Yan-Kui Song
Closed subsets of absolutely star-Lindelöf spaces II


Persistent URL: http://dml.cz/dmlcz/119389

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 2003

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://dml.cz
Closed subsets of absolutely star-Lindelöf spaces II

Yan-Kui Song

Abstract. In this paper, we prove the following two statements: (1) There exists a discretely absolutely star-Lindelöf Tychonoff space having a regular-closed subspace which is not CCC-Lindelöf. (2) Every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented in a Hausdorff (regular, Tychonoff) absolutely star-Lindelöf space as a closed $G_δ$ subspace.

Keywords: star-Lindelöf space, absolutely star-Lindelöf space

Classification: 54D20, 54B10, 54D55

1. Introduction

By a space, we mean a topological space. A space $X$ is absolutely star-Lindelöf (see [1]) (discretely absolutely star-Lindelöf (see [8])) if for every open cover $U$ of $X$ and every dense subspace $D$ of $X$, there exists a countable subset $F \subseteq D$ such that $\text{St}(F, U) = X$ (respectively, $F$ is discrete and closed in $X$ and $\text{St}(F, U) = X$), where $\text{St}(F, U) = \bigcup \{U \in U : U \cap F \neq \emptyset\}$.

A space $X$ is star-Lindelöf (see [4], [8] – under different name) if for every open cover $U$ of $X$, there exists a countable subset $F$ of $X$ such that $\text{St}(F, U) = X$. It is clear that very separable space is star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A space $X$ is centered-Lindelöf (linked-Lindelöf, CCC-Lindelöf) (see [2], [3]) if every open cover has a $\sigma$-centered ($\sigma$-linked, CCC, respectively) subcover. A family of sets is centered (linked) if every finite subfamily (every two elements, respectively) has non-empty intersection and a family is $\sigma$-centered ($\sigma$-linked) if it can be represented as the union of countably many centered-subfamilies (linked-subfamilies, respectively). A family of nonempty sets is a CCC-family if there is no uncountable pairwise disjoint subfamily.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf, every absolutely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf, every centered-Lindelöf space is linked-Lindelöf and every linked-Lindelöf space is CCC-Lindelöf.

The author is supported by NSFC project 10271056.
Bonanzinga and Matveev [2] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. They asked if every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed \( G_\delta \) subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. Song [8] gave a positive answer to their question. Moreover, Song and Shi [9] showed that every centered-Lindelöf Tychonoff space can be represented as a closed \( G_\delta \) subspace in a Tychonoff absolutely star-Lindelöf space. But their construction does not work in the classes of Hausdorff and regular spaces. Thus, it is natural for us to consider the following more general question:

**Question.** Is it true that every Tychonoff (Hausdorff, regular) star-Lindelöf (centered-Lindelöf, linkered-Lindelöf) space can be embedded into some Tychonoff (Hausdorff, regular, respectively) absolutely star-Lindelöf space as a closed subspace? And can it be embedded as a closed \( G_\delta \) subspace?

Throughout the paper, the cardinality of a set \( A \) is denoted by \(|A|\). For a cardinal \( \kappa \), \( \kappa^+ \) denotes the smallest cardinal greater than \( \kappa \). Let \( \varepsilon \) denote the cardinality of the continuum and \( \omega \) the first infinite cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols will be used as in [5].

### 2. Closed subspaces of absolutely star-Lindelöf spaces

In [1], Bonanzinga showed that a regular-closed subspace of a star-Lindelöf space need not be star-Lindelöf, and in [9], Song and Shi showed that a regular-closed subspace of an absolutely star-Lindelöf space need not be absolutely star-Lindelöf. In the following, we give a stronger example to show that a regular-closed subspace of a discretely absolutely star-Lindelöf space need not be CCC-Lindelöf.

Recall that the Alexandroff duplicate of a space \( X \), denoted by \( A(X) \), is constructed in the following way: the underlying set of \( A(X) \) is \( X \times \{0, 1\} \) and each point of \( X \times \{1\} \) is isolated; a basic neighborhood of a point \( \langle x, 0 \rangle \in X \times \{0\} \) is a set of the from \( (U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\}) \), where \( U \) is a neighborhood of \( x \) in \( X \). It is well-known that \( A(X) \) is Hausdorff (regular, Tychonoff, normal) iff so is \( X \) and \( A(X) \) is compact iff so is \( X \).

Recall from [6] that a space \( X \) is \emph{absolutely countably compact} if for every open cover \( \mathcal{U} \) of \( X \) and every dense subspace \( D \) of \( X \), there exists a finite subset \( F \subseteq D \) such that \( \text{St}(F, \mathcal{U}) = X \). Vaughan [10] proved that every countably compact GO-space is absolutely countably compact. Thus, every cardinal with uncountable cofinality is absolutely countably compact. In the next example we use the following lemma from [11].

**Lemma 2.1.** If \( X \) is countably compact, then \( A(X) \) is acc.
Given a Tychonoff space $X$, let $\beta X$ denote the Čech-Stone compactification of $X$.

**Example 2.2.** There exists a discretely absolutely star-Lindelöf Tychonoff space $X$ having a regular-closed subspace which is not CCC-Lindelöf.

**Proof:** Let $R$ be a maximal almost disjoint family of infinite subsets of $\omega$ with $|R| = \mathfrak{c}$. Define $S_1 = (\mathfrak{c}^+ \times \omega) \cup R$. We topologize $X$ as follows: $\mathfrak{c}^+ \times \omega$ has the usual product topology and is an open subspace of $S_1$, and a basic neighborhood of $r \in R$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}^+\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}^+$ and a finite subset $K$ of $\omega$. Then, the space $S_1$ is Tychonoff and $e(S_1) = \mathfrak{c}$, because $R$ is discrete and closed in $X$. Now, we show that $S_1$ is discretely absolutely star-Lindelöf. For this end, let $U$ be an open cover of $S_1$. Let $S$ be the set of all isolated points of $\mathfrak{c}^+$ and let $T = S \times \omega$. Then, $T$ is dense in $X$ and every dense subspace of $X$ includes $T$. Thus, it suffices to show that there exists a countable subset $F \subseteq T$ such that $F$ is discrete closed in $X$ and $\text{St}(F, U) = S_1$. For each $n < \omega$, since $\mathfrak{c}^+ \times \{n\}$ is absolutely countably compact, there exists a finite subset $F_n \subseteq S \times \{n\}$ such that $\mathfrak{c}^+ \times \{n\} \subseteq \text{St}(F_n, U)$. Let $F' = \bigcup\{F_n : n \in \omega\}$. Then, $\mathfrak{c}^+ \times \omega \subseteq \text{St}(F', U)$. For each $x \in R$, take $U_x \in U$ with $x \in U_x$, and fix $\alpha_x < \mathfrak{c}^+$ and $n_x \in \omega$ such that $\{\langle n_x, \alpha_x \rangle : \alpha_x < \alpha < \mathfrak{c}^+\} \subseteq U_x$. For each $n \in \omega$, let $X_n = \{x \in R : n_x = n\}$ and choose $\beta_n \in S$ with $\beta_n > \sup\{\alpha_x : x \in X_n\}$. Then, $X_n \subseteq \text{St}(\langle \beta_n, n\rangle, U)$. It is quicker to choose $\beta \in S$ such that $\beta_n < \beta$ for all $n \in \omega$. Thus, if we put $F'' = \{\langle \beta, n\rangle : n \in \omega\}$, then $R \subseteq \text{St}(F'', U)$. Let $F = F' \cup F''$. Then, $F$ is a countable subset of $D$ such that $S_1 = \text{St}(F, U)$. Since $F \cap (\mathfrak{c}^+ \times \{n\})$ is finite for each $n < \omega$, $F$ is discrete and closed in $S_1$, which shows that $S_1$ is discretely absolutely star-Lindelöf.

Let $D$ be a discrete space of cardinality $\mathfrak{c}$ and let

$$S_2 = A((\beta D \times (\mathfrak{c}^+ + 1))) \setminus ((\beta D \setminus D) \times \{\mathfrak{c}^+\}) \setminus ((D \times \{\mathfrak{c}^+\}) \times \{1\}).$$

We show that $S_2$ is not CCC-Lindelöf. Since $|D| = \mathfrak{c}$, we can enumerate $D$ as $\{d_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{(d_\alpha) \times [0, \mathfrak{c}^+]\} \setminus \{\langle d_\alpha, \mathfrak{c}^+\rangle, 0\}.$$

Let us consider the open cover

$$U = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{[0, \mathfrak{c}] \times \beta D\} \times \{0, 1\}$$

of $S_2$. This contains an uncountable pairwise disjoint subfamily $\{U_\alpha : \alpha < \mathfrak{c}\}$ and since $U_\alpha$ is the only element of $U$ containing $\langle \mathfrak{c}^+, d_\alpha \rangle, 0\rangle$, $S_2$ is not CCC-Lindelöf.
We assume that $S_1 \cap S_2 = \emptyset$. Let $\varphi : \mathcal{R} \to \{c^+\} \times D \times \{0\}$ be a bijection. Let $X$ be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $r$ with $\varphi(r)$ for each $r \in \mathcal{R}$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map. It is easy to check that $\pi(S_2)$ is a regular-closed subset of $X$, however, it is not CCC-Lindelöf, since it is homeomorphic to $S_2$.

Now, we show that $X$ is discretely absolutely star-Lindelöf. For this end, let $U$ be an open cover of $X$. Let $S' = \pi((D \times S \times \{0\}) \cup (\beta D \times [0, c^+] \times \{1\})) \cup \pi(T)$, where $S$ is the set of all isolated point of $c^+$. Then, $S'$ is dense in $X$ and every dense subspace of $X$ includes $S'$. Thus, it is suffices to show that there exists a countable subset $F \subseteq S'$ such that $F$ is discrete and closed in $X$ and $St(F, U) = X$. Since $\pi(S_1)$ is homeomorphic to $S_1$, $\pi(S_1)$ is absolutely discretely star-Lindelöf, and there is a countable subset $F_1$ of $S'$ such that $F_1$ is discrete and closed in $\pi(S_1)$ and $\pi(S_1) \subseteq St(F_1, U)$. Since $\pi(S_1)$ is a closed subset of $X$, $F_1$ is discrete and closed in $X$. On the other hand, since $c^+$ is locally compact and countably compact, it follows from [4, Theorem 3.10.13] that $\beta D \times c^+$ is countably compact. Thus $\pi(A(\beta D \times c^+))$ is acc by Lemma 2.1. Hence, there exists a finite subset $F_2$ of $S'$ such that $\pi(A(\beta D \times c^+)) \subseteq St(F_2, U)$.

If we put $F = F_1 \cup F_2$, then $F$ is a countable subset of $S'$ such that $X = St(F, U)$. Since $F_1$ is discrete and closed in $X$ and $F_2$ is finite, $F$ is discrete and closed in $X$, which completes the proof.  


Remark 2. Example 2.2 also shows that regular-closed subspaces of discretely absolutely star-Lindelöf (absolutely star-Lindelöf, star-Lindelöf, centered-Lindelöf, linked-Lindelöf and CCC-Lindelöf) spaces need not be discretely absolutely star-Lindelöf (absolutely star-Lindelöf, star-Lindelöf, centered-Lindelöf and linked-Lindelöf and CCC-Lindelöf, respectively).

Song and Shi [9] proved that every centered-Lindelöf Tychonoff space is representable as a closed $G_\delta$ subspace of a star-Lindelöf Tychonoff space. The following theorem is a generalization of this fact.

**Theorem 2.3.** Every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed $G_\delta$ subspace in a Hausdorff (regular, Tychonoff, respectively) absolutely star-Lindelöf space.

**Proof:** First, we state a construction from [8, Theorem 2.1] which is a minor improvement of [2, Theorem 1]. Let $X$ be a linked-Lindelöf space, $T(X)$ the
topology of $X$ and $\mathcal{L}$ the collection of all linked subfamilies of $\mathcal{T}(X)$, and consider $\mathcal{L}$ as a discrete space. Let $\mathcal{A} = \beta \mathcal{L} \times \omega$ and define $S(X) = X \cup \mathcal{A}$. We topologize $S(X)$ as follows: the subspace $\mathcal{A}$ has the usual product topology and is an open subspace of $S(X)$; and a basic neighborhood of $x \in X$ in $S(X)$ is a set of the form

$$G_{U,n} = U \cup (\text{cl}_{\beta \mathcal{L}} L(U) \times \{m : n < m < \omega\})$$

for an open neighborhood $U$ of $x$ in $X$ and $n < \omega$, where

$$L(U) = \{U \in \mathcal{L} : (\exists V \in U)(V \subseteq U)\}.$$

Then, it was proved in [2, Theorem 1] and [8, Theorem 2.1] that $S(X)$ is Hausdorff (regular, Tychonoff) iff so is $X$ and $S(X)$ is star-Lindelöf if $X$ is linked-Lindelöf.

Now, we modify the above construction. Let

$$\mathcal{R}(S(X)) = A(S(X)) \setminus (X \times \{1\}).$$

Let $X_\omega = X \times \{0\}$ and $X_n = (\beta \mathcal{L} \times \{n\}) \times \{0, 1\}$ for each $n < \omega$. Then

$$\mathcal{R}(S(X)) = X_\omega \cup \bigcup_{n<\omega} X_n.$$

Then, $X$ can be represented as $\mathcal{R}(S(X))$ as a closed-$G_\delta$ subspace, since $X$ is homeomorphic to $X_\omega$ and $A(S(X))$ is Hausdorff (regular, Tychonoff) if $X$ is Hausdorff (regular, Tychonoff). Thus it suffices to show that $A(S(X))$ is absolutely star-Lindelöf. Let $\mathcal{U}$ be an open cover of $\mathcal{R}(S(X))$. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open sets. Let

$$D_n = (\mathcal{L} \times \{n\}) \times \{0\}) \cup (\mathcal{L} \times \{n\}) \times \{1\}$$

for each $n < \omega$ and let

$$D = \bigcup_{n<\omega} D_n.$$

Then, every dense subspace of $\mathcal{R}(S(X))$ includes $D$. Thus, it suffices to show that there exists a countable subset $F \subseteq D$ such that $\text{St}(F, \mathcal{U}) = \mathcal{R}(S(X))$. Let $\mathcal{U}_X = \{U \cap X_\omega : U \in \mathcal{U}\}$. Then, $\mathcal{U}_X$ is an open cover of $X_\omega$. Since $X_\omega$ is homeomorphic to $X$ and $X$ is linked-Lindelöf, $\mathcal{U}_X$ has a $\sigma$-linked open refinement $\bigcup\{\mathcal{U}_m : m < \omega\}$. Let

$$F' = \{(\mathcal{U}_m, n, i) : m < \omega, n < \omega, i = 0, 1\} \subseteq D.$$

To show that $X_\omega \subseteq \text{St}(F', \mathcal{U})$, let $x \in X$ be fixed. Then there exist $m, n < \omega$, $V \in \mathcal{U}_m$ and $U \in \mathcal{T}(X)$ such that

$$\langle x, 0 \rangle \in V \subseteq G_{U,n} = (U \times \{0\}) \cup (\text{cl}_{\beta \mathcal{L}} L(U) \times \{m : n < m < \omega\} \times \{0, 1\}) \in \mathcal{U}. $$
Since \( \langle U_m, n+1 \rangle, 1 \rangle \in F' \cap G'_{U, n}, \langle x, 0 \rangle \in St(F', U) \). Hence, \( X_\omega \subseteq St(F', U) \). On the other hand, since \( X_n \) is compact for each \( n < \omega \), it is not difficult to find a finite subset \( F_n \subseteq D_n \) such that \( X_n \subseteq St(F_n, U) \). If we put \( F = F' \cup \bigcup_{n<\omega} F_n \), then \( F \) is a countable subset of \( D \) and \( R(S(X)) = St(F, U) \), which completes the proof. \( \square \)

Since every star-Lindelöf space is centered-Lindelöf and every centered-Lindelöf space is linked-Lindelöf, the next corollary follows from Theorem 2.3.

**Corollary 2.4** (Song and Shi [9]). Every Hausdorff (regular, Tychonoff) centered-Lindelöf (star-Lindelöf) space can be embedded in a Hausdorff (regular, Tychonoff, respectively) absolutely star-Lindelöf space as a closed-\( G_\delta \) subspace.

**Acknowledgments.** This paper was written while the author was studying at Nanjing University as a post-doctor. He would like to use the opportunity to express his gratitude to Prof. W.-X. Shi for his valuable suggestions. He is also most grateful to the referee for his helpful comments.

**References**


Department of Mathematics, Nanjing Normal University, Nanjing, 210097, P.R. China

and

Department of Mathematics, Nanjing University, Nanjing, 210093, P.R. China

E-mail: songyankui@email.njnu.edu.cn

(Received October 8, 2002, revised January 13, 2003)