \leq_{RF} q$ if there is an embedding $f : \omega \rightarrow \beta(\omega)$ such that $\hat{f}(p) = q$.

It is known that $\leq_{RF} \subset \leq_{RK}$, and $p \approx q$ iff $p \leq_{RK} q$ and $q \leq_{RK} p$, for $p, q \in \omega^*$. For $p, q \in \omega^*$, $p <_{RK} q$ will mean that $p \leq_{RK} q$ and $p \not\approx q$. For $p \in \omega^*$, we let $P_{RK}(p) = \{q \in \omega^* : q \leq_{RK} p\}$ and $S_{RF}(p) = \{q \in \omega^* : p <_{RF} q\}$. Z. Frolík [Fro] proved that $|S_{RF}(p)| = 2^c$, for all $p \in \omega^*$.

Lemma 3.1. *There is a family $\{X_\mu : \mu < \omega_1\}$ of subsets of ω^* and a set $\{p_\mu : \mu < \omega_1\}$ of points in ω^* such that:*

- i.* p_μ and p_ν are *RK-incomparable ultrafilters* for distinct $\mu, \nu < \omega_1$;
- ii.* $X_\mu = \{\hat{f}(p_\mu) : f : \omega \rightarrow \bigcup_{\nu < \mu} X_\nu \text{ is an embedding}\} \subseteq S_{RF}(p_\mu)$, for every $0 < \mu < \omega_1$;
- iii.* $|X_\mu| \leq c$, for every $\mu < \omega_1$;
- iv.* $X_\mu \cap X_\nu = \emptyset$, whenever $\mu < \nu < \omega_1$.

PROOF: We know that there is a set W of size 2^c consisting of pairwise *RK-incomparable weak P -points* in ω^* (see [Ku] and [Si]). In virtue of Theorem 16.16 of [CN], we have that $S_{RF}(s) \cap S_{RF}(t) = \emptyset$ for distinct $s, r \in W$. Take $p_0 \in W$ and let $X_0 = T(p_0)$. Now, assume that p_μ and X_μ have been defined satisfying conditions *i-iv*, for each $\mu < \theta < \omega_1$. Put $X = \bigcup_{\mu < \theta} X_\mu$. It follows from *iii* that $|X| \leq c$. Then, choose $p_\theta \in W \setminus \{p_\mu : \mu < \theta\}$. Then, we have that $S_{RF}(p_\theta) \cap X = \emptyset$. Thus, we define $X_\theta = \{\hat{f}(p_\theta) : f : \omega \rightarrow \bigcup_{\mu < \theta} X_\mu \text{ is an embedding}\} \subseteq S_{RF}(p_\theta)$. □

For $A \subseteq \omega$, let $A \nearrow^\omega = \{f \in A^\omega : f \text{ is strictly increasing}\}$.

Example 3.2. *For every $1 \leq n < \omega$, there exists a space X such that X^n is countably compact and X^{n+1} is not a \mathcal{G} -space.*

PROOF: Let $\{X_\mu : \mu < \omega_1\}$ and $\{p_\mu : \mu < \omega_1\}$ be subsets of ω^* satisfying all the properties given in Lemma 3.1. We remark, by Theorem 16.16 of [CN], that $S_{RF}(p_\mu) \cap S_{RF}(p_\nu) = \emptyset$ whenever $\mu < \nu < c$. For $\emptyset \neq I \subseteq \omega_1$, we define $X_I = \omega \cup (\bigcup_{\mu \in I} X_\mu)$. We need the following fact which is Theorem *D* of [Fro]:

(*) Let $\{I_n : n < \omega\} \subseteq \mathcal{P}(\omega_1)$ be nonempty sets. If $\bigcap_{n < \omega} I_n$ is unbounded in ω_1 , then $\prod_{n < \omega} X_{I_n}$ is countably compact. If $\bigcap_{n < \omega} I_n = \emptyset$, then $\prod_{n < \omega} X_{I_n}$ is not countably compact.

Fix $1 \leq n < \omega$. For each $k \leq n$, let $I_k = \{\mu < \omega_1 : \mu \not\equiv k \pmod{n+1}\}$. Let us consider the topological sum $X = \bigoplus_{k \leq n} X_{I_k}$. Since $\bigcap_{1 \leq i \leq n} I_{k_i}$ is unbounded in

ω_1 , by $(*)$, $X_{I_{k_1}} \times \cdots \times X_{I_{k_n}}$ is countably compact, for every $k_1, \dots, k_n \in n + 1$. It follows that $X^n = \bigcup_{k_1, \dots, k_n \in n+1} X_{I_{k_1}} \times \cdots \times X_{I_{k_n}}$ is countably compact. To prove that X^{n+1} cannot be a \mathcal{G} -space it suffices to show that $X_{I_0} \times X_{I_1} \times \cdots \times X_{I_n}$ is not a \mathcal{G} -space. For every $k \leq n$, fix a non-isolated point $x_k \in X_{I_k}$. Now, we will prove that player II has a winning strategy in the $\mathcal{G}((x_0, \dots, x_n), X_{I_0} \times \cdots \times X_{I_n})$ -game. Indeed, let $\sigma = \{\sigma_m : (X_{I_0} \times \cdots \times X_{I_n})^{m+1} \rightarrow \mathcal{N}((x_0, \dots, x_n)) : m < \omega\}$ be a strategy for player I . It is not hard to prove that, for every $k \leq n$, player II may choose $f_k \in \omega^{\omega}$ so that

$$(f_1(m+1), \dots, f_n(m+1)) \in \sigma_m((f_1(0), \dots, f_n(0)), \dots, (f_1(m), \dots, f_n(m))),$$

for every $m < \omega$ and for every $k \leq n$. Suppose that there is $r \in \omega^*$ such that $\hat{f}_k(r) \in X_{I_k}$, for all $k \leq n$. Then, for each $k \leq n$, we have that $\hat{f}_k(r) \in X_{\mu_k}$, for some $\mu_k \in I_k$. By definition, we know that $p_{\mu_k} \leq_{RF} \hat{f}_k(r) \approx r$, for every $k \leq n$. Theorem 16.16 of [CN] implies that $p_{\mu_0} = p_{\mu_1} = \cdots = p_{\mu_n}$ and so $\mu_0 = \mu_1 = \cdots = \mu_n \in \bigcap_{k \leq n} I_k = \emptyset$, which is a contradiction. Thus, the set $\{(f_1(m), \dots, f_n(m)) : m < \omega\}$ does not have any accumulation point in $X_{I_0} \times \cdots \times X_{I_n}$. Therefore, $X_{I_0} \times \cdots \times X_{I_n}$ is not a \mathcal{G} -space. \square

Our next example shows that the product of a \mathcal{G}_p -space and a \mathcal{G} -space is not in general a \mathcal{G} -space. First, we prove some preliminary results.

Lemma 3.3. *Let $p, q \in \omega^*$ and let $f, g : \omega \rightarrow \omega$ be two one-to-one functions. If $p \not\approx q$, then (p, q) is not an accumulation point of $\{(f(n), g(n)) : n < \omega\}$ in $\beta(\omega) \times \beta(\omega)$.*

PROOF: Suppose the contrary. Then, there is $r \in \omega^*$ such that $p = \hat{f}(r)$ and $q = \hat{g}(r)$. By a fact quoted above [CN, Theorem 9.2(b)], we must have that $p \approx r$ and $q \approx r$. So $p \approx q$, but this is a contradiction. \square

Lemma 3.4. *Let $\omega \subseteq X, Y \subseteq \beta(\omega)$ be two non-discrete spaces. If $X \cap (\bigcup\{T(p) : p \in Y \cap \omega^*\}) = \emptyset = Y \cap (\bigcup\{T(q) : q \in X \cap \omega^*\})$, then $X \times Y$ cannot be a \mathcal{G} -space.*

PROOF: Let $x \in X$ and $y \in Y$ be accumulation points of X and Y , respectively. We will prove that player II has a winning strategy in the $\mathcal{G}((x, y), X \times Y)$ -game. Indeed, suppose that $(V_n \times W_n)_{n < \omega}$ is a sequence of basic neighborhoods of (x, y) in $X \times Y$. Then, we may find two strictly increasing functions $f, g : \omega \rightarrow \omega$ such that $f(n) \in V_n$ and $g(n) \in W_n$, for every $n < \omega$, and $f[\omega] \cap g[\omega] = \emptyset$. By Lemma 3.3, the countable set $\{(f(n), g(n)) : n < \omega\}$ does not have an accumulation point in $X \times Y$. This shows that player II has a winning strategy. Therefore, $X \times Y$ is not a \mathcal{G} -space. \square

Theorem 3.5. *Let $M \in [\omega^*]^{\leq c}$. If $\omega \subseteq X \subseteq \beta(\omega)$ satisfies $|X| \leq c$, then there are $p \in \omega^*$ and a countably compact \mathcal{G}_p -space Y such that $X \times Y$ is not a \mathcal{G} -space and $r <_{RK} p$, for every $r \in M$.*

PROOF: By Theorem 10.9 of [CN], we can find $q \in \omega^*$ so that $r <_{RK} q$, for every $r \in M \cup (X \cap \omega^*)$. Choose $p \in \omega^*$ so that $q <_{RK} p$ and let $\Gamma_q = \beta(\omega) \setminus P_{RK}(q)$. We know that Γ_q is countably compact. Theorem 2.1 from [GG] assures that Γ_q is a \mathcal{G}_p -space. Suppose that $X \times \Gamma_q$ is a \mathcal{G} -space. So, by Lemma 3.4, either $X \cap \{T(s) : s \in \Gamma_q \cap \omega^*\} \neq \emptyset$ or $\Gamma_q \cap \{T(t) : t \in X \cap \omega^*\} \neq \emptyset$, but this is impossible. Therefore, $X \times \Gamma_q$ cannot be a \mathcal{G} -space and $r <_{RK} p$, for all $r \in M$. \square

Corollary 3.6. *For every $p \in \omega^*$, there are $q \in \omega^*$, a \mathcal{G}_p -space X and a countably compact \mathcal{G}_q -space Y such that $X \times Y$ is not a \mathcal{G} -space and $p <_{RK} q$.*

PROOF: Let $p \in \omega^*$. We apply Theorem 3.5 to the p -compactification $\beta_p(\omega)$ of ω and $M = \{p\}$. \square

Theorem 3.7. *Let $p, q \in \omega^*$. If $q \in R(p) = \{\hat{f}(p) : f \in \omega^\omega \text{ and } \exists A \in p(f|_A \text{ is strictly increasing})\}$, then the product of a \mathcal{G}_p -space and a \mathcal{G}_q -space is a \mathcal{G}_q -space.*

PROOF: This theorem is a direct consequence of Theorem 2.6 and Theorem 2.4 of [GG]. \square

Example 3.8. *Let $p, q \in \omega^*$. If $q \in T(p) \setminus R(p)$, then there are a \mathcal{G}_p -space X and a \mathcal{G}_q -space Y such that $X \times Y$ is not a \mathcal{G}_q -space.*

PROOF: Let $\Omega(p)$ the space defined in Theorem 2.3 from [GG]. We know that $\Omega(p)$ is a \mathcal{G}_p -space that is not a \mathcal{G}_q -space. Thus, $\Omega(p) \times \beta_q(\omega)$ is a \mathcal{G}_p -space that is not a \mathcal{G}_q -space and $\beta_q(\omega)$ is a \mathcal{G}_q -space. \square

It was proved in [HST] that $p \in \omega^*$ is a Q -point iff $T(p) = R(p)$. Example 3.8 shows that the condition “ $q \in R(p)$ ” given in Theorem 3.7 is essential. Next, we will give some relationships between the game and some of the orderings on ω^* .

Theorem 3.9. *Let $p, q \in \omega^*$ be RK -incomparable. Then there are a \mathcal{G}_p -space X and a \mathcal{G}_q -space Y such that $X \times Y$ is not a \mathcal{G} -space.*

PROOF: Our spaces are $X = S_{RF}(p) \cup T(p) \cup \omega$ and $Y = S_{RF}(q) \cup T(q) \cup \omega$. It is not hard to see that X is a \mathcal{G}_p -space and Y is a \mathcal{G}_q -space. Fix $(x, y) \in X \times Y \setminus (\omega \times \omega)$. Let us see that player II has a winning strategy in the $\mathcal{G}((x, y), X \times Y)$ -game. Indeed, suppose that $\sigma = \{\sigma_n : (X \times Y)^{n+1} \rightarrow \mathcal{N}((x, y)) : n < \omega\}$ is a strategy for player I . Then, player II can always choose two functions $f, g \in \omega^{<\omega}$ so that $((f(n), g(n)))_{n < \omega}$ is a σ -sequence with $f[\omega] \cap g[\omega] = \emptyset$. Suppose that $(s, t) \in X \times Y$ is an accumulation point for $\{(f(n), g(n)) : n < \omega\}$. By Lemma 3.3, we must have that $s \approx t$. On the other hand, by definition, there are two embeddings $e : \omega \rightarrow X$ and $h : \omega \rightarrow Y$ for which $\hat{e}(p) = s$ and $\hat{h}(q) = t$. Since $p \leq_{RF} s, q \leq_{RF} t$ and $s \approx t$, by Theorem 16.16 of [CN], either $p \leq_{RK} q$ or $q \leq_{RK} p$, but this is a contradiction. Therefore, $X \times Y$ is not a \mathcal{G} -space. \square

W.W. Comfort introduced in [G] the following order on ω^* :

We say that $p \leq_C q$ if every q -compact space is p -compact.

It is known that $\leq_{RK} \subset \leq_C$ and they are different from each other. For $p, q \in \omega^*$, $p \approx_C q$ will mean that $q \leq_C p$ and $p \leq_C q$. For $p \in \omega^*$, $T_C(p) = \{q \in \omega^* : p \approx_C q\}$ is the *Comfort type* of p . It is not hard to see that $\Delta_p = \omega \cup T_C(p)$ is a \mathcal{G}_p -space, for each $p \in \omega^*$. The next theorem follows directly from Theorem 2.6 and Lemma 3.4.

Theorem 3.10. *Let $p, q \in \omega^*$. Then, $\Delta_p \times \Delta_q$ is a \mathcal{G} -space iff $p \approx_C q$.*

The following is a direct consequence of Theorems 3.9 and 3.10.

Corollary 3.11. *Let $p, q \in \omega^*$. If the product of a \mathcal{G}_p -space and a \mathcal{G}_q -space is a \mathcal{G} -space, then p and q are *RK-comparable* and $p \approx_C q$.*

We know that $S_{RF}(p)$ is a \mathcal{G}_p -space, for every $p \in \omega^*$.

Theorem 3.12. *Let $p, q \in \omega^*$. If $S_{RF}(p) \times S_{RF}(q)$ is a \mathcal{G} -space, then either*

- i. p and q are *RF-comparable*; or*
- ii. there is $r \in \omega^*$ such that $r <_{RF} p$ and $r <_{RF} q$.*

PROOF: Fix $(x, y) \in X = S_{RF}(p) \times S_{RF}(q)$. Let $\sigma = \{\sigma_n : n < \omega\}$ be a winning strategy for player I in the $\mathcal{G}((x, y), X)$ -game. Then, we may choose a σ -sequence $((f(n), g(n)))_{n < \omega}$ so that $f : \omega \rightarrow S_{RF}(p)$ and $g : \omega \rightarrow S_{RF}(q)$ are embeddings. By assumption, there is $r \in \omega^*$ such that $\hat{f}(r) \in S_{RF}(p)$ and $\hat{g}(r) \in S_{RF}(q)$. It then follows that $r <_{RF} \hat{f}(r)$, $r <_{RF} \hat{g}(r)$, $p <_{RF} \hat{f}(r)$ and $q <_{RF} \hat{g}(r)$. According to Theorem 16.16 from [CN], r and p are *RF-comparable* and also r and q are *RF-comparable*. The conclusion then follows from these relations. \square

Next, we shall prove that the first clause *i* of the conclusion of Theorem 3.14 suffices to get the converse of the same theorem.

Theorem 3.13. *Let $p, q \in \omega^*$. If $p \leq_{RF} q$, then $S_{RF}(p) \times S_{RF}(q)$ is a \mathcal{G}_q -space.*

PROOF: Let $f : \omega \rightarrow S_{RF}(q)$ and $g : \omega \rightarrow S_{RF}(p)$ be two embeddings. By the transitivity of the *RF-order*, we get that $\hat{f}(q) \in S_{RF}(p)$ and $\hat{g}(q) \in S_{RF}(q)$. So, $(\hat{f}(q), \hat{g}(q))$ is an accumulation point of $\{(f(n), g(n)) : n < \omega\}$. This shows that $S_{RF}(p) \times S_{RF}(q)$ is a \mathcal{G}_q -space. \square

To give more examples we need the following notion:

The *tensor product* of two ultrafilters $p, q \in \omega^*$ is the ultrafilter

$$p \otimes q = \{A \subseteq \omega \times \omega : \{n < \omega : \{m < \omega : (n, m) \in A\} \in q\} \in p\}$$

on $\omega \times \omega$. For $p, q \in \omega^*$, $p \otimes q$ can be viewed as an ultrafilter on ω via a fixed bijection between $\omega \times \omega$ and ω . It is know that $p <_{RF} p \otimes q$ and $q <_{RK} p \otimes q$, for every $p, q \in \omega^*$. We list some relevant properties of the tensor product:

1. It was proved in [G] that if r and s are *RK-incomparable free ultrafilters* on ω and $r <_{RK} p$ and $s <_{RK} p$, then $p \otimes r \approx_C p \approx_C p \otimes s$ and $p \otimes r$ and $p \otimes s$ are *RK-incomparable* too.

2. Let s and t be two RF -minimal and RK -incomparable free ultrafilters on ω (see [Ku]), and let $r \in \omega^*$ be such that $s <_{RK} r$ and $t <_{RK} r$. Put $p = s \otimes r$ and $q = t \otimes (s \otimes r)$. Then, we have that $p <_{RK} q$, $p \approx_C q$ (for this fact see [G]), p and q are RF -incomparable and do not have a common RF -predecessor. From Theorem 3.12 we get that $S_{RF}(p) \times S_{RF}(q)$ is not a \mathcal{G} -space. This shows that the converse of Corollary 3.11 fails.

We will see that the second condition *ii* of Theorem 3.14 implies its converse under some additional conditions.

Theorem 3.14. *Let $p, q \in \omega^*$. Suppose that there are $r \in \omega^*$ and two embeddings $f, g : \omega \rightarrow \omega^*$ such that $\hat{f}(r) = p$, $\hat{g}(r) = q$ and $f(n) \leq_{RF} p$ and $g(n) \leq_{RF} q$, for every $n < \omega$. Then, $S_{RF}(p) \times S_{RF}(q)$ is a \mathcal{G}_r -space.*

PROOF: Let $e : \omega \rightarrow S_{RF}(p)$ and $h : \omega \rightarrow S_{RF}(q)$ be two embeddings. Since $f(n) <_{RF} e(n)$ and $g(n) <_{RF} h(n)$, for every $n < \omega$, by Lemma 2.20 from [Boo], $p = \hat{f}(r) <_{RF} \hat{e}(r)$ and $q = \hat{g}(r) <_{RF} \hat{h}(r)$. Then, $(\hat{e}(r), \hat{h}(r)) \in S_{RF}(p) \times S_{RF}(q)$ is an accumulation point of the set $\{(e(n), h(n)) : n < \omega\}$. This shows that $S_{RF}(p) \times S_{RF}(q)$ is a \mathcal{G}_r -space. \square

As a consequence of Theorem 3.14, $S_{RF}(r \otimes p) \times S_{RF}(r \otimes q)$ is a \mathcal{G} -space, for every $p, q, r \in \omega^*$. But, the next example shows that the second clause *ii* of the conclusion of Theorem 3.14 does not imply its converse.

Example 3.15. *Let $r \in \omega^*$ and let $f, g : \omega \rightarrow \omega^*$ be two embeddings such that $r, f(n)$ and $g(n)$, for every $n < \omega$, are all pairwise RK -incomparable and RF -minimal. If $\hat{f}(r) = p$ and $\hat{g}(r) = q$, then $S_{RF}(p) \times S_{RF}(q)$ is not a \mathcal{G} -space.*

PROOF: It is not hard to see, by Lemma 2.20 of [Boo], that p and q are RF -incomparable, and notice that $r <_{RF} p$ and $r <_{RF} q$. Let $e : \omega \rightarrow S_{RF}(p)$ and $h : \omega \rightarrow S_{RF}(q)$ be two embeddings. First, suppose that there is $s \in T(r)$ such that $\hat{e}(s) \in S_{RF}(p)$ and $\hat{h}(s) \in S_{RF}(q)$. Fix a bijection $\sigma : \omega \rightarrow \omega$ such that $\hat{\sigma}(r) = s$. Then, we have that $p = \hat{f}(r) <_{RF} \hat{e}(s) = \hat{e}(\hat{\sigma}(r))$ and $q = \hat{g}(r) <_{RF} \hat{h}(s) = \hat{h}(\hat{\sigma}(r))$. According to Lemma 2.20 from [Boo], $A = \{n < \omega : f(n) <_{RF} \hat{e}(\sigma(n))\} \cap \{n < \omega : g(n) <_{RF} \hat{h}(\sigma(n))\} \in r$. Take two distinct points $m, n \in A$. So, we have that $f(n) <_{RF} \hat{e}(\sigma(n))$, $f(m) <_{RF} \hat{e}(\sigma(m))$, $p <_{RF} \hat{e}(\sigma(n))$ and $p <_{RF} \hat{e}(\sigma(m))$. Then, by Theorem 16.16 from [CN] and our hypothesis, $f(n) <_{RF} p$ and $f(m) <_{RF} p$. Hence, by Theorem 16.16 of [CN], we conclude that $f(n)$ and $f(m)$ are RF -comparable, but this is a contradiction. A similar contradiction is obtained if we replace f by g . This proves that for every $s \in T(r)$ we have that $\hat{e}(s) \notin S_{RF}(p)$ and $\hat{h}(s) \notin S_{RF}(q)$. Now, let us assume that there is $t \in \omega^*$ such that $r <_{RF} t <_{RF} p$ and $r <_{RF} t <_{RF} q$. Choose an embedding $j : \omega \rightarrow \omega^*$ such that $\hat{j}(r) = t$. By Lemma 2.20 of [Boo], we have that $B = \{n < \omega : j(n) <_{RF} f(n)\} \cap \{n < \omega : j(n) <_{RF} g(n)\} \in r$, which is impossible since $f(n)$ and $g(n)$ are RF -minimal, for every $n < \omega$. Therefore,

there is no $t \in \omega^*$ with $r <_{RF} t <_{RF} p$ and $r <_{RF} t <_{RF} q$. By Theorem 3.14, we obtain that $S_{RF}(p) \times S_{RF}(q)$ is not a \mathcal{G} -space. \square

Now, we give a necessary condition for the product of two subspaces of ω^* to fail be a \mathcal{G} -space. For our purposes we need a lemma:

Lemma 3.16. *Let $p, q \in \omega^*$ and let $f, g : \omega \rightarrow \omega^*$ be two embeddings. If p and q do not have a common RF -predecessor, then (p, q) is not an accumulation point of $\{(f(n), g(n)) : n < \omega\}$.*

PROOF: Suppose the contrary. Then, there is $r \in \omega^*$ such that $(p, q) = r - \lim_{n \rightarrow \omega} (f(n), g(n))$. Hence, $p = r - \lim_{n \rightarrow \omega} f(n)$ and $q = r - \lim_{n \rightarrow \omega} g(n)$. Since f and g are embeddings, $r <_{RF} p$ and $r <_{RF} q$, which is a contradiction. \square

Theorem 3.17. *If $X, Y \subseteq \omega^*$ satisfy that $P_{RK}(p) \cap P_{RK}(q) = \emptyset$ for every $p \in X$ and for every $q \in Y$, then $X \times Y$ is not a \mathcal{G} -space.*

PROOF: We apply an argument similar to the one used in the proof of Lemma 3.4 by using Lemma 3.16. \square

We end by listing some open questions that the authors were unable to respond.

Question 3.18. *Are there spaces X and Y such that X is a \mathcal{G}_p -space, for all $p \in \omega^*$, and Y is a \mathcal{G} -space, but $X \times Y$ is not a \mathcal{G} -space?*

Question 3.19. *If $p, q \in \omega^*$ and $q \approx_C p <_{RF} q$, are there a \mathcal{G}_p -space and a \mathcal{G}_q -space whose product is not a \mathcal{G} -space?*

We point out that Lemma 3.12.10 of [En] implies that if X is a \mathcal{G} - k -space and Y is a \mathcal{G} -space, then $X \times Y$ is a \mathcal{G} -space.

Question 3.20. *For $n < \omega$, is there a topological group G such that G^n is a \mathcal{G} -space but G^{n+1} is not a \mathcal{G} -space?*

Question 3.21. *Is there a topological group G that is a \mathcal{G} -space and it is not a \mathcal{G}_p -space for any $p \in \omega^*$?*

Question 3.22. *Let $p, q \in \omega^*$ be RK -incomparable. Are there a \mathcal{G}_p -topological group G and a \mathcal{G}_q -topological group H such that $G \times H$ is not a \mathcal{G} -space?*

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(Received January 22, 2002)