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Perfect sets and collapsing continuum

MIROSLAV REPICKÝ

Abstract. Under Martin’s axiom, collapsing of the continuum by Sacks forcing \( S \) is characterized by the additivity of Marczewski’s ideal (see [4]). We show that the same characterization holds true if \( d = \aleph_1 \) proving that under this hypothesis there are no small uncountable maximal antichains in \( S \). We also construct a partition of \( \omega^2 \) into \( \aleph_1 \) perfect sets which is a maximal antichain in \( S \) and show that \( s^0 \)-sets are exactly (subsets of) selectors of maximal antichains of perfect sets.

Keywords: Sacks forcing, Marczewski’s ideal, cardinal invariants

Classification: Primary 03E40; Secondary 03E17

1. General remarks

Let \( (P, \leq) \) be a partial order. We say that elements (conditions) \( p, q \in P \) are compatible and write \( p \land q \neq 0 \) if there is \( r \in P \) such that \( r \leq p \) and \( r \leq q \). Otherwise \( p \) and \( q \) are incompatible and we write \( p \land q = 0 \). A family of pairwise incompatible elements is called an antichain. For \( p \in P \), \( P \upharpoonright p = \{ q \in P : p \leq q \} \).

Let us recall some cardinal invariants for \( P \):

\[
\begin{align*}
\pi(P) &= \min\{|X| : X \text{ is a dense subset of } P\}, \\
sat(P) &= \min\{\kappa : \text{every antichain has size } < \kappa\}, \\
a(\kappa, P) &= \min\{\pi(P) \cup \{|A| : A \subseteq P \text{ is a maximal antichain with } |A| \geq \kappa\}\}, \\
\cf_\pi(P) &= \min\{\kappa : \Vdash_{P} \cf(\pi^V(P)) \leq \kappa\}.
\end{align*}
\]

The hereditary version of a cardinal invariant \( \kappa(\cdot) \) for partial orders is defined by

\[
\h\kappa(P) = \min\{\kappa(P \upharpoonright p) : p \in P\}.
\]

The symbols \( \h\pi(P) \), \( \hsat(P) \), \( \ha(\kappa, P) \) denote the hereditary versions of the cardinals \( \pi(P) \), \( \sat(P) \), \( a(\kappa, P) \), respectively.

A matrix on \( P \) is a sequence of antichains in \( P \) (the antichains may be maximal). Let \( \mathcal{A} \) be a matrix on \( P \). A matrix \( \mathcal{A} \) is shattering if for every \( p \in P \) there exists an antichain \( A \in \mathcal{A} \) such that \( |\{q \in A : p \land q \neq 0\}| \geq \pi(P) \). A matrix \( \mathcal{A} \) is weakly shattering if \( \sum_{A \in \mathcal{A}} |\{q \in A : p \land q \neq 0\}| \geq \pi(P) \) for every \( p \in P \). A matrix is a base matrix if \( \bigcup \mathcal{A} \) is a dense subset of \( P \). The following two theorems contain some well known basic facts about all these notions.

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Theorem 1.1. (1) A shattering matrix is weakly shattering.
(2) There exists a base matrix on $\mathbb{P}$ of size $\pi(\mathbb{P})$.
(3) If $h\pi(\mathbb{P}) = \pi(\mathbb{P})$, then every base matrix on $\mathbb{P}$ is weakly shattering.
(4) There exists a shattering matrix on $\mathbb{P}$ if and only if $hsat(\mathbb{P}) = \pi(\mathbb{P})^+$. 
(5) If there is a weakly shattering matrix on $\mathbb{P}$ of size $< \pi(\mathbb{P})$, then $hsat(\mathbb{P}) = \pi(\mathbb{P})^+$. 
(6) For every weakly shattering matrix there exists a weakly shattering base matrix of the same size. 
(7) If $hsat(\mathbb{P}) = \pi(\mathbb{P})^+$, then for every base matrix on $\mathbb{P}$ there exists a shattering base matrix on $\mathbb{P}$ of the same size. 
(8) If $hsat(\mathbb{P}) = \pi(\mathbb{P})^+$, then there exists a shattering matrix on $\mathbb{P}$ of size $\text{cf}(\pi(\mathbb{P}))$.

Proof: The assertions (1)–(5) are easy to see. For the rest of the proof let us fix a dense set $D \subseteq \mathbb{P}$ with $|D| = \pi(\mathbb{P})$.

(6) Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be a weakly shattering matrix on $\mathbb{P}$. There exists a one-to-one mapping $\varphi : D \to \bigcup_{\alpha < \kappa} \{\alpha\} \times A_\alpha$, $\varphi = (\varphi_1, \varphi_2)$, such that $p \land \varphi_2(p) \neq 0$ for every $p \in D$. For every $p \in D$ let us fix an element $r(p) \in P$ below $p$ and $\varphi_2(p)$ and let $A'_\alpha = \{r(p) : \varphi_1(p) = \alpha\}$. The matrix $\mathcal{A} = \{A'_\alpha : \alpha < \kappa\}$ is a weakly shattering base matrix on $\mathbb{P}$.

(7) For $p \in \mathbb{P}$ let $B_p$ be an antichain below $p$ of size $\pi(\mathbb{P})$. If $\mathcal{A}$ is a base matrix on $\mathbb{P}$, then the matrix $\mathcal{A}' = \bigcup_{p \in \mathcal{A}} B_p : A \in \mathcal{A}$ is a shattering base matrix on $\mathbb{P}$.

(8) Let $D = \bigcup \{D_\alpha : \alpha < \text{cf}(\pi(\mathbb{P}))\}$ with $|D_\alpha| < \pi(\mathbb{P})$. By the Balcar-Vojt\'a\v{s}'s Theorem (see [1] or [6]) for each $\alpha$ there is a disjoint refinement $A_\alpha$ of $D_\alpha$. Therefore $\{A_\alpha : \alpha < \text{cf}(\pi(\mathbb{P}))\}$ is a base matrix on $\mathbb{P}$ and by assertion (7) there exists a shattering matrix on $\mathbb{P}$ of the same size. \hfill \Box

From now on we assume that $h\pi(\mathbb{P}) = \pi(\mathbb{P})$ and we define:

$$sh(\mathbb{P}) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a weakly shattering matrix on } \mathbb{P}\},$$
$$sh_\lambda(\mathbb{P}) = \min\{|\pi(\mathbb{P})| \cup \{\kappa : r.o.(\mathbb{P}) \text{ is } (\kappa, \pi(\mathbb{P}), \lambda)-\text{nowhere distributive}\}\}.$$

We use the definition of the three-parameter distributivity from [2]. Clearly, $sh(\mathbb{P}) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a base matrix on } \mathbb{P}\} = \min\{|\pi(\mathbb{P})| \cup \{|\mathcal{A}| : \mathcal{A} \text{ is a shattering matrix on } \mathbb{P}\}\} = sh_\pi(\mathbb{P})(\mathbb{P})$. Again, $hsh(\mathbb{P})$ denotes the hereditary version of the cardinal $sh(\mathbb{P})$.

Theorem 1.2. Let us assume that $h\pi(\mathbb{P}) = \pi(\mathbb{P})$.

(1) If $r.o.(\mathbb{P})$ is $(\kappa, \lambda, \lambda)$-nowhere distributive, then $r.o.(\mathbb{P})$ is $(\kappa, \text{cf} \lambda, \text{cf} \lambda)$-nowhere distributive.
(2) If $r.o.(\mathbb{P})$ is $(\kappa, \text{cf} \lambda, \text{cf} \lambda)$-nowhere distributive, then $r.o.(\mathbb{P})$ is $(\kappa, \lambda, \text{cf} \lambda)$-nowhere distributive.
(3) If $\kappa < \text{cf} \lambda$, then $r.o.(\mathbb{P})$ is $(\kappa, \text{cf} \lambda, \text{cf} \lambda)$-nowhere distributive if and only if $\|P\| \leq \kappa$. 

Proof: The assertions (1) and (2) are easy.

(3) Let \( \{ \lambda_\xi : \xi < \text{cf} \lambda \} \) be an increasing cofinal sequence in \( \lambda \) and let \( \kappa < \text{cf} \lambda \). Let \( \hat{f} \) be a \( \mathbb{P} \)-name of an unbounded function from \( \kappa \) to \( \lambda \). For \( \alpha < \kappa \) let \( A_\alpha = \{ \Vert \hat{f}(\alpha) \in [\lambda_\xi, \lambda_{\xi+1}] : \xi < \text{cf} \lambda \} \setminus \{0\} \). The matrix \( \{ A_\alpha : \alpha < \kappa \} \) witnesses the \((\kappa, \text{cf} \lambda, \text{cf} \lambda)\)-nowhere distributivity of \( r.o.\mathbb{P} \). Conversely, if \( \{ A_\alpha : \alpha < \kappa \} \) is a matrix on \( r.o.\mathbb{P} \) with \( A_\alpha = \{ a_{\alpha, \xi} : \xi < \text{cf} \lambda \} \) witnessing the \((\kappa, \text{cf} \lambda, \text{cf} \lambda)\)-nowhere distributivity of \( r.o.\mathbb{P} \), then the formula \( \Vert \hat{f}(\alpha) = \lambda_\xi \Vert = a_{\alpha, \xi} \) defines a \( \mathbb{P} \)-name of an unbounded function from \( \kappa \) to \( \lambda \).

(4) Let us assume that \( p \) and \( \mu \) are such that \( p \Vdash \mathbb{P} \mid \pi^V(\mathbb{P}) \mid = \mu \). Let \( \hat{f} \) be a \( \mathbb{P} \)-name of a function from \( \mu \) onto \( \pi(\mathbb{P}) \) and for \( \alpha < \mu \) let \( A_\alpha \) be a maximal antichain in \( \mathbb{P} \mid p \) consisting of \( q \in \mathbb{P} \mid p \) deciding \( \hat{f}(\alpha) \). Since every \( q \in \mathbb{P} \mid p \) forces that \( \hat{f} \) is onto \( \pi(\mathbb{P}) = \pi(\mathbb{P}) \), easily, it can be verified that \( \{ A_\alpha : \alpha < \mu \} \) is a weakly shattering matrix on \( \mathbb{P} \mid p \). Therefore \( \text{sh}(\mathbb{P}) = \text{sh}(\mathbb{P} \mid p) \leq \mu \) and \( p \Vdash \text{sh}^V(\mathbb{P}) \leq \mid \pi^V(\mathbb{P}) \mid \).

Let \( \text{sh}(\mathbb{P}) = \kappa \). If \( \text{sh}(\mathbb{P}) = \pi(\mathbb{P}) \), then clearly, \( p \Vdash \mathbb{P} \mid \pi^V(\mathbb{P}) \mid \leq \text{sh}^V(\mathbb{P}) \). Let us assume that \( \text{sh}(\mathbb{P}) < \pi(\mathbb{P}) \). Then by Theorem 1.1(5), \( \text{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+ \). For every \( q \in \mathbb{P} \) let us fix a maximal antichain \( \{ (q)_\xi : \xi < \pi(\mathbb{P}) \} \) below \( q \). As \( \text{sh}(\mathbb{P}) = \kappa \), there is a base matrix \( \mathcal{A} = \{ A_\alpha : \alpha < \kappa \} \) (with all antichains maximal). We define a \( \mathbb{P} \)-name \( \hat{f} \) of a function from \( \kappa \) onto \( \pi^V(\mathbb{P}) \) by \( \forall \alpha \in \kappa \). Therefore \( p \Vdash \mathbb{P} \mid \pi^V(\mathbb{P}) \mid \leq \text{sh}^V(\mathbb{P}) \).

(5) Clearly, \( p \Vdash \mathbb{P} \mid \pi(\mathbb{P}) \mid \leq \mid \pi^V(\mathbb{P}) \mid \). Let \( p \) and \( \kappa \) be such that \( p \Vdash \mathbb{P} \mid \pi(\mathbb{P}) \mid = \kappa \) and \( \text{hsh}(\mathbb{P} \mid p) = \text{sh}(\mathbb{P} \mid p) \). Let \( \hat{f} \) be a \( \mathbb{P} \)-name of a function from \( \kappa \) into \( \mathbb{P} \) such that \( p \Vdash (\forall q \in \mathbb{P} \exists \kappa < \alpha \xi : \alpha \xi \hat{f}(\alpha) \leq q \). Let \( A_\alpha, \alpha < \kappa \), be a maximal antichain of conditions below \( p \) deciding \( \hat{f}(\alpha) \). For \( q \leq p \) let \( B_{r, q} = \{ r \in \mathbb{P} : q \wedge r \neq 0 \} \) and \( B'_{r, q} = \{ s \in \mathbb{P} : (\exists r \in B_{r, q}) r \Vdash \hat{f}(\alpha) = s \} \). The set \( \bigcup_{\alpha} B'_{r, q} \) is a dense subset of \( \mathbb{P} \) for every \( q \leq p \) and \( |B_{r, q}| \geq |B'_{r, q}| \). Therefore \( \sum_{\alpha \leq \kappa} |B_{r, q}| \geq |(r, q) | \) and hence the matrix \( \{ A_\alpha : \alpha < \kappa \} \) is weakly shattering on \( \mathbb{P} \). Hence \( \text{sh}(\mathbb{P} \mid p) \leq \kappa \) and by (4) we have \( p \Vdash \mathbb{P} \mid \pi^V(\mathbb{P}) \mid \leq \pi(\mathbb{P}) \). A density argument proves that \( p \Vdash \mathbb{P} \mid \pi^V(\mathbb{P}) \mid \leq \pi(\mathbb{P}) \).

(6) By (1)–(3) we easily obtain the inequalities \( \min \{ \text{sh}(\mathbb{P}) \mid \pi(\mathbb{P}) \mid \leq \text{cf}(\mathbb{P}) \leq \min \{ \text{sh}(\mathbb{P}), \text{cf}(\mathbb{P}) \} \} \). If \( \text{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+ \), then, by Theorem 1.1(8), \( \text{sh}(\mathbb{P}) \leq \text{cf}(\mathbb{P}) \). Since \( \text{sh}(\mathbb{P}) \leq \text{sh}(\mathbb{P}) \), by (5), \( \text{sh}(\mathbb{P}) \leq \text{cf}(\mathbb{P}) \). If
hsat(\mathbb{P}) \leq \pi(\mathbb{P})$, then $\text{sh}(\mathbb{P}) = \pi(\mathbb{P})$ by Theorem T1.1(5)

(7) immediately follows by (6), and (8) can be obtained by an easy computation.

In the case $\text{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$, in some special cases (e.g., if $\pi(\mathbb{P})$ is regular, or $\text{a}(\text{cf}(\pi(\mathbb{P}))), \mathbb{P} = \pi(\mathbb{P})$, etc., see Theorem 1.2(7) or (8)), $\text{sh}(\mathbb{P})$ is regular (even in $V^{\text{r.o.}}(\mathbb{P})$). But in general it is not clear whether $\text{sh}(\mathbb{P})$ is a regular cardinal.

We use the standard terminology. By $\mathcal{M}$ and $\mathcal{N}$ we denote the ideal of meager sets and the ideal of null sets, respectively, $\mathfrak{b}$ is the least cardinality of an unbounded family and $\mathfrak{d}$ is the least cardinality of a dominating family of functions in the ordering $\leq^*$ on $\omega^\omega$ defined for $f, g \in \omega^\omega$ by $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. $\text{add}(I)$ is the additivity of an ideal $I$, $\text{cov}(I)$ is the least size of a set $I_0 \subseteq I$ such that $\bigcup I_0 = \bigcup I$, $\text{non}(I)$ is the least size of a subset of $\bigcup I$ not belonging to $I$, and $\text{cof}(I)$ is the least size of a set $I_0 \subseteq I$ which is cofinal in $(I, \subseteq)$. Sacks forcing $\mathbb{S}$ is the set of perfect trees $p \subseteq \omega^2$ where $p$ is stronger than $q$, $p \leq q$, if $p \subseteq q$. For $p \in \mathbb{S}$ and $s \in \omega^2$ we denote $p_s = \{t \in p : s \subseteq t \text{ or } t \subseteq s\}$, $[p] = \{x \in \omega^2 : \forall n x \in p\}$, $[s] = \{x \in \omega^2 : s \subseteq x\}$. Every perfect set in $\omega^2$ is of the form $[p]$ for some $p \in \mathbb{S}$.

2. Maximal antichains in $\mathbb{S}$

Important is the question what the possible sizes of small maximal antichains in Sacks forcing are. By the next well-known theorem, $\text{a}(\omega_1, \mathbb{S}) \geq \text{cov}(\mathcal{M})$ and we prove in Theorem 2.5 below that $\text{a}(\omega_1, \mathbb{S}) \geq \mathfrak{d}$.

**Theorem 2.1.** For a cardinal $\kappa$ the following conditions are equivalent:

1. $\kappa < \text{cov}(\mathcal{M})$;
2. for every family $B$ of perfect sets such that $|B| \leq \kappa$ and $\omega^2 \setminus \bigcup C$ is uncountable for every $C \in [B]^{\leq \omega}$, $\omega^2 \setminus \bigcup B \neq \emptyset$;
3. for every family $B$ of perfect sets such that $|B| \leq \kappa$ and $\omega^2 \setminus \bigcup C$ is uncountable for every $C \in [B]^{\leq \omega}$, $\omega^2 \setminus \bigcup B$ contains a perfect set.

**Proof:** The implications (3) $\rightarrow$ (2) $\rightarrow$ (1) are obvious. We prove (1) $\rightarrow$ (3).

Let $\kappa < \text{cov}(\mathcal{M})$ and let $B$ be a family of perfect sets such that $|B| \leq \kappa$ and $\omega^2 \setminus \bigcup C$ is uncountable for every $C \in [B]^{\leq \omega}$. Let $q$ be the set of all $s \in \omega^2$ such that $[s] \setminus \bigcup C$ is uncountable for every $C \in [B]^{\leq \omega}$. By the assumption, $\emptyset \in q$ and it follows that $q$ is a perfect tree and for every perfect set $[p] \in B$, $[p] \cap [q]$ is nowhere dense in $[q]$. As $\kappa < \text{cov}(\mathcal{M})$, $\text{MA}_\kappa(\text{countable})$ implies the existence of a perfect tree $r \leq q$ such that $[r] \cap [p] = \emptyset$ for all $[p] \in B$ (using a countable forcing for adding a perfect set of Cohen reals).
We need the following special case of Exercise 7.13 in [5]:

**Lemma 2.2.** If $G$ is a dense $G_δ$ subset of $\omega^2$ such that $\omega^2 \setminus G$ is dense in $\omega^2$, then there exists a homeomorphism $f$ from $G$ onto $\omega^ω$.

**Proof:** By the assumptions no relatively clopen subset of $G$ is compact. Let $U_n$, $n ∈ \omega$, be open sets in $\omega^2$ such that $G = \bigcap_{n ∈ \omega} U_n$ and $U_{n+1} ⊆ U_n$ for all $n$. For $s ∈ ω^ω$ let us define $t_s ∈ ω^2$ by induction on $|s|$ so that $s ⊆ t'_s$ if and only if $t_s ⊆ t_s'$, $t_∅ = ∅$, and $[t_s] ∩ U_{n+1} = \bigcup_{i ∈ ω} [t_{s'} \setminus (\langle i \rangle)]$ for $|s| = n$. Then for $x ∈ G$ we let $f(x)$ be the unique element $y ∈ ω^ω$ such that $t_y|n ⊆ x$ for all $n ∈ ω$. □

**Theorem 2.3.** If $B$ is a family of perfect sets in $\omega^2$ and $|B| < ω$, then the set $\omega^2 \setminus \bigcup B$ is either at most countable or contains a perfect set.

**Proof:** Let us assume that $|B| < ω$ and the set $X = \omega^2 \setminus \bigcup B$ is uncountable. Let $Y$ be a countable subset of $X$ without isolated points. Let $q ∈ S$ be such that $[q] = \overline{Y}$. By Lemma 2.2 there is a homeomorphism $f$ from $G = [q] \setminus Y$ onto $ω^ω$. For $F ∈ B$, $F ∩ Y = ∅$ and hence $F ∩ G = F ∩ [q]$. It follows that $f^ω(F ∩ G)$ is compact and hence bounded in $ω^ω$. As $|B| < ω$, there is an $y ∈ ω^ω$ not dominated by any member of the set $\bigcup_{F ∈ B} f^ω(F ∩ G)$. Then the set $E = f^{-1}(\{x ∈ ω^ω : ∀n x(n) ≥ y(n)\})$ is an uncountable relatively closed subset of $G$ disjoint from $\bigcup B$. □

If $ω = κ$, then using Theorem 2.3 one can construct a partition of $\omega^2$ into $κ$ perfect sets.

In the next theorem we prove that partitions of $\omega^2$ into $κ$ perfect sets exist in ZFC. We shall use the following notation:

Let $p ∈ S$ and $x ∈ [p]$. Let $\{k_n : n ∈ ω\}$ be the increasing enumeration of the set $\{k ∈ ω : (x|k) \setminus (0) ∈ p$ and $(x|k) \setminus (1) ∈ p\}$ and let $x \in ω^2$ be such that $\bar{x}(n) ≠ x(n)$ for all $n ∈ ω$. Let us define $τ(p, x, n) = p(x|k_n) \setminus (\bar{x}(k_n)) = \{s ∈ p : s ⊆ (x|k_n) \setminus (\bar{x}(k_n)) \text{ or } (x|k_n) \setminus (\bar{x}(k_n)) ⊆ s\}$. Then the system $[τ(p, x, n)], n ∈ ω$, is a partition of $[p] \setminus \{x\}$. In particular, $[τ(\langle ω^2, x, n\rangle)], n ∈ ω$, is a partition of $ω^2 \setminus \{x\}$ into clopen sets.

For $A ⊆ S$ let $B_A = \{[p] : p ∈ A\}$ and let $\bigvee A$ denote the Boolean sum of $A$ in r.o.(S). In the Boolean sums we will consider only those $A ⊆ S$ for which $\bigvee A ∈ S$. Notice that $\bigvee_n τ(p, x, n) = \bigcup_n τ(p, x, n) = p$.

**Theorem 2.4.** Let $D$ be a dense subset of $S$.

1. There exists a maximal antichain $A ⊆ D$ such that the family $B_A$ is disjoint and for every $p ∈ S$ with $[p] \subseteq ∪ B_A$ there exists $C ∈ [B_A] < κ$ such that $[p] ⊆ ∪ C$.

2. There exist maximal antichains $A ⊆ D$ and $A ∈ S$, both of size $κ$, such that $B_A$ is a disjoint family, $B_A$ is a partition of $ω^2$, and the following conditions are satisfied:

   a. for every $q ∈ A \setminus A$ the set $A_q = \{p ∈ A : p ≤ q\}$ is countable, $q = \bigvee A_q$, and $|[q] \setminus ∪ B_{A_q}| = 1$;
(b) For every $q \in S$, if $|[q] \cup BA| < c$, then $|\{p \in A : [q] \cap [p] \neq \emptyset\}| < c$;
(c) for every $q \in S$, $|\{p \in A : q \cap p \neq 0\}| < c$ if and only if $|\{p \in A : [q] \cap [p] \neq \emptyset\}| < c$.

In particular, by (b), $|\omega^2 \setminus BA| = c$.

PROOF: The assertion (1) is Lemma 1.1 in [4] and it clearly follows from (2). The following proof of (2) is a modification of the proof in [4].

Let $\{q_{\alpha} : \alpha < c\}$ be an enumeration of $S$ such that for each $q \in S$, $q = q_\alpha$ for $\alpha$ many $\alpha$’s, and let $\{y_\alpha : \alpha < c\}$ be an enumeration of $\omega^2$ without repetitions.

Let $A'$ be a maximal antichain in $S$ such that the set $\{[p] \cap [s] : p \in A'\}$ has size $c$ for every $s \in <\omega^2$ (for example, find pairwise disjoint perfect sets $[p_s] \subseteq [s]$, $s \in <\omega^2$ and split each $[p_s]$ into $c$ many disjoint perfect sets). Without loss of generality we can assume that $D \subseteq \{p : \exists q \in A' \ p \leq q\}$. By induction on $\alpha < c$ we construct $p_{\alpha} \in D$, countable $A'_{\alpha} \subseteq D$, and $x_{\alpha} \in \omega^2$. Let us assume that $p_{\beta}$, $A'_{\beta}$, $x_{\beta}$ for $\beta < \alpha$ have been constructed and that the set $A''_{\alpha} = \bigcup_{\beta < \alpha} A'_{\beta} \cup \{p_{\beta}\}$ is an antichain.

If the set $[q_{\alpha}] \setminus (\{x_\beta : \beta < \alpha\} \cup BA'_{\alpha})$ is nonempty, then let $x_\alpha$ be its element; otherwise let $x_\alpha = x_0$.

If $q_\alpha$ is compatible with some $p \in A''_{\alpha}$, then we set $p_{\alpha} = p_0$. Otherwise the set

$$X_\alpha = \{x_\beta : \beta \leq \alpha\} \cup \{y_\beta : \beta < \alpha\} \cup ([q_{\alpha}] \cup BA'_{\alpha})$$

$$\cup \{[q_\beta] \cap [q_{\alpha}] : \beta < \alpha \text{ and } q_\beta \wedge q_\alpha = 0\}$$

has size $< c$ and let $p_{\alpha} \in D$, $p_\alpha \leq q_\alpha$, be such that $[p_\alpha] \cap X_\alpha = \emptyset$. Notice that if $p_\alpha \neq p_0$, then $x_\alpha \neq x_{\beta}$ for all $\beta < \alpha$.

If $y_\alpha \in \bigcup BA''_{\alpha} \cup \{p_{\alpha}\}$, then we set $A'_\alpha = \{p_0\}$. Assume that $y_\alpha \notin \bigcup BA''_{\alpha} \cup \{p_{\alpha}\}$. By the assumption put on $D$ the antichain $A''_{\alpha} \cup \{p_{\alpha}\}$ is nowhere locally maximal and for every $n \in \omega$ there is $r'_{\alpha,n}$ such that $p \wedge r'_{\alpha,n} = 0$ for $p \in A''_{\alpha} \cup \{p_{\alpha}\}$. The set

$$X_{\alpha,n} = \{x_\beta : \beta \leq \alpha\} \cup \{y_\beta : \beta \leq \alpha\} \cup ([r'_{\alpha,n}] \cup BA''_{\alpha} \cup \{p_{\alpha}\})$$

$$\cup \{[q_\beta] \cap [r'_{\alpha,n}] : \beta < \alpha \text{ and } q_\beta \wedge r'_{\alpha,n} = 0\}$$

has size $< c$. Let $r_{\alpha,n} \in D$, $r_{\alpha,n} \leq r'_{\alpha,n}$, be such that $[r_{\alpha,n}] \cap X_{\alpha,n} = \emptyset$ and let $A'_{\alpha} = \{r_{\alpha,n} : n \in \omega\}$. Then $r_{\alpha,n} = \tau(\bigvee A'_{\alpha}, y_\alpha, n)$ and $[\bigvee A'_{\alpha}] = \{y_\alpha\} \cup \bigcup_{n \in \omega} [r_{\alpha,n}]$.

By the construction it is clear that $A = \bigcup A$ is a maximal antichain in $S$ refining the antichain $A'$. It follows that its size is $c$. Let $\{A_{\alpha} : \alpha < c\}$ be an enumeration of the family $A$ without repetitions and let $\bar{A} = \{\bigvee A_{\alpha} : \alpha < c\}$. Then $\bar{A}$ is a maximal antichain in $S$. $BA$ is a disjoint family and as $A''_{\alpha} \neq \{p_0\}$ if and only if $y_\alpha \notin \bigcup BA$, $[\bigvee A''_{\alpha}] = \{y_\alpha\} \cup \bigcup BA''_{\alpha}$ whenever $A''_{\alpha} \neq \{p_0\}$. Therefore $BA_{\bar{A}}$ is a partition of $\omega^2$ and condition (a) is satisfied. We prove conditions (b) and (c). Let $q \in S$ be arbitrary.
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(b) If the set \( \{ p \in A : [p] \cap [q] \neq \emptyset \} \) has size \( c \), then, for every \( \alpha \) such that \( q_\alpha = q \), the set \([q_\alpha] \setminus \bigcup B_{A_\alpha''} \) has size \( c \) and hence \( x_\alpha \neq x_\beta \) for all \( \beta < \alpha \). Therefore the set \( \{ x_\alpha : q_\alpha = q \} \) has size \( c \) and is a subset of \([q] \setminus \bigcup B_A \).

(c) There is \( \beta < c \) such that \( q = q_\beta \). Let us assume that the set \( B = \{ p \in A : q \land p \neq 0 \} \) has size \( < c \). Let \( \gamma > \beta \) be such that \( B \subseteq A''_\gamma \). We prove that the set \( \{ p \in A : [q] \cap [p] \neq \emptyset \} \) is a subset of \( A''_\gamma \) and hence it has size \( < c \).

For every \( \alpha \geq \gamma \), if \( p_\alpha \notin A''_\gamma \), then \( p_\alpha \neq p_0 \) and \( q_\beta \land q_\alpha = 0 \). Therefore \( p_\alpha \leq q_\alpha \) is such that \([q_\beta] \cap [p_\alpha] = \emptyset \).

For every \( \alpha \geq \gamma \), if \( A'_\alpha \setminus A''_\gamma = \emptyset \), then \( A'_\alpha \neq \{p_0\} \) and \( A'_\alpha = \{ r_{\alpha,n} : n \in \omega \} \) where \( r_{\alpha,n} \leq r'_{\alpha,n} \) and \( p \land r'_{\alpha,n} = 0 \) for all \( p \in A''_\alpha \supseteq A''_\gamma \), \( n \in \omega \). It follows that \( q_\beta \land r'_{\alpha,n} = 0 \) and hence \( r_{\alpha,n} \leq r'_{\alpha,n} \) is such that \([r_{\alpha,n}] \) is disjoint from \([q_\beta] \). So, if \( A'_\alpha \neq \{p_0\} \), then \([q_\beta] \cap [p] = \emptyset \) for all \( p \in A'_\alpha \).

Let us consider the following families:

\[
\mathcal{A}_1 = \{ A : A \text{ is a maximal antichain in } S \text{ and } B_A \text{ is a disjoint family} \},
\mathcal{A}_2 = \{ B : B \text{ is a partition of } \omega 2 \text{ into closed sets} \},
\mathcal{A}_3 = \{ A : A \text{ is a maximal antichain in } S , B_A \text{ is a disjoint family, and the set } \omega 2 \setminus \bigcup B_A \text{ has size } c \},
\mathcal{A}_4 = \{ A : A \text{ is a maximal antichain in } S , B_A \text{ is a disjoint family, and the set } \omega 2 \setminus \bigcup B_A \text{ is uncountable} \}.
\]

By Theorem 2.4 all these families are nonempty and by Theorem 2.3 the families \( \mathcal{A}_3 \) and \( \mathcal{A}_4 \) do not contain countable antichains. Let us define the cardinals:

\[
\begin{align*}
\alpha_i &= \min \{|A| : X \in \mathcal{A}_i \text{ and } |A| \geq \omega_1 \}, \quad i = 1, 2, 3, 4, \\
\tilde{\alpha}_i &= \sup \{|A| : A \in \mathcal{A}_i \text{ and } |A| < c \} \cup \{ \omega_1 \}, \quad i = 1, 2, 3, 4. \\
\text{cov}_1 &= \min \{|B| : B \text{ is a family of perfect sets such that the set } \omega 2 \setminus \bigcup B \text{ is uncountable and does not contain a perfect set} \}, \\
\text{cov}_2 &= \min \{|B| : B \text{ is a family of perfect sets such that the set } \omega 2 \setminus \bigcup B \text{ has size } c \text{ and does not contain a perfect set} \}.
\end{align*}
\]

**Theorem 2.5.**

(1) \( \vartheta = \text{cov}_1 \leq a(\omega_1, S) \leq a_1 = a_4 \leq \min \{a_2, a_3\} \); \( \tilde{\alpha}_1 = \tilde{\alpha}_4 \).

(2) \( \text{cov}_1 \leq \text{cov}_2 \leq a_3 \).

(3) For every \( i, \tilde{\alpha}_i \leq \alpha_i \) if and only if \( \tilde{\alpha}_i = \omega_1 \) if and only if \( \alpha_i = c \).

(4) For every \( i, \tilde{\alpha}_1 \leq \alpha_i \) if and only if \( \alpha_i = c \).

(5) If \( \alpha_1 = c \), then, for all \( i, \alpha_i = c \) and \( \tilde{\alpha}_i = \omega_1 \).

(6) If \( \alpha_2 = c \), then \( \alpha_1 = a_3 \) and \( \tilde{\alpha}_1 = \tilde{\alpha}_3 \).

(7) If \( \alpha_3 = c \), then \( \alpha_1 = c \) if and only if \( \alpha_2 = c \).

(8) \( \tilde{\alpha}_1 = \max \{\tilde{\alpha}_2, \tilde{\alpha}_3\} \).
**Proof:** (1) The inequality \( d \leq \text{cov}_1 \) is Theorem 2.3. To prove \( \text{cov}_1 \leq d \), without loss of generality let us assume that \( c > d \). Let \( X = \{x_\alpha, y_\alpha : \alpha < \omega_1\} \subseteq \omega_2 \) be a Hausdorff gap (see [3]), i.e., \( x_\alpha \leq^* x_\beta \leq^* y_\beta \leq^* y_\alpha \) for \( \alpha \leq \beta < \omega_1 \), and for every \( x \in \omega_2 \) there is \( \alpha < \omega_1 \) such that \( x_\alpha \not\leq^* x \) or \( x \not\leq^* y_\alpha \). Let \( K_\alpha = \{x \in \omega_2 : x_\alpha \not\leq^* x \) or \( x \not\leq^* y_\alpha\} \) for \( \alpha < \omega_1 \). Then \( K_\alpha \subseteq K_\beta \) for \( \alpha \leq \beta \), \( K_\alpha \cap X \) is countable, and consequently, the sets \( K_\alpha \setminus X \), \( \alpha < \omega_1 \), are \( G_\delta \) sets covering \( \omega_2 \setminus X \). The Baire space \( \omega \omega \) is a union of \( d \) many compact sets and as every Polish space is a continuous image of \( \omega \omega \), every Polish space is a union of \( \leq d \) compact sets. It follows that every set \( K_\alpha \setminus X \) a union of \( \leq d \) compact sets and hence \( \omega_2 \setminus X \) is a union of \( \leq d \) compact sets. Considering the perfect kernels of these compacts (obtained by removing countable sets) we obtain a family of \( \leq d \) perfect subsets of \( \omega_2 \) whose union has uncountable complement of size \( < c \) and hence \( \text{cov}_1 \leq d \).

Let us assume that \( a(\omega_1, S) < \text{cov}_1 \) and we prove a contradiction. Let \( A \subseteq S \) be a maximal antichain of size \( a(\omega_1, S) \). The set \( X = \bigcup \{[p] \cap [q] : p, q \in A, p \neq q\} \) has size \( < c \). For every \( p \in A \) let \( x_p \in [p] \setminus X \) be arbitrary. The family \( A' = \{\tau(p, x_p, n) : p \in A \) and \( n \in \omega\} \) is a maximal antichain in \( S \) because if \([p] \cap [q] \) is uncountable for some \( p \in A \), then \( [\tau(p, x_p, n)] \cap [q] \) is uncountable for some \( n \). The set \( Y = \omega_2 \setminus \bigcup B_{A'} \) is uncountable as it contains the set \( \{x_p : p \in A\} \) and as \( a(\omega_1, S) \leq \text{cov}_1 \), there is a perfect set \( [q] \subseteq Y \). But \([p] \cap [q] \subseteq \{x_p\} \) for all \( p \in A \) which contradicts the assumption that \( A \) is maximal. Therefore \( \text{cov}_1 \leq a(\omega_1, S) \).

The inequality \( a_4 \leq a_1 \) can be easily proved by the same argument. Therefore \( a_1 = a_4 \) and by the same proof we obtain \( \tilde{a}_1 = \tilde{a}_4 \). The other inequalities are trivial.

(2) is an easy consequence of definitions.

(3–4) The implications from the right to the left are obvious. Let us assume that \( a_i < c \) for some \( i \). Then \( a_i < a_i^+ \leq \tilde{a}_i \) and \( \tilde{a}_i \leq \tilde{a}_1 \).

(5) By (1), for all \( i \), \( a_i = c \) and by (3), \( \tilde{a}_1 = \omega_1 \).

(6) If there is a maximal antichain \( A \subseteq S \) of size \( < c \) such that the family \( B_A \) is disjoint and the set \( X = \omega_2 \setminus \bigcup B_A \) has size \( < c \), then the partition \( B = B_A \cup \{\{x\} : x \in X\} \) has size \( < c \).

(7) Let \( a_3 = c \). If \( a_2 = c \), then, by (6), \( a_1 = a_3 = c \).

(8) \( \tilde{a}_1 \geq \tilde{a}_2 \) and \( \tilde{a}_1 \geq \tilde{a}_3 \). Let us assume that \( \tilde{a}_3 < \tilde{a}_1 \). For any \( \kappa \) with \( \tilde{a}_3 \leq \kappa < \tilde{a}_1 \) there is an antichain \( A \subseteq A_1 \setminus \mathcal{A}_3 \) of size \( < c \) and \( \geq \kappa \). Then the partition \( B_A \cup \{\{x\} : x \in \omega_2 \setminus \bigcup B_A \} \) has size \( < c \) and \( \geq \kappa \). Therefore \( \tilde{a}_2 > \kappa \) and so \( \tilde{a}_2 = \tilde{a}_1 \). \( \Box \)

Clearly, \( a(\omega, S) = \omega \). There are known several constructions of small uncountable antichains in \( S \). J. Stern and independently K. Kunen (for the proof see [8]) under CH constructed a partition of \( \omega_2 \) into \( \omega_1 \) compact sets. L. Newelski [9] pointed out that under MA the same construction produces a partition into \( \mathfrak{c} \) compact sets which is preserved by forcing with measure algebras and he proved that after adding \( \omega_1 \) dominating reals, the Baire space \( \omega \omega \) (and hence, by Lemma 2.2,
also the Cantor space $\omega^2$ can be partitioned into $\omega_1$ disjoint compact perfect sets. A. Rosłanowski and S. Shelah [10], by a finite support iteration of c.c.c. forcing notions of length $\omega_1$, constructed a maximal antichain $A$ such that the family $B_A$ is disjoint and every tree $p \in A$ has on each level at most one branching node. Moreover, the set $\bigcup B_A$ does not contain any ground model reals and therefore $a_3 = \omega_1$ holds in the extension.

We say that a set $a \subseteq \omega^2$ is saturated if for every $s, t \in \omega^2$ whenever $s \subseteq t$ and $t \in a$, then $s \in a$. Easily, it can be observed that $a_2$ is the minimal size of a family $A$, maximal with respect to the inclusion, such that $A$ is an uncountable almost disjoint family of infinite saturated sets. Notice that such a family $A$ cannot be a maximal almost disjoint family of infinite subsets of $\omega^2$. To see this, let $a \in A$ be such that the set of all infinite branches in $a$ is nowhere dense in $\omega^2$ and let $x \in a$ be arbitrary. For every $n$ choose $s_n \in \omega^2$ such that $x|n \subseteq s_n$ and $s_n \notin a$. Then the set $\{s_n : n \in \omega\}$ has a finite intersection with every $b \in A$. The similarity of this characterization of $a_2$ with maximal almost disjoint families suggests the question whether there is some relation between $a_2$ and $a$ (the minimal size of a maximal almost disjoint family of subsets of $\omega$).}

3. Marczewski’s ideal and the collapse by Sacks forcing

A subset $X$ of $\omega^2$ is an $s^0$-set if for every $p \in S$ there is $q \leq p$ such that $[q] \cap X = \emptyset$. This notion is due to E. Marczewski [7]. It is known that $\omega_1 \leq \text{add}(s^0) \leq \text{cov}(s^0) \leq \text{cf}(c) \leq \text{non}(s^0) = c < \text{cf}(\text{cof}(s^0))$ (see [4]) and $\text{add}(s^0) \leq b$ (in fact $\text{sh}(S) \leq b$ see [11]; this is not true for $\text{cov}(s^0)$ because in the iterated Sacks forcing model $\text{cov}(s^0) = \omega_2$ see [4] but $b = \text{cof}(\mathcal{N}) = \omega_1$).

Notice that $\text{add}(I) \leq \text{cf}(\text{non}(I))$ for each ideal $I$. If $y \in \omega^2$ is a new real, then the perfect set $A_y = \{x \in \omega^2 : (\forall n) x(2n) = y(n)\}$ does not contain old reals. This explains why in iterations of length $\omega_1$ the set of old reals is an $s^0$-set and $\text{cov}(s^0) = \omega_1$.

To see that there are $s^0$-sets of size $\omega$ (see also [4]), take any maximal antichain $\{p_\alpha : \alpha < \omega\}$ of size $\omega$ in $S$ so that the system of perfect sets $B_A = \{[p_\alpha] : \alpha < c\}$ is disjoint and clearly, every selector of this system is an $s^0$-set. By Theorem 2.4(2) every $s^0$-set has this form. If $B_A$ is not disjoint, then its selectors need not be $s^0$-sets (observe that the system $\{A_y : y \in \omega^2\}$ has a perfect selector).

The next theorem refines Theorem 1.1 in [4].

**Theorem 3.1.** (1) $\text{sh}_{a_3}(S) \leq \text{add}(s^0) \leq \text{sh}_{a_2}(S) \leq \text{sh}(S) \leq \text{min}\{\text{cf } c, b\}$.

(2) $\text{sh}_{\omega_1}(S) = \text{sh}_{a_1}(S) = \text{min}\{\text{sh}_{a_2}(S), \text{add}(s^0)\} \leq \text{sh}_{a_3}(S)$.

(3) $\text{sh}_{a_2}(S) \leq \max\{\text{sh}_{a_3}(S), \text{add}(s^0)\} = \text{sh}_{a_1}(S) = \text{sh}(S)$.

(4) $\text{sh}_{\omega_1}(S) \leq \text{sh}_{\text{cf } c}(S) \leq \text{cf}(S) \leq \text{sh}(S)$.

(5) $\text{sh}_{\text{cf } c}(S) \leq \text{sh}(S)$, and if $\text{sh}(S)$ is singular, then $\text{sh}_\kappa(S) < \text{sh}(S) \leq \kappa$ for $\kappa < c$,

$\alpha_1 = \alpha_3 = c$, and $c$ is singular.

(6) If $\max\{a_1, a_2, a_3\} = c$, then $\text{add}(s^0) = \text{sh}_{a_3}(S) = \text{sh}_{a_2}(S)$.

(7) If $a_1 = c$, then, for every $\kappa$ with $\omega_1 \leq \kappa < c$, $\text{add}(s^0) = \text{sh}_\kappa(S) = \text{cf}_\kappa(S)$. 


(8) If $a_2 = c$, then $\text{add}(s^0) = sh_{\omega_1}(S)$.
(9) If $a_3 = c$, then $\text{add}(s^0) = \text{sh}(S)$.
(10) If $a(c, f, S) = c$, then $\text{sh}(S) = \text{cf}_\pi(S) = \text{sh}_{cf, c}(S)$.

In particular, if $\varnothing = c$, then the assumptions of (6)–(10) are satisfied, and if $c$ is regular, then the assumption of (10) is satisfied.

Here is the picture of the inequalities between the cardinals:

\[ \text{sh}_{\omega_1}(S) = \text{sh}_{\alpha}(S) \rightarrow \text{sh}_{\alpha_3}(S) \rightarrow \text{add}(s^0) \rightarrow \text{sh}_{\alpha_2}(S) \rightarrow \text{sh}_{\alpha_1}(S) = \text{sh}(S) \rightarrow \text{sh}_{cf, c}(S) \rightarrow \text{cf}_\pi(S) \]

Proof: (1) $\text{sh}_{\alpha_2}(S) \leq \text{sh}(S)$ because $\alpha_2 \leq c$, $\text{sh}(S) \leq \text{cf} c$ by Theorem 1.1(8). We shall sketch a proof of the inequality $\text{sh}(S) \leq b$ which a little simplifies the proof presented in [11]. Let us recall some notation.

For $p \in S$ let $f_p \in \omega \omega$ be such that for every $n$ and every $s \in f_p(n)2$ there is a splitting node $t \in \lfloor f_p(n+1)2 \rfloor$ above $s$ in $p$. For $p \in S$ and $a \subseteq \omega$, $p[a]$ is a subtree of $p$ defined by induction: (i) $\emptyset \in p[a]$; (ii) Let $s \in p[a]$ and $\text{dom} s = n$. If $n \in a$, then, for $i = 0, 1, s \searrow i \in p[a]$ if and only if $s \searrow i \in p$. If $n \notin a$, then, for $i = 0, 1, s \searrow i \in p[a]$ if and only if $i = 0$ and $s \searrow 0 \in p$ or $i = 1$ and $s \searrow 0 \notin p$.

If $p, q \in S$ and $a, b \subseteq \omega$, then $p[a] \cap q[b] = (p \cap q)[a \cap b]$, and if $[f_p(n), f_p(n + 1)] \subseteq a$ for infinitely many $n$, then $p[a] \in S$.

We shall construct a base matrix on $S$ of size $b$ using the fact that $\text{h} \leq b$ where $\text{h}$ is the minimal size of a base matrix on $\mathcal{P}(\omega)/\text{fin}$ (see [2]). Let $\mathcal{F} \subseteq \omega \omega$ be an unbounded family of increasing functions and let $\{B_\alpha : \alpha < \text{h}\}$ be a base matrix on $\mathcal{P}(\omega)/\text{fin}$. If $p \in S$, then there is an $f \in \mathcal{F}$ such that the set $x_p = \{n : \|f(n), f(n + 1)\} \cap \text{rng} f_p \geq 2\}$ is infinite and so there is $\alpha < \text{h}$ and $a \in B_\alpha$ such that $a \subseteq^* x_p$. Now for $f \in \mathcal{F}$ and $a \in \bigcup_{\alpha < \text{h}} B_\alpha$ let $S_{f, a}$ be the set of all $p \in S$ such that $\|f(n), f(n + 1)\} \cap \text{rng} f_p \geq 2\}$ for all but finitely many $n \in a$. As $S_{f, a}$ has size $\leq \text{c}$, we can assign, in a one-to-one way, for each $p \in S_{f, a}$ an infinite set $b_{f, a, p} \subseteq a$ so that the system $\{g_{f, a, p} : p \in S_{f, a}\}$ is almost disjoint. Let $c_{f, a, p} = \bigcup \{f(n), f(n + 1)\} : n \in b_{f, a, p}\}$. Then $\{c_{f, a, p} : a \in B_\alpha \}$ is an almost disjoint family and hence the system $A_{f, a, \alpha} = \{p[c_{f, a, p}] : a \in B_\alpha \}$ is an antichain in $S$ refining $\bigcup_{\alpha \in B_\alpha} S_{f, a}$.

Therefore $\{A_{f, a, \alpha} : f \in \mathcal{F}$ and $\alpha < \text{h}\}$ is a base matrix on $S$.

$\text{sh}_{\alpha_3}(S) \leq \text{add}(s^0)$: Let $\kappa < \text{sh}_{\alpha_3}(S)$ and let $X_\alpha, \alpha < \kappa$, be $s^0$-sets. We prove that the set $X = \bigcup_{\alpha \in \kappa} X_\alpha$ is an $s^0$-set and hence $\kappa < \text{add}(s^0)$. Let $A_\alpha, \alpha < \kappa,$
be maximal antichains in $S$ such that $X_\alpha \cap B_{A_\alpha} = \emptyset$. By Theorem 2.4(1) we can assume that for every $\alpha < \kappa$, $B_{A_\alpha}$ is a disjoint family. Let $q \in S$ be arbitrary. By $(\kappa, c, a_3)$-distributivity of $r.o.(S)$ there is $q' \leq q$ such that for every $\alpha$ the set $A'_\alpha = \{ p \in A_\alpha : q' \land p \neq 0 \}$ has size $< a_3$. By the definition of $a_3$ it follows that every set $Y_\alpha = [q'] \cup B_{A'_\alpha}$ has size $< c$ and as $\alpha < cf \cdot c$ the set $X \cap [q'] \subseteq \bigcup_{\alpha < \kappa} Y_\alpha$ has size $< c$. Therefore there is $r < q'$ such that $X \cap [r] = \emptyset$.

add$(s^0) \leq sh_{\tilde{a}_2}(S)$: Let $\kappa < add(s^0)$ and let $\{A_\alpha : \alpha < \kappa\}$ be a system of maximal antichains in $S$. We prove that for every $q \in S$ there is $r \leq q$ such that for every $\alpha < \kappa$ the set $\{ p \in A_\alpha : r \land p \neq 0 \}$ has size $< a_2$ and hence $\kappa < sh_{\tilde{a}_2}(S)$. By refining the antichains, if necessary, we can assume without loss of generality that they all satisfy the conditions in Theorem 2.4(1). By the additivity assumption, the set $X = \bigcup_{\alpha < \kappa}(\omega^2 \setminus B_{A_\alpha})$ is an $s^0$-set. Let $q \in S$. There is $r \leq q$ such that $X \cap [r] = \emptyset$ and hence for every $\alpha$, $[r] \subseteq B_{A_\alpha}$. By Theorem 2.4(1) then, for every $\alpha$, $C_\alpha = \{ p \in A_\alpha : [r] \cap [p] \neq \emptyset \}$ has size $< c$ and by the definition of $\tilde{a}_2$ we have $|C_\alpha| < \tilde{a}_2$.

(2) We prove only $\min\{sh_{\tilde{a}_2}(S), add(s^0)\} \leq sh_{\omega_1}(S)$; all the remaining inequalities of this part of the theorem hold due to the monotonicity of the invariants $sh_\kappa(S)$ and part (1).

Let $\kappa < \min\{sh_{\tilde{a}_2}(S), add(s^0)\}$ and let $A_\alpha, \alpha < \kappa$, be maximal antichains in $S$. We show that for every $q \in S$ there is $r \leq q$ such that for every $\alpha < \kappa$ the set $\{ p \in A_\alpha : r \land p \neq 0 \}$ is countable. Without loss of generality we can assume that all the antichains $A_\alpha$ satisfy conditions in Theorem 2.4(2). Given $q \in S$ by the $\kappa$-additivity of $s^0$ and $(\kappa, c, a_2)$-distributivity of $r.o.(S)$ there is $q' \leq q$ such that for each $\alpha < \kappa$, $[q'] \subseteq \bigcup B_{A_\alpha}$ and the set $\{ p \in A_\alpha : q' \land p \neq 0 \}$ has size $< a_2$. By condition $(c)$ in Theorem 2.4(2), as $\kappa < cf \cdot c$, the set $X = \bigcup_{\alpha < \kappa} \bigcup [q'] \cap [p] : p \in A_\alpha$ and $q' \land p = 0$ has size $< c$. Let $r \leq q'$ be such that $X \cap [r] = \emptyset$. Then for each $\alpha < \kappa$ the set $\{ p \in A_\alpha : [r] \cap [p] \neq \emptyset \}$ has size $< a_2$ and therefore it is countable.

(3) It is clear that $sh_{\tilde{a}_2}(S) \leq sh(S) = sh_{\tilde{a}_1}(S)$. Let $\kappa_1 = sh_{\tilde{a}_2}(S)$ and $\kappa_2 = add(s^0)$. We prove that $\max\{\kappa_1, \kappa_2\} = sh(S)$. We know that the inequality $\leq$ holds true. Let us assume that $\kappa_1, \kappa_2 \in sh(S)$ and we prove a contradiction. Let $\{A'_\alpha : \alpha < \kappa_1\}$ be a system of maximal antichains in $S$ witnessing the $(\kappa, c, \tilde{a}_3)$-nowhere distributivity of $r.o.(S)$ and let $X_\beta : \beta < \kappa_2$ be a system of $s^0$-sets such that for every $q \in S$, $[q] \cap \bigcup_{\beta < \kappa} X_\beta$ has size $c$. For each pair $(\alpha, \beta) \in \kappa_1 \times \kappa_2$ let $A_{\alpha, \beta}$ be a maximal antichain in $S$ such that $A_{\alpha, \beta}$ refines $A'_\alpha$ and $X_\beta \cap \bigcup B_{A_{\alpha, \beta}} = \emptyset$. We can find $A_{\alpha, \beta}$'s so that the conditions in Theorem 2.4(2) are satisfied. We claim that the system $\{A_{\alpha, \beta} : (\alpha, \beta) \in \kappa_1 \times \kappa_2\}$ is a witness for the $(\kappa_1 \times \kappa_2, c, c)$-nowhere distributivity of $r.o.(S)$ which contradicts the inequality $\kappa_1 \cdot \kappa_2 < sh(S)$. To see this let $q \in S$ be arbitrary. As $\kappa_1 \cdot \kappa_2 < sh(S)$ there is $r \leq q$ such that for every $(\alpha, \beta) \in \kappa_1 \times \kappa_2$ the set $A'_{\alpha, \beta} = \{ p \in A_{\alpha, \beta} : r \land p = 0 \}$ has size $< c$. As $[r] \cap \bigcup_{\beta < \kappa_2} X_\beta$ has size $c$ and $\kappa_2 < cf \cdot c$ there is $\beta < \kappa_2$ such that $[r] \cap X_\beta$ has size $c$. As for every $\alpha$ the antichain $A_{\alpha, \beta}$ refines the antichain $A'_\alpha$, there is $\alpha < \kappa_1$.
such that $|A'_{\alpha,\beta}| \geq \tilde{a}_3$. Now $[r] \cap X_\beta$ is disjoint from $\bigcup B_{A'_{\alpha,\beta}}$ and $|A'_{\alpha,\beta}| < c$. It follows that $\tilde{a}_3 \geq |A'_{\alpha,\beta}|^+$ while $|A'_{\alpha,\beta}| \geq \tilde{a}_3$. A contradiction.

(4) The inequalities hold true by Theorem 1.2(6) because $\text{sh}_{\omega_1}(S) \leq \text{sh}_{\text{cf}}(S) \leq \text{sh}(S) \leq cf c$.

(5) The inequalities hold true by Theorem 1.2(8) by which $\text{sh}_\kappa(S)$ is regular for $\kappa$ regular. Hence if $\text{sh}(S)$ is singular, then $c$ is singular, and as $\text{add}(s^0)$ is regular, by (3), $\text{sh}_{\tilde{a}_3}(S) = \text{sh}_{\tilde{a}_1}(S) = \text{sh}(S)$. Therefore, $\tilde{a}_1 = \tilde{a}_3 = c$.

(6)–(9) are easy consequences of the above proved inequalities using the fact that $a_i = c$ if and only if $\tilde{a}_i = \omega_1$.

(10) follows by (4) since under the assumption $\text{sh}(S) = \text{sh}_{\text{cf}}(S)$. □

By Theorem 3.1(10), if the continuum is regular, then it is collapsed to a regular cardinal of the extension. MA(countable) does not imply the continuum is regular. Anyway, by Theorem 3.1(7), under MA(countable) (even under $d = c$) Sacks forcing collapses the continuum to a regular cardinal in $\mathcal{V}_{r.0}(S)$. We think that it is an open question whether Sacks forcing can collapse the continuum to a singular cardinal.

Under some hypotheses (see Theorem 3.1), there is $\kappa \leq c$ such that $\text{add}(s^0) = \text{sh}_\kappa(S)$. We do not know whether the same is true in ZFC.

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