

Sadoon Ibrahim Othman; Victor Anandam
Biharmonic Green domains in a Riemannian manifold

Commentationes Mathematicae Universitatis Carolinae, Vol. 44 (2003), No. 2, 359--365

Persistent URL: <http://dml.cz/dmlcz/119392>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2003

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

Biharmonic Green domains in a Riemannian manifold

S.I. OTHMAN, V. ANANDAM

Abstract. Let R be a Riemannian manifold without a biharmonic Green function defined on it and Ω a domain in R . A necessary and sufficient condition is given for the existence of a biharmonic Green function on Ω .

Keywords: biharmonic Green functions

Classification: 31C12, 31B30

1. Introduction

In a Riemannian manifold R , we say that a domain Ω is a biharmonic Green domain if there exists a positive solution $Q_y(x)$ for the equation $\Delta^2 Q_y(x) = \delta_y(x)$ in Ω , where y is some point in Ω and Δ is the Laplace-Beltrami operator in R . Some necessary and sufficient conditions for R to be a biharmonic Green space are given in Sario et al. [8, Chapter VIII]. In this note we give a necessary and sufficient condition for a domain Ω in R to be a biharmonic Green domain when R itself is not a biharmonic Green space.

2. Preliminaries

Let R be an oriented Riemannian manifold of dimension $n \geq 2$ with local parameters $x = (x^1, \dots, x^n)$ and a C^∞ metric tensor g_{ij} such that $g_{ij}x^i x^j$ is positive definite. If D is the determinant of g_{ij} , denote the volume element by $dx = D^{\frac{1}{2}} dx^1 \dots dx^n$; $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator acting on R in the sense of distributions; in the Euclidean case, Δ reduces to the form $\Delta u = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. A continuous function h on an open set is harmonic, by definition, if $\Delta h = 0$. To every open set w in R , let $H(w)$ denote the class of harmonic functions on w . Then these harmonic functions have the *sheaf property*, solve locally the *Dirichlet problem* and possess the *Harnack property*; that is, they satisfy the axioms 1, 2, 3 of Brelot in the axiomatic potential theory ([5, pp. 13–14]). Consequently, we can use all the notions and the results of this axiomatic theory in the context of a Riemannian manifold; some of these are the following:

- (1) Let w be a regular open set in R , that is w is relatively compact in R and each boundary point of w is regular for the Dirichlet problem. A compact set k in w is said to be *outerregular* if $w \setminus k$ is a regular open set. Given

a compact set k and a domain w , one can construct a regular domain w_0 and an outerregular compact set k_0 such that $k \subset \overset{\circ}{k_0} \subset k_0 \subset w_0 \subset w$ (see Loeb [6]).

- (2) (See [5, pp. 37, 38 and 47]). If $s > 0$ is a superharmonic function on a domain $\Omega \subset R$ and if e is a subset of Ω , the *reduced function* by definition is

$$R_s^e(x) = \inf\{t(x) : t \geq 0 \text{ superharmonic on } \Omega \text{ and } t \geq s \text{ on } e\};$$

and its l.s.c. regularization is the *balayage* $\widehat{R}_s^e(x) = \liminf_{y \rightarrow x} R_s^e(y)$. In a domain Ω with a positive potential, a set e is *polar* if and only if $R_1^e(x) = 0$ at some point x , or, equivalently $\widehat{R}_1^e \equiv 0$.

- (3) If there is a positive potential on Ω , we define on Ω the Green function $G(x, y) = G_y(x)$ with pole $y \in \Omega$, so that $\Delta G_y = \delta_y$. Then for any potential p on Ω , $\Delta p = \mu$ is a Radon measure and $p(x) = \int_{\Omega} G(x, y) d\mu(y)$. Also it is proved in [7] that given a Radon measure $\mu \geq 0$ on Ω , $\int_{\Omega} G(x, y) d\mu(y)$ is a potential if and only if

$$\int_{\Omega} \widehat{R}_1^w(x) d\mu(x) < \infty \text{ for some nonempty open set } w \text{ in } \Omega.$$

- (4) More generally, we have the following result in [3]: Let Ω be a domain in R with or without positive potentials. Let $\mu \geq 0$ be a Radon measure on Ω . Then there exists a superharmonic function s on Ω such that $\Delta s = \mu$. This result is in fact a simple generalization of a classical result of Brelot [4] in R^n .

Lemma 2.1. *Let Ω be a domain in R such that Ω has the Green function $G(x, y)$ defined on it. Then for a Radon measure $\mu \geq 0$ on Ω , $\int_{\Omega} G(x, y) d\mu(y)$ is a potential if and only if for one (and hence any) nonpolar compact set k in Ω , $\int_{\Omega} R_1^k d\mu < \infty$.*

PROOF: This is a more useful reformulation of Theorem 3.1 [7]. First note that R_1^k is μ -measurable. For $R_1^k = \inf_n R_1^{w_n}$ where w_n is a decreasing sequence of relatively compact open sets such that $k = \bigcap w_n$. Since each $R_1^{w_n} = \widehat{R}_1^{w_n}$ is l.s.c., it is μ -measurable and hence R_1^k is μ -measurable.

- (1) Suppose $\int_{\Omega} G(x, y) d\mu(y)$ is a potential on Ω and k is a nonpolar compact set in Ω . Then for some $x_0 \in k$, $\int_{\Omega} G(x_0, y) d\mu(y) < \infty$. If $G(x_0, y) \geq a$ on k , $G(x_0, y) \geq aR_1^k$ on Ω and hence $\int_{\Omega} R_1^k d\mu < \infty$.
- (2) Conversely, suppose $\int_{\Omega} R_1^k d\mu < \infty$ for some nonpolar compact set k . Since $R_1^k = \inf_n R_1^{w_n}$, we can find an open set w and an outerregular compact set A such that $k \subset \overset{\circ}{A} \subset A \subset w$ and $\int_{\Omega} R_1^w d\mu < \infty$. Now $p(x) =$

$\int_A G(x, y) d\mu(y)$ is a potential on Ω ; hence $p(x_0) < \infty$ for some $x_0 \in k$. If $a \leq G(x_0, y) \leq b$ on ∂A , then $aR_1^k(y) \leq G(x_0, y) \leq bR_1^A(y)$ on $\Omega \setminus A$ and hence $\int_{\Omega \setminus A} G(x_0, y) d\mu(y)$ is finite, which implies that $\int_{\Omega} G(x_0, y) d\mu(y)$ is finite and hence $\int_{\Omega} G(x, y) d\mu(y)$ is a potential on Ω . □

The following form of Lemma 2.1, without an explicit reference to the reduced functions, is convenient for applications.

Lemma 2.2. *Let Ω be a domain in R with the Green function $G(x, y)$ defined on it; $\mu \geq 0$ is a Radon measure on Ω . Then the following are equivalent:*

- (1) *There exists a superharmonic function $s > 0$ on Ω such that $\int_{\Omega} s d\mu < \infty$.*
- (2) *$p(x) = \int_{\Omega} G(x, y) d\mu(y)$ is a potential on Ω .*
- (3) *For any locally bounded potential $q(x)$ with compact harmonic support on Ω , $\int_{\Omega} q d\mu < \infty$.*

PROOF: (1) \Rightarrow (2): Let k be a nonpolar compact subset of Ω . If $s \geq \alpha > 0$ on k , then $\alpha R_1^k \leq s$ on Ω and hence $\int_{\Omega} R_1^k d\mu \leq \frac{1}{\alpha} \int_{\Omega} s d\mu < \infty$. Hence by Lemma 2.1, $p(x) = \int_{\Omega} G(x, y) d\mu(y)$ is a potential on Ω .

(2) \Rightarrow (3): Let q be a locally bounded potential on Ω , with compact harmonic support A . Let k be an outerregular compact set such that $A \subset \overset{\circ}{k}$. Then $R_q^k = q$ on $\Omega \setminus k$. For, $\widehat{R}_q^k \leq q$ on Ω and hence $t = q - \widehat{R}_q^k$ on $\Omega \setminus k$ extended by 0 on k is a positive subharmonic function less or equal to q on Ω ; hence $t \leq 0$, so that $q = \widehat{R}_q^k = R_q^k$ on $\Omega \setminus k$. Consequently, if $q \leq \alpha$ on k , then $q \leq \alpha R_1^k$ on k ; also on $\Omega \setminus k$, $q = R_q^k \leq R_{\alpha}^k = \alpha R_1^k$. Thus $q \leq \alpha R_1^k$ on Ω . Now assumption (2) along with Lemma 2.1 shows that $\int_{\Omega} R_1^k d\mu < \infty$. Hence $\int_{\Omega} q d\mu < \infty$.

(3) \Rightarrow (1): Let k be a nonpolar compact set. Let $s = \widehat{R}_1^k$ on Ω . Then $s > 0$ is a superharmonic function that is bounded on Ω and has compact harmonic support. Hence by (3), $\int_{\Omega} s d\mu < \infty$. □

3. Biharmonic Green domains

Let Ω be a domain in R . Given $y \in \Omega$, let w be a regular domain for the Dirichlet problem such that $y \in w \subset \overline{w} \subset \Omega$. Let $v_w(x, y)$ be the biharmonic Green function on w with biharmonic singularity y , that is $\Delta^2 v_w(x, y) = \delta_y(x)$, and with boundary conditions $v_w/\partial w = 0$ and $\Delta v_w/\partial w = 0$. Then v_w increases with w . Write $v_{\Omega}(x, y) = \lim_{w \rightarrow \Omega} v_w(x, y)$ if the limit exists for some regular exhaustion $\{w\}$. $v_{\Omega}(x, y)$ is called the *biharmonic Green function* on Ω and its existence is independent of the regular exhaustion $\{w\}$ and the choice of the singular point y (see Sairo et al. [8, pp. 300–307]). When $v_{\Omega}(x, y)$ exists on Ω , it can be written as $v_{\Omega}(x, y) = \int_{\Omega} G(x, z)G(z, y) dz$.

Definition 3.1. A domain Ω in R is said to be a *biharmonic Green domain* if and only if the biharmonic Green function $v_\Omega(x, y)$ exists on Ω .

The following theorem is a collection of known results about $v_\Omega(x, y)$.

Theorem 3.2. Let Ω be a domain in R , carrying the harmonic Green function $G(x, y)$. Then the following are equivalent:

- (1) Ω is a biharmonic Green domain.
- (2) For one (and hence any) $y \in \Omega$, there exists a potential $q_y(x)$ on Ω such that $\Delta^2 q_y = \delta_y$.
- (3) There exists a potential $Q(x) > 0$ on Ω such that $\Delta Q(x)$ is a superharmonic function.
- (4) There exist potentials p and q on Ω such that $\Delta q = p$. (q is called a *bipotential*.)

PROOF: (1) \Rightarrow (2): Let $v(x, y)$ be the biharmonic Green function on Ω . Since $v(x, y) = \int_\Omega G(x, z)G(z, y) dz$, for fixed y , $v_y(x) = v(x, y)$ is a potential on Ω and $\Delta v_y(x) = G_y(x)$ (see [8, p. 300]); hence $\Delta^2 v_y = \delta_y$.

(2) \Rightarrow (3): For some potential q on Ω , let $\Delta^2 q = \delta_y$. Since $\Delta^2 q = \Delta G_y$, $\Delta q(x) = G_y(x) +$ (a harmonic function) on Ω . That is, $\Delta q = s$ is a superharmonic function on Ω ; note that $s > 0$ since q is a potential > 0 .

(3) \Rightarrow (4): See Theorem 3.2 in [1].

(4) \Rightarrow (1): This is a consequence of Theorem 4.2 in [1]. □

Theorem 3.3. A domain Ω in R is a biharmonic Green domain if and only if there exists a superharmonic function $s > 0$ on Ω such that $\int_\Omega s^2 dx < \infty$.

PROOF: (1) Let Ω be a biharmonic Green domain. Then there exist potentials $p > 0$ and $q > 0$ on Ω such that $\Delta q = p$. This means that if $G(x, y)$ is the Green function on Ω with $\Delta G_y = \delta_y$, $q(x) = \int_\Omega G(x, y)p(y) dy$ since q is a potential with the associated measure $d\mu(x) = (\Delta q)dx = p dx$ in the Riesz representation. This implies (by Lemma 2.1) that for any nonpolar compact set k in Ω , $\int_\Omega R_1^k(y)p(y) dy < \infty$. Moreover, since p is a potential on Ω , for some $\lambda > 0$, $R_1^k \leq \lambda p$ on Ω . Consequently, with $s = \widehat{R}_1^k$ we have $\int_\Omega s^2 dx < \infty$.

(2) Conversely, let $s > 0$ be superharmonic on Ω such that $\int_\Omega s^2 dx < \infty$. Since for a nonpolar compact k in Ω , $R_1^k \leq \lambda s$ for some $\lambda > 0$, $\int_\Omega R_1^k(y)\widehat{R}_1^k(y) dy < \infty$. This implies (Lemma 2.1) that $q(x) = \int_\Omega G(x, y)\widehat{R}_1^k(y) dy$ is a potential on Ω so that $\Delta q = \widehat{R}_1^k$. Since \widehat{R}_1^k is a potential on Ω , we conclude that Ω is a biharmonic Green domain. □

Corollary 1. Any domain in \mathbb{R}^n , $n \geq 5$, is a biharmonic Green domain; and \mathbb{R}^n for $2 \leq n \leq 4$ is not a biharmonic Green space. (Sario et al. [8, pp. 300–302] and [2, Theorem 5.5]).

PROOF: (a) Let Ω be a domain \mathbb{R}^n , $n \geq 5$. Note that $s(x) = |x|^{2-n}$ is a positive superharmonic function such that $\int_{\Omega} s^2 dx \leq \infty$. Hence Ω is a biharmonic Green domain.

(b) Suppose \mathbb{R}^n , $2 \leq n \leq 4$, is a biharmonic Green space. Then there exists a superharmonic function $s > 0$ in \mathbb{R}^n such that $\int_{\mathbb{R}^n} s^2 dx < \infty$. If B is the closed unit ball in \mathbb{R}^n , then for some $\lambda > 0$, $R_1^B \leq \lambda s$ and hence $\int_{\mathbb{R}^n} (R_1^B)^2 dx < \infty$. But $R_1^B = |x|^{2-n}$ on $\mathbb{R}^n \setminus B$. Hence we should have $\int_1^\infty \int_{\partial B} r^{4-2n} r^{n-1} dr dw$ is finite, that is, $\int_1^\infty r^{3-n} dr$ is finite, a contradiction when $2 \leq n \leq 4$. □

Corollary 2. *Suppose the Riemannian manifold R is not a biharmonic Green space. If Ω is a biharmonic Green domain in R , then $e = R \setminus \Omega$ is not compact.*

PROOF: Suppose e is compact. Let k be an outerregular compact set such that $e \subset \overset{\circ}{k} \subset k$. Since Ω is a biharmonic Green domain there exists $s > 0$ superharmonic on Ω such that $\int_{\Omega} s^2 dx < \infty$. Suppose $\inf_{\partial k} s(x) = \lambda$. Then $\lambda \widehat{R}_1^k \leq s$ on $\Omega \setminus k = R \setminus k$ and hence $\int_{\Omega \setminus k} (\widehat{R}_1^k)^2 dx < \infty$; also $\int_k (\widehat{R}_1^k)^2 dx < \infty$, and hence $\int_R (\widehat{R}_1^k)^2 dx < \infty$. This means that R is a biharmonic Green space, contradicting the hypothesis. □

Corollary 3 ([2, Theorem 5.4]). *If Ω is a biharmonic Green domain in \mathbb{R}^n , $2 \leq n \leq 4$, then $e = \mathbb{R}^n \setminus \Omega$ is neither locally polar nor compact.*

PROOF: (a) Since \mathbb{R}^n , $2 \leq n \leq 4$, is not a biharmonic Green space, by the above corollary, e is not compact.

(b) Suppose e is locally polar. Since Ω is a biharmonic Green domain, there exists a superharmonic function $s > 0$ such that $\int_{\Omega} s^2 dx < \infty$. Now $e = \mathbb{R}^n \setminus \Omega$ being locally polar by the assumption, $\int_e dx = 0$ and s extends as a superharmonic function $u > 0$ on \mathbb{R}^n . Hence $\int_{\mathbb{R}^n} u^2 dx < \infty$ which means that \mathbb{R}^n , $2 \leq n \leq 4$, is a biharmonic Green space, a contradiction. □

4. Biharmonic potentials and quasiharmonic potentials

If there exists a nonconstant positive harmonic function on Ω , then we can define the harmonic Green function $G(x, y)$ on Ω . However, we know that this sufficient condition for the existence of the harmonic Green function is not a necessary condition, as for example in \mathbb{R}^n , $n \geq 3$. A corresponding result for the biharmonic Green function is the following:

Proposition 4.1. *Suppose that there exists a biharmonic function which is a positive potential on Ω . Then Ω is a biharmonic Green domain.*

PROOF: Let b be a biharmonic function which is a positive potential on Ω . Since b is a potential such that Δb is harmonic, by Theorem 3.2(3), Ω is a biharmonic Green domain. □

In view of the above proposition, we propose the following terminology.

Definition 4.2. In a domain Ω in R , let $u > 0$ be a potential.

- (1) u is said to be a *biharmonic potential* if and only if $\Delta^2 u = 0$ on Ω .
- (2) u is said to be a *quasiharmonic potential* if and only if $\Delta u = 1$ on Ω .

Remark. Let Ω be a harmonic Green domain in R . Then there exists a quasiharmonic potential on Ω if and only if $p(x) = \int_{\Omega} G(x, y) dy$ is a potential on Ω . For, suppose $p(x)$ is a potential. Then $\Delta p = 1$ so that $p(x)$ is a quasiharmonic potential on Ω . Conversely, suppose q is a quasiharmonic potential on Ω . Since q is a potential and $\Delta q = 1$, $q(x) = \int_{\Omega} G(x, y) \Delta q(y) dy = \int_{\Omega} G(x, y) dy$.

Theorem 4.3. Let Ω be a harmonic Green domain in R . Then there exists a biharmonic (resp. quasiharmonic) potential on Ω if and only if there are a superharmonic function $s > 0$ and a harmonic function $h > 0$ on Ω such that $\int_{\Omega} s(x)h(x) dx < \infty$ (resp. $\int_{\Omega} s(x) dx < \infty$).

PROOF: Let $G(x, y)$ be the Green function on Ω . By Lemma 2.2, $\int_{\Omega} s(x)h(x) dx$ (resp. $\int_{\Omega} s(x) dx$) is finite if and only if $Q(x) = \int_{\Omega} G(x, y)h(y) dy$ (resp. $Q(x) = \int_{\Omega} G(x, y) dy$) is a potential on Ω which is equivalent to saying that Q is a biharmonic (resp. quasiharmonic) potential on Ω , since $\Delta Q = h$ (resp. $\Delta Q = 1$).

□

Corollary 1. Let Ω be a domain in R . If there exists a quasiharmonic potential on Ω , then for any potential p on Ω with compact harmonic support, $\int_{\Omega} p dx < \infty$. Consequently, there exists a unique bipotential q on Ω such that $\Delta q = p$.

PROOF: Since Ω has a quasiharmonic potential, there exists a superharmonic function $s > 0$ such that $\int_{\Omega} s dx < \infty$. Let p be a potential with compact harmonic support k . Let A be an outerregular compact set such that $k \subset \overset{\circ}{A} \subset A$. Then $p = B_{AP}$ on $\Omega \setminus A$ where B_{AP} denotes the Dirichlet solution with boundary values p on ∂A and 0 at infinity. Hence $p \leq \lambda s$ on $\Omega \setminus A$ for some $\lambda > 0$ so that $\int_{\Omega \setminus A} p dx < \infty$. Since p is locally integrable on Ω , $\int_A p dx < \infty$. Hence $\int_{\Omega} p dx < \infty$. Consequently, for a nonpolar compact k , $\int_{\Omega} R_1^k(x)p(x) dx < \infty$. Hence $q(x) = \int_{\Omega} G(x, y)p(y) dy$ is a potential on Ω such that $\Delta q = p$ (Lemma 2.1). If q_1 is another bipotential on Ω such that $\Delta q_1 = p$, then $q_1 = q +$ (a harmonic function h) on Ω . Note $h \equiv 0$ by the uniqueness of the Riesz representation. □

Corollary 2. Let Ω be a domain in \mathbb{R}^n , $n \geq 2$, such that $\mathbb{R}^n \setminus \Omega$ is compact. If $u \geq 0$ is superharmonic on Ω and if Δu is constant, then u is harmonic and hence is of the form

$$u(x) = \begin{cases} \alpha \log |x - a| + b(x) & \text{if } n = 2, a \notin \Omega \\ \alpha + b(x) & \text{if } n \geq 3, \end{cases}$$

where $\alpha \geq 0$ and $b(x)$ is harmonic on Ω such that $|b(x)| \leq \beta|x|^{2-n}$ near infinity.

PROOF: First we note that there is no quasiharmonic potential on Ω . For, suppose Ω has a quasiharmonic potential. Then there exists a superharmonic function

$s > 0$ on Ω such that $\int_{\Omega} s \, dx < \infty$. Suppose $\mathbb{R}^n \setminus \Omega = e \subset \{x : |x| < a\}$. Let $\lambda = \inf_{|x|=a} s(x)$. Then $s(x) \geq \lambda \left|\frac{x}{a}\right|^{2-n}$ on $|x| > a$ by the minimum principle and hence $\int_a^\infty \int_{\partial B} \lambda \left(\frac{r}{a}\right)^{2-n} r^{n-1} \, dr \, dw \leq \int_{\Omega} s(x) \, dx < \infty$. This implies that $\int_a^\infty r \, dr < \infty$, a contradiction.

Now write $u = p + h$ on Ω where p is a potential and h is harmonic on Ω . Since $\Delta p = \Delta u$ is constant and since there is no quasiharmonic potential on Ω , $p \equiv 0$. Hence u is harmonic ≥ 0 outside a compact set. Then, applying an inversion in the unit ball to the classical representation of Bocher's, we get the stated expression for u . \square

Remarks. (1) The above corollary implies that if a positive superharmonic function u on \mathbb{R}^n , $n \geq 3$, is biharmonic, then u is constant. Apparently, it generalizes the result that every positive harmonic function on \mathbb{R}^n is constant.

(2) $\Omega = \{x : |x| \geq 1\}$ in \mathbb{R}^n , $n \geq 5$, is an example of a domain in which there exists a biharmonic potential but no quasiharmonic potential. For, if $s(x) = h(x) = |x|^{2-n}$, then $\int_{\Omega} sh \, dx < \infty$ and hence by Theorem 4.3, there exists a biharmonic potential on Ω . But there is no quasiharmonic potential on Ω . For, suppose $Q(x)$ is a potential > 0 on Ω such that $\Delta Q = 1$; then by the above Corollary 2, $Q(x)$ should be harmonic, a contradiction.

REFERENCES

- [1] Anandam V., *Biharmonic Green functions in a Riemannian manifold*, Arab J. Math. Sc. **4** (1998), 39–45.
- [2] Anandam V., Damlakhi M., *Biharmonic Green domains in \mathbb{R}^n* , Hokkaido Math. J. **27** (1998), 669–680.
- [3] Anandam V., *Biharmonic classification of harmonic spaces*, Rev. Roumaine Math. Pures Appl. **45** (2000), 383–395.
- [4] BreLOT M., *Fonctions sousharmoniques associées à une mesure*, Stud. Cerc. Şti. Mat. Iaşi **2** (1951), 114–118.
- [5] BreLOT M., *Axiomatique des fonctions harmoniques*, Les presses de l'Université de Montréal, 1966.
- [6] Loeb P.A., *An axiomatic treatment of pairs of elliptic differential equations*, Ann. Inst. Fourier **16** (1966), 167–208.
- [7] Othman S.I., Anandam V., *Liouville-Picard theorem in harmonic spaces*, Hiroshima Math. J. **28** (1998), 501–506.
- [8] Sario L., Nakai M., Wang C., Chung L.O., *Classification theory of Riemannian manifolds*, Lecture Notes in Math. 605, Springer-Verlag, 1977.

DEPARTMENT OF MATHEMATICS, KING SAUD UNIVERSITY, P.O.BOX 2455, RIYADH 11451, SAUDI ARABIA

E-mail: sadoon@ksu.edu.sa
vanandam@ksu.edu.sa

(Received November 16, 2001, revised October 17, 2002)