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Commentationes Mathematicae Universitatis Carolinae, Vol. 44 (2003), No. 2, 347--358

Persistent URL: <http://dml.cz/dmlcz/119391>

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Mittag-Leffler type expansions of $\bar{\partial}$ -closed (0, n - 1)-forms in certain domains in \mathbb{C}^n

TELEMACHOS HATZIAFRATIS

Abstract. In this paper we will prove a Mittag-Leffler type theorem for $\bar{\partial}$ -closed (0, n - 1)-forms in \mathbb{C}^n by addressing the question of constructing such differential forms with prescribed periods in certain domains.

Keywords: Mittag-Leffler type expansions, $\bar{\partial}$ -closed forms, Bochner-Martinelli kernel

Classification: 32A25

1. Introduction

Let us recall that given a sequence $c_k, k = 0, 1, 2, \dots$, of complex numbers, there exists a holomorphic function $f(z)$ defined for $z \in \mathbb{C} - \{0\}$ so that

$$\int_{|z|=r} z^k f(z) dz = c_k, \quad k = 0, 1, 2, \dots \quad (r > 0),$$

if and only if

$$\sum_{k=0}^{\infty} |c_k| s^k < \infty, \quad \text{for every } s > 0,$$

and that, moreover, such a function is of the form

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} c_k \frac{1}{z^{k+1}} + \text{ a holomorphic function in } \mathbb{C}.$$

More generally, if $D \subset \mathbb{C}$ is an open set, $A = \{\alpha^l : l = 1, 2, 3, \dots\}$ is a discrete subset of D and if for each l we are given a sequence c_k^l of complex numbers which satisfies the condition

$$\sum_{k=0}^{\infty} |c_k^l| s^k < \infty, \quad \text{for every } s > 0,$$

then there exists $f \in \mathcal{O}(D - A)$ so that

$$\int_{|z-\alpha^l|=r_l} (z - \alpha^l)^k f(z) dz = c_k^l, \quad k = 0, 1, 2, \dots, \quad l = 1, 2, 3, \dots,$$

where $r_l > 0$ are sufficiently small. And, moreover, such a function f is unique up to a holomorphic function in D .

In \mathbb{C}^n , we may consider systems (f_1, \dots, f_n) of C^∞ functions, which satisfy the differential equation

$$\sum_{j=1}^n (-1)^{j-1} \frac{\partial f_j}{\partial \bar{z}_j} = 0.$$

This means that the $(0, n - 1)$ -form

$$\theta = \sum_{j=1}^n f_j d\bar{z}_1 \wedge \dots \wedge (j) \dots \wedge d\bar{z}_n$$

is $\bar{\partial}$ -closed, and therefore

$$d[\theta(z) \wedge \omega(z)] = 0,$$

where $\omega(z) = dz_1 \wedge \dots \wedge dz_n$.

By Stokes's theorem, this implies that

$$\int_{\mathcal{S}_1} \theta(z) \wedge \omega(z) = \int_{\mathcal{S}_2} \theta(z) \wedge \omega(z),$$

where \mathcal{S}_1 and \mathcal{S}_2 are $(2n - 1)$ -dimensional closed surfaces, homotopic in the domain where θ is defined and $\bar{\partial}$ -closed.

Thus such $\bar{\partial}$ -closed $(0, n - 1)$ -forms play, in certain cases, roles of holomorphic functions.

Also, again by Stokes's theorem,

$$\int_{\mathcal{S}} \theta(z) \wedge \omega(z) = 0,$$

if the $(0, n - 1)$ -form θ is $\bar{\partial}$ -exact in a neighborhood of the $(2n - 1)$ -dimensional closed surface \mathcal{S} . Thus the $\bar{\partial}$ -exact $(0, n - 1)$ -forms are, in a sense, negligible, at least as far as their periods are concerned.

As for the notation, we will denote by $Z_{\bar{\partial}}^{(0,q)}(D)$ the set of $\bar{\partial}$ -closed $(0, q)$ -forms with C^∞ coefficients in an open set D and by $\mathcal{O}(D)$ the set of holomorphic

functions in D . Also we will denote by $B_{\bar{\partial}}^{(0,q)}(D)$ the set of $(0, q)$ -forms which are $\bar{\partial}$ -exact in D and $H_{\bar{\partial}}^{(0,q)}(D) = Z_{\bar{\partial}}^{(0,q)}(D)/B_{\bar{\partial}}^{(0,q)}(D)$.

In this paper we will prove a Mittag-Leffler type theorem for $\bar{\partial}$ -closed $(0, n - 1)$ -forms in \mathbb{C}^n by addressing the question of constructing such differential forms with prescribed periods in certain domains. More precisely we will prove the following theorems.

Theorem 1. *Suppose that for each $k = (k_1, \dots, k_n)$, where k_j are non-negative integers, we are given a complex number $c_k = c_{k_1, \dots, k_n}$. Then there exists $\theta \in Z_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\})$ with*

$$\int_{|z|=r} z_1^{k_1} \dots z_n^{k_n} \theta(z) \wedge \omega(z) = c_{k_1, \dots, k_n}, \quad \text{for every } k$$

(where $r > 0$), if and only if the sequence c_{k_1, \dots, k_n} satisfies the condition

$$(*) \quad \sum_{k_1, \dots, k_n \geq 0} |c_{k_1, \dots, k_n}| s_1^{k_1} \dots s_n^{k_n} < \infty, \quad \text{for every } s_1, \dots, s_n > 0.$$

Theorem 2. *Let D be an open set in \mathbb{C}^n and A a discrete subset of D . Suppose that for each $\alpha \in A$, we are given a sequence $c_k^\alpha \in \mathbb{C}$ which satisfies condition $(*)$. Then there exists $\theta \in Z_{\bar{\partial}}^{(0, n-1)}(D - A)$ so that*

$$(**) \quad \int_{|z-\alpha|=r_\alpha} (z_1 - \alpha_1)^{k_1} \dots (z_n - \alpha_n)^{k_n} \theta(z) \wedge \omega(z) = c_{k_1, \dots, k_n}^\alpha, \quad \text{for every } k \text{ and } \alpha,$$

where $r_\alpha > 0$ are sufficiently small.

If, moreover, D is pseudoconvex, the differential form θ which satisfies $(**)$ is unique up to a $\bar{\partial}$ -exact $(0, n - 1)$ -form in $D - A$.

Before we prove Theorem 1, let us observe that the sequence c_{k_1, \dots, k_n} satisfies condition $(*)$ if and only if

$$(1) \quad \sum_{k_1, \dots, k_n \geq 0} \frac{n(n+1) \dots (n+k_1 + \dots + k_n - 1)}{k_1! \dots k_n!} |c_{k_1, \dots, k_n}| s_1^{k_1} \dots s_n^{k_n} < \infty$$

for every $s_1, \dots, s_n > 0$.

Indeed, first, (1) implies $(*)$ because of the inequalities

$$\frac{n \dots (n+k_1 - 1)}{k_1!} \geq 1, \dots, \frac{(n+k_1 + \dots + k_{n-1}) \dots (n+k_1 + \dots + k_n - 1)}{k_n!} \geq 1.$$

To prove the other direction, let us set $N = n + k_1 + \dots + k_n - 1$ and notice that

$$\begin{aligned}
 & \frac{n(n+1)\cdots(n+k_1+\dots+k_n-1)}{k_1!\dots k_n!} \\
 (2) \quad & \leq \sum_{0 \leq p_1, \dots, p_n \leq N} \frac{N!}{p_1! \dots p_n! (N - p_1 - \dots - p_n)!} \\
 & = (n+1)^N = (n+1)^{n-1} (n+1)^{k_1} \dots (n+1)^{k_n}.
 \end{aligned}$$

This gives that the sum in (1) is

$$\leq (n+1)^{n-1} \sum_{k_1, \dots, k_n \geq 0} |c_{k_1, \dots, k_n}| [(n+1)s_1]^{k_1} \dots [(n+1)s_n]^{k_n}.$$

Therefore (*) implies (1).

2. Proof of Theorem 1

For the one direction, let us consider a sequence c_k of complex numbers which satisfies (*). We will construct a $\theta \in Z_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\})$ so that

$$\int_{|z|=r} z_1^{k_1} \dots z_n^{k_n} \theta(z) \wedge \omega(z) = c_k, \quad \text{for every } k.$$

For $z \neq w$, set

$$M(z, w) = \frac{\beta_n}{|z-w|^{2n}} \sum_{j=1}^n (-1)^{j-1} (\bar{z}_j - \bar{w}_j) d\bar{z}_1 \wedge \dots \wedge (j) \dots d\bar{z}_n,$$

where $\beta_n = (-1)^{n(n-1)/2} (n-1)! / (2\pi i)^n$, and, as in [1], for each $k = (k_1, \dots, k_n)$, define

$$\begin{aligned}
 \eta_k(z) &= \left. \frac{\partial^{k_1+\dots+k_n} M(z, w)}{\partial w_1^{k_1} \dots \partial w_n^{k_n}} \right|_{w=0} \\
 &= \beta_n n(n+1)\cdots(n+k_1+\dots+k_n-1) \frac{\bar{z}_1^{k_1} \dots \bar{z}_n^{k_n}}{|z|^{2(n+k_1+\dots+k_n)}} \times \\
 & \quad \times \sum_{j=1}^n (-1)^{j-1} \bar{z}_j d\bar{z}_1 \wedge \dots \wedge (j) \dots \wedge d\bar{z}_n.
 \end{aligned}$$

Then $\eta_k \in Z_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\})$. Also, by the Bochner-Martinelli formula, for $f \in \mathcal{O}(\mathbb{C}^n)$,

$$(3) \quad \int_{|z|=r} f(z)\eta_{k_1, \dots, k_n}(z) \wedge \omega(z) = \left. \frac{\partial^{k_1 + \dots + k_n} f(w)}{\partial w_1^{k_1} \dots \partial w_n^{k_n}} \right|_{w=0}.$$

But the sequence c_k satisfies (1), since as we pointed out, (*) implies (1). Writing the factor $\frac{\bar{z}_1^{k_1} \dots \bar{z}_n^{k_n}}{|z|^{2(n+k_1+\dots+k_n)}}$ of $\eta_{k_1, \dots, k_n}(z)$ in the form

$$\frac{1}{|z|^{2n}} \left(\frac{\bar{z}_1}{|z|^2} \right)^{k_1} \dots \left(\frac{\bar{z}_n}{|z|^2} \right)^{k_n},$$

we see that (1) implies that the series

$$\theta(z) = \sum_{k_1, \dots, k_n \geq 0} \frac{c_{k_1, \dots, k_n}}{k_1! \dots k_n!} \eta_{k_1, \dots, k_n}(z)$$

converges and defines a $\bar{\partial}$ -closed $(0, n - 1)$ -form with C^∞ coefficients in $\mathbb{C}^n - \{0\}$. Also (1) implies that

$$\begin{aligned} \int_{|z|=r} z_1^{p_1} \dots z_n^{p_n} \theta(z) \wedge \omega(z) \\ = \sum_{k_1, \dots, k_n \geq 0} \frac{c_{k_1, \dots, k_n}}{k_1! \dots k_n!} \int_{|z|=r} z_1^{p_1} \dots z_n^{p_n} \eta_{k_1, \dots, k_n}(z) \wedge \omega(z). \end{aligned}$$

Applying (3) with $f(z) = z_1^{p_1} \dots z_n^{p_n}$, we find that

$$(4) \quad \int_{|z|=r} z_1^{p_1} \dots z_n^{p_n} \eta_{k_1, \dots, k_n}(z) \wedge \omega(z) = \begin{cases} p_1! \dots p_n! & \text{if } (k_1, \dots, k_n) = (p_1, \dots, p_n) \\ 0 & \text{otherwise.} \end{cases}$$

This gives that

$$\int_{|z|=r} z_1^{p_1} \dots z_n^{p_n} \theta(z) \wedge \omega(z) = c_{p_1, \dots, p_n}$$

and completes the proof of the theorem in this direction.

To prove the other direction of the theorem, we consider (as in [3]) the function $F(\zeta)$ defined by the integral

$$F(\zeta) = \int_{|z|=r} e^{\langle \zeta, z \rangle} \theta(z) \wedge \omega(z), \quad \zeta \in \mathbb{C}^n,$$

where $\langle \zeta, z \rangle = \sum \zeta_j z_j$. It is easy to see that F is an entire holomorphic function and that

$$c_k = \int_{|z|=r} z_1^{k_1} \dots z_n^{k_n} \theta(z) \wedge \omega(z) = \left. \frac{\partial^{k_1+\dots+k_n} F(\zeta)}{\partial \zeta_1^{k_1} \dots \partial \zeta_n^{k_n}} \right|_{\zeta=0}.$$

Since r (in the definition of F) can be arbitrarily small, it follows that the function F satisfies the following: For every $\delta > 0$ there exists a constant $C_\delta > 0$ so that

$$|F(\zeta)| \leq C_\delta e^{\delta|\zeta|} \quad \text{for every } \zeta \in \mathbb{C}^n.$$

Therefore, by Cauchy's inequalities, applied to the entire function $F(\zeta)$, the coefficients c_k satisfy the inequality: For $\delta > 0$,

$$\frac{|c_{k_1, \dots, k_n}|}{k_1! \dots k_n!} \leq C_\delta \frac{e^{\delta(R_1 + \dots + R_n)}}{R_1^{k_1} \dots R_n^{k_n}}, \quad \text{for every } R_1, \dots, R_n > 0.$$

Applying this inequality with $R_1 = k_1/\delta, \dots, R_n = k_n/\delta$ we obtain that for every $\delta > 0$,

$$\frac{|c_{k_1, \dots, k_n}|}{k_1! \dots k_n!} \leq C_\delta \frac{(\delta e)^{k_1 + \dots + k_n}}{k_1^{k_1} \dots k_n^{k_n}}, \quad \text{for all } k_1, \dots, k_n.$$

(In case some $k_j = 0$, the above inequality also holds with the convention $k_j^{k_j} = 1$.) Thus

$$|c_{k_1, \dots, k_n}| \leq C_\delta (\delta e)^{k_1 + \dots + k_n}, \quad \text{for all } k_1, \dots, k_n, \quad \text{and for all } \delta > 0.$$

Therefore

$$\sum_{k_1, \dots, k_n \geq 0} |c_{k_1, \dots, k_n}| s_1^{k_1} \dots s_n^{k_n} \leq C_\delta \sum_{k_1, \dots, k_n \geq 0} (\delta e s_1)^{k_1} \dots (\delta e s_n)^{k_n} < \infty,$$

provided that $\delta < \min\{1/(e s_j) : j = 1, \dots, n\}$. Thus the sequence c_k satisfies (*). The proof of the theorem is now complete.

3. Remark

According to Theorem 1, to each $\theta \in Z_{\delta}^{(0, n-1)}(\mathbb{C}^n - \{0\})$, we may associate an entire function T_θ defined by the formula:

$$T_\theta(\zeta) = \sum_{k_1, \dots, k_n \geq 0} c_{k_1, \dots, k_n} \zeta_1^{k_1} \dots \zeta_n^{k_n}, \quad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n,$$

where

$$c_{k_1, \dots, k_n} = \int_{|z|=r} z_1^{k_1} \dots z_n^{k_n} \theta(z) \wedge \omega(z).$$

Then the transform $T : Z_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\}) \rightarrow \mathcal{O}(\mathbb{C}^n)$, $\theta \rightarrow T\theta$, is linear and its kernel is the set of $\bar{\partial}$ -exact forms, i.e.,

$$\ker T = B_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\}).$$

(This follows from Lemma 2, below).

Thus T induces an isomorphism of linear spaces:

$$\tilde{T} : H_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\}) \rightarrow \mathcal{O}(\mathbb{C}^n), \quad \text{defined by } \tilde{T}([\theta]) = T(\theta),$$

for $[\theta] \in H_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\})$.

In particular we may transfer, in a natural way, the multiplication structure from $\mathcal{O}(\mathbb{C}^n)$ to $H_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\})$:

$$[\theta_1] \cdot [\theta_2] = \tilde{T}^{-1}(T(\theta_1) \cdot T(\theta_2)), \quad \text{for } [\theta_1], [\theta_2] \in H_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{0\}).$$

According to this multiplication,

$$[\eta_{k_1, \dots, k_n}] \cdot [\eta_{p_1, \dots, p_n}] = \frac{k_1! \dots k_n! p_1! \dots p_n!}{(k_1 + p_1)! \dots (p_n + k_n)!} [\eta_{k_1 + p_1, \dots, k_n + p_n}].$$

This follows from (4) and the fact that

$$(\zeta_1^{k_1} \dots \zeta_n^{k_n}) \cdot (\zeta_1^{p_1} \dots \zeta_n^{p_n}) = \zeta_1^{k_1 + p_1} \dots \zeta_n^{k_n + p_n}.$$

For the proof of Theorem 2, we will need the following lemmas. The proof of Lemma 1 is based on a classical idea of a “patching” process, using a partition of unity, and a “correction” process, using a solution of an appropriate differential equation. (The case $n = 1$ is in [4, p. 13]). Lemma 2 is a generalization of a case of [2, Lemma 5].

Lemma 1. *Let D be an open set in \mathbb{C}^n and V_l , $l = 1, 2, 3, \dots$, a sequence of open subsets of D with $D = \bigcup_l V_l$. Suppose that, for each pair $(l, m) \in \mathbb{N} \times \mathbb{N}$, we are given a differential form $\theta_{lm} \in Z_{\bar{\partial}}^{(0, n-1)}(V_l \cap V_m)$ (here we assume that $Z_{\bar{\partial}}^{(0, n-1)}(\emptyset) = \{0\}$) in such a way that*

$$\theta_{lm} + \theta_{mp} + \theta_{pl} = 0 \quad \text{in } V_l \cap V_m \cap V_p, \quad \text{for every } l, m, p \in \mathbb{N}.$$

Then there exist $\theta_l \in Z_{\bar{\partial}}^{(0,n-1)}(V_l)$, $l \in \mathbb{N}$, so that

$$\theta_l - \theta_m = \theta_{lm} \text{ in } V_l \cap V_m, \text{ for every } l, m \in \mathbb{N}.$$

PROOF: First, let us notice that the assumptions, imposed on θ_{lm} , imply that

$$\theta_{ll} = 0 \text{ in } V_l \text{ and } \theta_{lm} + \theta_{ml} = 0 \text{ in } V_l \cap V_m, \text{ for every } l, m \in \mathbb{N}.$$

Then, let us consider a partition of unity subordinate to the cover $\{V_l\}$, i.e., we consider functions $\chi_l \in C^\infty(D)$, with the following properties: $0 \leq \chi_l \leq 1$, $\text{supp}(\chi_l) \subset V_l$, the family $\{\text{supp}(\chi_l) : l \in \mathbb{N}\}$ should be locally finite, and $\sum \chi_l = 1$ in D .

For $l \in \mathbb{N}$, we define the $(0, n - 1)$ -forms

$$\Theta_l = \sum_{m \in \mathbb{N}} \chi_m \theta_{lm}, \text{ with } C^\infty \text{ coefficients in } V_l.$$

Here, the differential form $\chi_m \theta_{lm}$ is defined to be 0 in $V_l - V_m$. Writing the set V_l as the union of the open sets $V_l \cap V_m$ and $V_l - \text{supp}(\chi_m)$, and observing that, according to the above definition of the differential form $\chi_m \theta_{lm}$, $\chi_m \theta_{lm} = 0$ in $V_l - \text{supp}(\chi_m)$, we see that, indeed, the sum $\sum_m \chi_m \theta_{lm}$ has C^∞ coefficients in V_l . A computation shows that

$$\begin{aligned} \Theta_l - \Theta_m &= \sum_{p \in \mathbb{N}} \chi_p \theta_{lp} - \sum_{p \in \mathbb{N}} \chi_p \theta_{mp} = \sum_{p \in \mathbb{N}} \chi_p (\theta_{lp} - \theta_{mp}) \\ &= \sum_{p \in \mathbb{N}} \chi_p \theta_{lm} = \theta_{lm}, \text{ in } V_l \cap V_m. \end{aligned}$$

But $\theta_{lm} \in Z_{\bar{\partial}}^{(0,n-1)}(V_l \cap V_m)$, i.e., $\bar{\partial}\theta_{lm} = 0$, and therefore

$$\bar{\partial}\Theta_l = \bar{\partial}\Theta_m, \text{ in } V_l \cap V_m.$$

Since $H_{\bar{\partial}}^{(0,n)}(D) = 0$, it follows that there exists a $(0, n - 1)$ -form Θ with C^∞ coefficients in D , so that

$$\bar{\partial}\Theta = \bar{\partial}\Theta_l, \text{ in } V_l.$$

Thus if we set $\theta_l = \Theta_l - \Theta$, we obtain differential forms $\theta_l \in Z_{\bar{\partial}}^{(0,n-1)}(V_l)$ which satisfy the equations

$$\theta_l - \theta_m = (\Theta_l - \Theta) - (\Theta_m - \Theta) = \theta_{lm} \text{ in } V_l \cap V_m, \text{ for every } l, m \in \mathbb{N}.$$

This completes the construction of the lemma. □

Lemma 2. *Let D be an open pseudoconvex set in \mathbb{C}^n , A a discrete subset of D and $\theta \in Z_{\bar{\partial}}^{(0, n-1)}(D - A)$. Then the following are equivalent:*

- (I) θ is $\bar{\partial}$ -exact in $D - A$;
- (II) $\int_{|z-\alpha|=r_\alpha} e^{\langle \zeta, z \rangle} \theta(z) \wedge \omega(z) = 0$, for every $\alpha \in A$ and $\zeta \in \mathbb{C}^n$ (where $r_\alpha > 0$ are sufficiently small);
- (III) $\int_{|z-\alpha|=r_\alpha} f(z)\theta(z) \wedge \omega(z) = 0$, for every $\alpha \in A$ and for every $f \in \mathcal{O}(\mathbb{C}^n)$;
- (IV) $\int_{|z-\alpha|=r_\alpha} (z_1 - \alpha_1)^{k_1} \dots (z_n - \alpha_n)^{k_n} \theta(z) \wedge \omega(z) = 0$, for every k and $\alpha \in A$.

PROOF: Since the set of linear combinations of the functions $e^{\langle \zeta, z \rangle}$, $\zeta \in \mathbb{C}^n$, is dense in $\mathcal{O}(\mathbb{C}^n)$, it is clear that (II) \Leftrightarrow (III).

Also, since every entire function can be expanded in a power series with center α , it follows that (III) \Leftrightarrow (IV).

The implication (I) \Rightarrow (III) follows from Stokes's theorem.

Thus it remains to show that (III) \Rightarrow (I). The proof of this is based on the Cauchy-Leray formula and it is similar to the proof of [2, Lemma 5].

First, let us notice that we may find a sequence $G_\nu \subset\subset D$, $\nu = 1, 2, 3, \dots$, of strictly pseudoconvex sets with smooth boundary, which exhaust the pseudoconvex set D , and in such a way that $(\partial G_\nu) \cap A = \emptyset$, for every ν . This is possible, since A is discrete in D . Then each set G_ν will contain finitely many points from the set A . It follows that the set $D - A$ can be exhausted by a sequence of compact sets of the form

$$K = \bar{G} - [B(\alpha^1, \varepsilon^1) \cup \dots \cup B(\alpha^N, \varepsilon^N)],$$

where $G \subset\subset D$ is strictly pseudoconvex with smooth boundary, $\alpha^l \in A$, $\varepsilon^l > 0$, $l = 1, 2, \dots, N$, and the closures of the balls

$$B(\alpha^l, \varepsilon^l) = \{z \in \mathbb{C}^n : |z - \alpha^l| < \varepsilon^l\}$$

are pair-wise disjoint.

Fixing such a set K , we consider the map $\gamma : (\partial K) \times \text{int}(K) \rightarrow \mathbb{C}^n$ as follows: For $(\zeta, z) \in (\partial K) \times \text{int}(K)$, $\{\gamma_j(\zeta, z)\}_{j=1}^n$ is defined to be a Henkin-Ramirez map of the strictly pseudoconvex set G , if $\zeta \in \partial G$, and

$$\gamma_j(\zeta, z) = \frac{\partial \rho_l}{\partial \zeta_j}(z) \quad \text{if } \zeta \in \{\rho_l = 0\},$$

where $\rho_l(\zeta) = |\zeta - \alpha^l|^2 - (\varepsilon^l)^2$.

(For exhaustions of pseudoconvex sets by strictly pseudoconvex domains and constructions of Henkin-Ramirez maps, see [5] and [6].)

Then

$$\sum_{j=1}^n (\zeta_j - z_j) \gamma_j(\zeta, z) \neq 0, \quad \text{for } (\zeta, z) \in (\partial K) \times \text{int}(K),$$

and therefore we may write down the Cauchy-Leray formula:

$$(5) \quad u = \bar{\partial}_z(T_{q-1}u) + T_q(\bar{\partial}u) + L_q^\gamma(u),$$

for $(0, q)$ -forms u in a neighborhood of K

(notation is as in [2, p. 912]).

Now if $\theta \in Z_{\bar{\partial}}^{(0, n-1)}(D-A)$ satisfies (III), it follows as in the proof of [2, Lemma 5] that $L_{n-1}^\gamma(\theta) = 0$, and therefore (5) gives

$$\theta = \bar{\partial}_z(T_{n-2}\theta), \quad \text{in } \text{int}(K).$$

Now the conclusion that θ is $\bar{\partial}$ -exact in $D - A$, follows from [2, Lemma 4], and this completes the proof of the implication (III) \Rightarrow (I).

The proof of the lemma is complete. □

4. Proof of Theorem 2

First, with D being an arbitrary open set in \mathbb{C}^n , we will use Lemma 1 in order to construct a $\theta \in Z_{\bar{\partial}}^{(0, n-1)}(D - A)$ which satisfies (**).

Let $\alpha^l, l = 1, 2, 3, \dots$, be an enumeration of the set A . By Theorem 1, for each $l = 1, 2, 3, \dots$, there exists $\theta_l \in Z_{\bar{\partial}}^{(0, n-1)}(\mathbb{C}^n - \{\alpha^l\})$ so that

$$\int_{|z-\alpha^l|=r_{\alpha^l}} (z_1 - \alpha_1^l)^{k_1} \dots (z_n - \alpha_n^l)^{k_n} \theta_l(z) \wedge \omega(z) = c_{k_1, \dots, k_n}^{\alpha^l}, \quad \text{for every } k.$$

Next we consider an open cover $\{V_0, V_1, V_2, \dots\}$ of D , which is of the form: $V_0 = D - A$, and, for $l \geq 1, V_l$ is a small ball centered at the point α^l , so that $V_l \cap V_m = \emptyset$ for $l \neq m, l, m \geq 1$.

For each pair (l, m) with $l, m \geq 0$, we define $\theta_{lm} \in Z_{\bar{\partial}}^{(0, n-1)}(V_l \cap V_m)$ in the following way:

$$\theta_{00} = 0, \theta_{lm} = 0 \text{ if } l, m \geq 1, \text{ and } \theta_{0l} = -\theta_{l0} = \theta_l$$

in $V_0 \cap V_l = V_l - \{\alpha^l\}$ for $l \geq 1$.

Then

$$\theta_{lm} + \theta_{mp} + \theta_{pl} = 0 \text{ in } V_l \cap V_m \cap V_p, \text{ for every } l, m, p \geq 0.$$

Therefore, from Lemma 1, there exist $\tilde{\theta}_l \in Z_{\bar{\partial}}^{(0, n-1)}(V_l), l \geq 0$, so that

$$\tilde{\theta}_l - \tilde{\theta}_m = \theta_{lm} \text{ in } V_l \cap V_m, \text{ for every } l, m \geq 0.$$

In particular,

$$\tilde{\theta}_0 - \tilde{\theta}_l = \theta_{0l} = \theta_l, \quad \text{in } V_0 \cap V_l = V_l - \{\alpha^l\}, \quad \text{for } l \geq 1.$$

Define $\theta = \tilde{\theta}_0 \in Z_{\bar{\partial}}^{(0, n-1)}(D - A)$. Since $\tilde{\theta}_l \in Z_{\bar{\partial}}^{(0, n-1)}(V_l)$, the $(0, n - 1)$ -form

$$(z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \tilde{\theta}_l(z)$$

is also $\bar{\partial}$ -closed in V_l .

It follows that the differential form $(z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \tilde{\theta}_l(z) \wedge \omega(z)$ is d -closed in V_l , and therefore, by Stokes's theorem,

$$\int_{|z - \alpha^l| = r_{\alpha^l}} (z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \tilde{\theta}_l(z) \wedge \omega(z) = 0, \quad \text{for every } k \quad \text{and } l \geq 1.$$

(Recall that the r_{α} 's are assumed sufficiently small.) Since for each $l \geq 1$,

$$\theta = \theta_l + \tilde{\theta}_l, \quad \text{in } V_l - \{\alpha^l\},$$

it follows that θ satisfies (**).

This completes the proof of the first part of the theorem.

Finally, assume that D is pseudoconvex and that two differential forms $\theta, \hat{\theta} \in Z_{\bar{\partial}}^{(0, n-1)}(D - A)$ satisfy (**). Then their difference $\theta - \hat{\theta}$ satisfies the following equations

$$\int_{|z - \alpha^l| = r_{\alpha^l}} (z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} [\theta(z) - \hat{\theta}(z)] \wedge \omega(z) = 0,$$

for every k and every $\alpha \in A$.

It follows from Lemma 2, that $\theta - \hat{\theta}$ is $\bar{\partial}$ -exact in $D - A$. This completes the proof of the theorem. □

We close with the following remark. In the case D is pseudoconvex, Theorem 2 establishes a bijection from the $\bar{\partial}$ -cohomology group $H_{\bar{\partial}}^{(0, n-1)}(D - A)$ to the set $[\mathcal{O}(\mathbb{C}^n)]^A$ of all maps $A \rightarrow \mathcal{O}(\mathbb{C}^n)$. Thus we have, in a natural way, an isomorphism of linear spaces:

$$H_{\bar{\partial}}^{(0, n-1)}(D - A) \approx [\mathcal{O}(\mathbb{C}^n)]^A.$$

REFERENCES

- [1] Hatziafratis T., *On a class of $\bar{\partial}$ -equations without solutions*, Comment. Math. Univ. Carolinae **39.3** (1998), 503–509.
- [2] Hatziafratis T., *Note on the Fourier-Laplace transform of $\bar{\partial}$ -cohomology classes*, Z. Anal. Anwendungen **17** (1998), 907–915.
- [3] Hatziafratis T., *Expansions of certain $\bar{\partial}$ -closed forms via Fourier-Laplace transform*, preprint.
- [4] Hörmander L., *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam, 1990.
- [5] Krantz S., *Function Theory of Several Complex Variables*, Wadsworth & Brooks/Cole, California, 1992.
- [6] Range R.M., *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer-Verlag, New York, 1986.

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(Received March 8, 2002, revised November 19, 2002)