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## Totally non-remote points in $\beta\mathbb{Q}$

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*Abstract.* Totally nonremote points in  $\beta\mathbb{Q}$  are constructed. The number of these points is  $2^{\mathfrak{c}}$ .

*Keywords:* totally nonremote point, far point, crowded point

*Classification:* 54D35

### 1. Introduction

All spaces considered are normal. We follow [2] in our terminology and notations. For a space  $X$  we identify a point  $a$  of the Čech-Stone remainder  $X^* = \beta X - X$  with

$$\{A \subseteq X : A \text{ closed in } X \text{ and } a \in \text{Cl}_{\beta X} A\}.$$

We call  $a \in X^*$

**far:** if no element of  $a$  is discrete;

**crowded:** if every element of  $a$  is dense-in-itself;

**remote:** if no element of  $a$  is nowhere dense; and

**totally nonremote:** if for every  $A \in a$  there is  $B \in a$  that is nowhere dense in  $A$ .

After the introduction of remote points in [1] and [3], they became one of the most intriguing in the theory of Čech-Stone compactifications. What about the existence of points with antipodal properties? E. van Douwen set the following question:

Does  $\mathbb{Q}$  have a crowded totally nonremote point ?

and showed, in particular, that CMA (Martin's Axiom for countable posets) implies yes [2]. Every totally nonremote point is, obviously, far.

We prove naively the following

**Theorem 1.1.** *There are totally nonremote points in  $\beta\mathbb{Q}$ . The number of these points is  $2^{\mathfrak{c}}$ .*

The question above remains open.

**2. Proofs**

In this paper  $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$  is the set of rational numbers. We will use  $S = \bigcup_{n=0}^{\infty} \mathbb{N}^{2n}$  as a set of indexes.

By recursion on  $s \in S$  we will choose  $p_s$  and  $O_s$  as follows:

Let  $O_{\emptyset} = \mathbb{Q}$  and  $p_{\emptyset} = q_1$ .

If  $O_s$  and  $p_s$  are found we let  $\{O_{s,i} : i \in \mathbb{N}\}$  be a strictly decreasing local base at  $p_s$ , consisting of clopen sets and with  $O_{s,1} = O_s$ . For every  $U_{s,i} = O_{s,i} - O_{s,i+1}$  we choose an infinite pairwise disjoint clopen cover  $\mathcal{U}_{s,i} = \{O_{s,i,j} : j \in \mathbb{N}\}$  of  $U_{s,i}$ . We let  $p_{s,i,j} \in O_{s,i,j}$  be the  $q_n$  with minimal index (in this way we get  $\{p_s : s \in S\} = \mathbb{Q}$ ). Finally,  $\mathcal{O} = \{O_s : s \in S\}$ .

We use  $\mathcal{F}$  to denote the set of maps  $f$  from  $S$  to  $\text{Exp } \mathcal{O}$  with the properties that  $f(s) \subset \bigcup_i \mathcal{U}_{s,i}$  and  $|f(s) \cap \mathcal{U}_{s,i}| \leq 1$  for all  $s$  and  $i$ . For every  $f \in \mathcal{F}$  we put  $\mathcal{O}(f) = \bigcup \{f(s) : s \in S\}$ .

We denote  $\mathcal{D}$  all closed and discrete subsets of  $\mathbb{Q}$ . For every  $s \in \mathbb{N}^{2n}$  and  $D \in \mathcal{D}$  we set

$$i(s, D) = \min\{i : O_{s,i} \cap D \subseteq \{p_s\}\} = \min\{i : (\forall j \geq i)(U_{s,j} \cap D = \emptyset)\}.$$

We say that  $s$  is  $D$ -good if  $s_{2k+1} < i(s \upharpoonright 2k, D)$  for all  $k < n$ , and we put

$$\mathcal{O}(D) = \{U_{s,i(s,D)} : s \in S \text{ is } D\text{-good}\}.$$

**Claim 1.** For any  $D \in \mathcal{D}$ ,  $\mathcal{O}(D)$  is locally finite in  $\mathbb{Q}$ .

PROOF: Let  $p_s \in \mathbb{Q}$ . If the index  $s$  is  $D$ -good, then the neighborhood  $O_{p_s} = O_{s,i(s,D)+1}$  does not intersect any member of  $\mathcal{O}(D)$ . Otherwise, we choose the maximal  $k$  such that  $t = s \upharpoonright 2k$  is  $D$ -good. If  $s_{2k+1} > i(t, D)$  then  $O_{p_s} = O_s$  meets no member of  $\mathcal{O}(D)$  and if  $s_{2k+1} = i(t, D)$  then  $U_{t,i(t,D)}$  is the unique member of  $\mathcal{O}(D)$  that  $O_s$  intersects.  $\square$

**Claim 2.**

$$\bigcap_{k=1}^n (\bigcup \mathcal{O}(D_k)) - \bigcup_{j=1}^n \bigcup \mathcal{O}(f_j) \neq \emptyset$$

for any  $n \in \mathbb{N}$ ,  $D_k \in \mathcal{D}$  and  $f_j \in \mathcal{F}$ .

PROOF: We shall construct an  $s \in S$  such that  $p_s$  belongs to the set in question. To begin, let  $s \upharpoonright 0 = \emptyset$  and  $F_0 = \{1, 2, \dots, n\}$ .

Assume  $a \upharpoonright 2m$  and  $F_m$  have been found with  $F_m \neq \emptyset$  and  $s \upharpoonright 2m$  a  $D_k$ -good sequence for  $k \in F_m$ . Let  $s_{2m+1} = \min\{(s \upharpoonright 2m, D_k) : k \in F_m\}$  and choose  $s_{2m+2}$  so large that  $O_{s,s_{2m+1},s_{2m+2}} \in \mathcal{U}_{s,s_{2m+1}} - \bigcup_{k \leq n} f_k(s)$ . Let  $F_{m+1} = \{k \in F_m : i(s \upharpoonright 2m, D_k) > s_{2m+1}\}$ ; observe that  $F_{m+1}$  is a proper subset of  $F_m$  and that  $s \upharpoonright (2m + 2)$  is  $D_k$ -good for  $k \in F_{m+1}$ .

There will be an  $m$  with  $F_m = \emptyset$ ; then  $s = s \upharpoonright 2m$  is as required.  $\square$

It follows that the point  $a$  in Claim 3 does really exist.

**Claim 3.** Every point  $a \in \mathbb{Q}^*$  such that

$$a \in \bigcap \{ \text{Cl}_{\beta\mathbb{Q}} \bigcup \mathcal{O}(D) : D \in \mathcal{D} \} - \bigcup \{ \text{Cl}_{\beta\mathbb{Q}} \bigcup \mathcal{O}(f) : f \in \mathcal{F} \}$$

is totally nonremote.

PROOF: Let  $A \in a$ . If  $\text{Cl}_X D \in a$  for  $D = \{q \in A : q \text{ is isolated in } A\}$ , then  $\text{Cl}_X D - D \in a$ , because  $a$  is a far point. Otherwise, if  $Oa \cap D = \emptyset$  for a clopen neighborhood of  $a$ , then  $G = Oa \cap A$  has no isolated points. Define  $f_G \in \mathcal{F}$  so that for any  $s \in S$ ,  $\bigcup f_G(s)$  meets every nonempty intersection  $U_{s,i} \cap G$ . Then  $G - \bigcup \mathcal{O}(f_G) \in a$  is nowhere dense in  $A$ .  $\square$

**Claim 4.** The number of totally nonremote points in  $\beta\mathbb{Q}$  is  $2^c$ .

PROOF: For every  $Q_j = (\sqrt{2}j, \sqrt{2}j + 1) \cap \mathbb{Q}$  we fix a totally nonremote point  $a_j \in Q_j^*$  and put  $A = \{a_j : j \in \mathbb{N}\}$ . Then  $Y = \mathbb{Q} \cup A$  is normal and  $\text{Cl}_{\beta\mathbb{Q}} Y$  is equivalent to  $\beta Y$ , because  $\mathbb{Q} \subset Y \subset \beta\mathbb{Q}$ . Hence  $\text{Cl}_{\beta\mathbb{Q}} A \subset \mathbb{Q}^*$  has cardinality  $2^c$ . Let  $a \in \text{Cl}_{\beta\mathbb{Q}} A$  and  $B \in a$ . For each  $a_j \in A$  there is  $G_j \subset Q_j$ , which belongs to  $a_j$  and has nowhere dense intersection (possibly empty) with  $B$ . Then  $G = \bigcup_{j \in \mathbb{N}} G_j$  belongs to  $a$  and has nowhere dense intersection with  $B$ . Our proof is complete.  $\square$

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