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Generalized Bochner-Riesz means on spaces generated by smooth blocks

JINCAI WANG

Abstract. We investigate generalized Bochner-Riesz means at the critical index on spaces generated by smooth blocks and give some approximation theorems.

Keywords: smooth blocks, generalized Bochner-Riesz means

Classification: 42B99, 41A35

1. Introduction and statement of result

The Bochner-Riesz multiplier of order α is defined by

$$(S_R^\alpha)^\wedge(x) = \left(1 - \frac{|x|^2}{R^2}\right)_+^\alpha \hat{f}(x), \quad f \in \varphi(\mathbb{R}^n),$$

where $\varphi(\mathbb{R}^n)$ is the Schwartz class.

It is known that for the Bochner-Riesz means at critical index there exists a function $f \in L^1(\mathbb{R}^n)$ such that $\limsup_{R \rightarrow \infty} (S_R^{(n-1)/2} f)(x) = \infty$, a.e. (see [1]).

Shanzhen Lu and Shiming Wang [2] introduced so-called *spaces generated by smooth blocks*, a subspace of $L^1(\mathbb{R}^n)$. On this space, the Bochner-Riesz means at the critical index $S_R^{(n-1)/2} f$ converge to f a.e. as $R \rightarrow \infty$.

Now let us turn to the definition of *smooth blocks*. A (q, λ) -block ($1 < q \leq \infty$) is a function b that is supported on a cube Q satisfying

$$\|b\|_{\mathcal{L}_\lambda^q} \leq |Q|^{\frac{1}{q}-1},$$

where $\mathcal{L}_\lambda^q(\mathbb{R}^n)$ denotes the Bessel potential space ([3]). We define the *spaces generated by smooth blocks* as

$$B_q^\lambda(\mathbb{R}^n) = \left\{ f : f = \sum_k m_k b_k, \quad b_k \text{ is } (q, \lambda)\text{-block, } N(\{m_k\}) < \infty \right\},$$

where $N(\{m_k\}) = \sum_k |m_k| \left(1 + \log \frac{\sum_l |m_l|}{|m_k|}\right)$, and $N_q(f) = \inf\{N(\{m_k\}) : f = \sum_k m_k b_k\}$ is a quasinorm on B_q^λ .

Shanzhen Lu and Shiming Wang [2] get the following result.

Theorem A. *If $f \in B_q^1(\mathbb{R}^n)(1 < q < \infty)$, then*

$$(S_R^{(n-1)/2} f)(x) - f(x) = o\left(\frac{1}{R}\right) \text{ a.e. as } R \rightarrow \infty.$$

Let α be complex number, $\text{Re } \alpha > -1, b > 0$. The generalized Bochner-Riesz multiplier of order α is defined by

$$(S_R^{\alpha,b} f)^\wedge(x) = \left(1 - \frac{|x|^b}{R^b}\right)_+^\alpha \hat{f}(x), \quad f \in \varphi(\mathbb{R}^n).$$

The main result of this paper is

Theorem 1. *If $f \in B_q^1(\mathbb{R}^n)(1 < q < \infty), b > 1$, then*

$$(S_R^{\frac{n-1}{2},b} f)(x) - f(x) = o(1/R) \text{ a.e. as } R \rightarrow \infty.$$

2. Proof of Theorem 1

Let

$$(M_\lambda^{\alpha,b})(x) = \sup_{R>0} |R^\lambda \{(S_R^{\alpha,b} f)(x) - f(x)\}|.$$

To prove Theorem 1, we need the following theorem.

Theorem 2. *Let $0 \leq \lambda \leq 2, 1 < p < \infty, b > 2, \alpha = \sigma + i\tau$ and $\sigma > \frac{n-1}{2}|\frac{2}{p} - 1|$. If $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, then*

$$\|M_\lambda^{\sigma,b} f\|_p \leq C \|f\|_{\mathcal{L}_\lambda^p},$$

where C is independent of f .

First we give some lemmas.

Lemma 1 (Xuean Zheng [6, p.1342]). *Let $\phi(t) = (1 - t^{\alpha_1})^{\delta_1} \dots (1 - t^{\alpha_s})^{\delta_s}, 0 \leq t < 1, \alpha$ be the minimum of $\alpha_1, \dots, \alpha_s$ except 2, $\delta = \delta_1 + \delta_2 + \dots + \delta_s$, and*

$$H_l^{\alpha,\delta}(t) = \int_0^1 \phi(u)(ut)^{l+1} J_\tau(ut) du, \quad \tau = \frac{1}{2}(n - 2).$$

Then

$$H_l^{\alpha,\delta}(t) = \left(\frac{\alpha_1}{2}\right)^{\delta_1} \dots \left(\frac{\alpha_s}{2}\right)^{\delta_s} H_l^\delta(t) + \sum_{k=1}^m C_k H_l^{\delta+k}(t) + R(t),$$

where $\alpha \neq 2$, $m > l + \alpha_1 + \dots + \alpha_s + 2 + \delta$ and m is a positive integer, $H_l^\delta(t) = H_l^{2,\delta}(t) = 2^\delta \Gamma(\delta + 1) J_{l+\delta+1}(t) t^{l-\delta}$, $|R(t)| \leq C t^{2l+1} (t \rightarrow 0)$, $|R(t)| \leq t^{-\alpha-1} (t \rightarrow \infty)$, $J_k(t)$ is the Bessel function (see [7]).

Setting

$$\phi_{\alpha,b}(x) = \begin{cases} (1 - |x|^b)^\alpha, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $\text{Re } \alpha > -1$, we have

$$\mathcal{F}(\phi_{\alpha,b})(y) = \frac{1}{(2\pi)^{n/2} |y|^{n/2-1}} \int_0^1 (1 - r^b)^\alpha r^{n/2} J_{n/2-1}(|y|r) dr$$

(see [9]), where \mathcal{F} denotes the Fourier transform.

It is known ([8]) that for $\alpha > \frac{n-1}{2}$ we have

$$\int_{\mathbb{R}^n} \mathcal{F}(\phi_{\alpha,b})(y) dy = 1.$$

Lemma 2. Let $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 2$. If $f \in L_2^p(\mathbb{R}^n)$, then

$$\|M_2^{\alpha,b} f\|_p \leq C e^{|\tau|^2} \|f\|_{L_2^p},$$

where C is independent of τ and f .

PROOF: Setting $\phi(t) = (1 - t^b)^\alpha$, $l = \frac{n}{2} - 1$ in Lemma 1, we have

$$(1) \quad H^{\alpha,b}(t) = \left(\frac{b}{2}\right)^\alpha H^\alpha(t) + \sum_{k=1}^m C_k H^{\alpha+k}(t) + R^{\alpha,b}(t),$$

where

$$(2) \quad \begin{aligned} H^{\alpha,b}(t) &= H_{\frac{n}{2}-1}^{\alpha,b}(t) = \int_0^1 (1 - u^b)^\alpha (ut)^{\frac{n}{2}} J_{\frac{n}{2}-1}(ut) du, \\ H^\alpha(t) &= H^{\alpha,2}(t), \quad m > \frac{n}{2} - 1 + b + 2 + \text{Re } \alpha \text{ is a positive integer,} \\ |R^{\alpha,b}(t)| &\leq C t^{n-1} (t \rightarrow 0), \quad |R^{\alpha,b}(t)| \leq C t^{-b-1} (t \rightarrow \infty). \end{aligned}$$

By (1), we have

$$(3) \quad \begin{aligned} \mathcal{F}(\phi_{\alpha,b})(y) &= \frac{1}{(2\pi)^{\frac{n}{2}} |y|^{\frac{n}{2}-1}} \int_0^1 (1 - r^b)^\alpha r^{\frac{n}{2}} J_{\frac{n}{2}-1}(|y|r) dr \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} |y|^{n-1}} H^{\alpha,b}(|y|) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \left\{ \left(\frac{b}{2}\right)^\alpha \frac{H^\alpha(|y|)}{|y|^{n-1}} + \sum_{k=1}^m C_k \frac{H^{\alpha+k}(|y|)}{|y|^{n-1}} + \frac{R^{\alpha,b}(|y|)}{|y|^{n-1}} \right\} \\ &= \left(\frac{b}{2}\right)^\alpha \mathcal{F}(\phi_\alpha)(y) + \sum_{k=1}^m C_k \mathcal{F}(\phi_{\alpha+k})(y) + \frac{R^{\alpha,b}(|y|)}{(2\pi)^{\frac{n}{2}} |y|^{n-1}}, \end{aligned}$$

where $\phi_\alpha(x) = \phi_{\alpha,2}$ and $\text{Re } \alpha > -1$. Denote $(S_R^\alpha f)(x) = (S_R^{\alpha,2} f)(x)$. We have

$$\begin{aligned}
 (S_R^{\alpha,b} f)(x) &= \int_{\mathbb{R}^n} f(y+x) R^n \mathcal{F}(\phi_{\alpha,b})(Ry) dy \\
 (4) \qquad &= \left(\frac{b}{2}\right)^\alpha (S_R^\alpha f)(x) + \sum_{k=1}^m C_k (S_R^{\alpha+k} f)(x) \\
 &\quad + \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy.
 \end{aligned}$$

Let $v_0 = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{R^{\alpha,b}(|y|)}{|y|^{n-1}} dy$ and note that

$$\int_{\mathbb{R}^n} \mathcal{F}(\phi_{\alpha,b})(y) dy = \int_{\mathbb{R}^n} \mathcal{F}(\phi_\alpha)(y) dy = 1.$$

Integrating both sides of (3), we get $v_0 = 1 - \sum_{k=1}^m C_k - (\frac{b}{2})^\alpha$. By (4), we have

$$\begin{aligned}
 &|R^2\{(S_R^{\alpha,b} f)(x) - f(x)\}| \\
 &\leq |R^2 \left(\frac{b}{2}\right)^\alpha \{(S_R^\alpha f)(x) - f(x)\}| + \sum_{k=1}^m |R^2 C_k \{(S_R^{\alpha+k} f)(x) - f(x)\}| \\
 &\quad + \left| \left\{ \frac{R^2}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy - R^2 v_0 f(x) \right\} \right|.
 \end{aligned}$$

By Lemma 1 of [2], to prove Lemma 2, we must set up the following inequality:

$$\left\| \sup_{R>0} \left\{ R^2 \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy - v_0 f(x) \right] \right\} \right\|_p \leq C e^{|\tau|^2} \|f\|_{L^2_p}.$$

In fact,

$$\begin{aligned}
 &R^2 \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy - v_0 f(x) \right] \\
 &= R^2 \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} [f(x+y) - f(x)] \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy \\
 &= R^2 \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\Sigma_{n-1}} [f(x+ty') + f(x-ty') - 2f(x)] dy' \cdot R^{\alpha,b}(Rt) R dt \\
 &= R^2 \frac{1}{(2\pi)^{n/2}} \int_0^\infty g(x, \frac{t}{R}) R^{\alpha,b}(t) dt,
 \end{aligned}$$

where $g(x, t) = \int_{\Sigma_{n-1}} [f(x + ty') + f(x - ty') - 2f(x)] dy'$. Denote $A(t) = \int_t^\infty R^{\alpha, b}(\tau) d\tau$, $B(t) = \int_t^\infty A(\tau) d\tau$. It is easy to see that $g(x, \frac{t}{R})|_{t=0} = 0$, $\frac{d}{dt}g(x, \frac{t}{R})|_{t=0} = 0$. Using integration by parts, we get

$$\begin{aligned} & R^2 \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x + y) \frac{R^{\alpha, b}(R|y|)}{|Ry|^{n-1}} R^n dy - v_0 f(x) \right] \\ &= \int_0^\infty R^2 \frac{d^2}{dt^2} g(x, \frac{t}{R}) B(t) dt. \end{aligned}$$

Let

$$\begin{aligned} g_{ij}(x, t) &= \int_{\Sigma_{n-1}} |D_{ij}f(x + ty')| dy', \text{ where } D_{ij}f(x) = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f(x), \\ (\mathcal{M}_2^{\alpha, b} f)(x) &= \sup_{R>0} R^2 \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x - y) \frac{R^{\alpha, b}(R|y|)}{|Ry|^{n-1}} R^n dy - v_0 f(x) \right|. \end{aligned}$$

Then

$$(\mathcal{M}_2^{\alpha, b} f)(x) \leq \sum_{i, j=1}^n \sup_{R>0} \int_0^\infty |g_{ij}(x, \frac{t}{R})| |B(t)| dt.$$

For $t \geq 1$,

$$\begin{aligned} |A(t)| &= \left| \int_t^\infty R^{\alpha, b}(\tau) d\tau \right| \leq C \int_t^\infty \tau^{-b-1} d\tau = Ct^{-b}, \\ |B(t)| &= \left| \int_t^\infty A(\tau) d\tau \right| \leq C \int_t^\infty \tau^{-b} = Ct^{-b+1} \quad (b > 1). \end{aligned}$$

For $t < 1$,

$$|A(t)| \leq C \left| \int_1^\infty R^{\alpha, b}(\tau) d\tau \right| + \left| \int_t^1 R^{\alpha, b}(\tau) d\tau \right| \leq C$$

and by (2),

$$|B(t)| \leq \left| \int_1^\infty A(\tau) d\tau \right| + \left| \int_t^1 A(\tau) d\tau \right| \leq \int_1^\infty C\tau^{-b} d\tau + \int_t^1 C d\tau \leq C.$$

By this, we have

$$\begin{aligned}
 & \int_0^1 g_{ij}(x, \frac{t}{R})|B(t)| dt \leq C \int_0^1 g_{ij}(x, \frac{t}{R}) dt \\
 & \leq CR \int_0^{\frac{1}{R}} \int_{\Sigma_{n-1}} |D_{ij}f(x + ty')| dy' dt \\
 & = CR \int_0^{\frac{1}{R}} t^{1-n} d \left[\int_0^t \left(\int_{\Sigma_{n-1}} |D_{ij}f(x + \tau y')| dy' \tau^{n-1} \right) d\tau \right] \\
 & \leq CR \left(\frac{1}{R} \right)^{1-n} \int_0^{\frac{1}{R}} \int_{\Sigma_{n-1}} |D_{ij}f(x + ty')| dy' t^{n-1} dt \\
 & \quad + CR \int_0^{\frac{1}{R}} t^{-n} \left[\int_0^t \left(\int_{\Sigma_{n-1}} |D_{ij}f(x + \tau y')| dy' \right) \tau^{n-1} d\tau \right] dt \\
 & \leq C \left(\frac{1}{R} \right)^{-n} \int_{|y| < \frac{1}{R}} |D_{ij}f(x + y)| dy \\
 & \quad + CR \int_0^{\frac{1}{R}} t^{-n} \left(\int_{|y| < t} |D_{ij}f(x + y)| dy \right) dt \\
 & \leq C \text{HL}(D_{ij}f)(x),
 \end{aligned}$$

where $\text{HL}(g)$ is the Hardy-Littewood maximal function of g . Thus, we get

$$\left\| \sup_{R>0} \int_0^1 g_{ij}(x, \frac{t}{R})|B(t)| dt \right\|_p \leq C \|f\|_{L_2^p} \leq C e^{|\tau|^2} \|f\|_{L_2^p},$$

where C is independent of τ . Meanwhile, we have

$$\int_1^\infty g_{ij}(x, \frac{t}{R})|B(t)| dt \leq \int_1^\infty g_{ij}(x, \frac{t}{R})t^{-b+1} dt = R^{-b+1} \int_{\frac{1}{R}}^\infty g_{ij}(x, t)t^{-b+1} R dt.$$

Using the inequality $\int_0^t \tau^{n-1} g_{ij}(x, \tau) d\tau \leq Ct^n \text{HL}(D_{ij}f)(x)$, we obtain

$$\begin{aligned}
 \int_1^\infty g_{ij}(x, \frac{t}{R})|B(t)| dt & \leq R^{-b+2} \int_{\frac{1}{R}}^\infty t^{-b+1-(n-1)} d \left(\int_0^t \tau^{n-1} g_{ij}(x, \tau) d\tau \right) \\
 & \leq R^{-b+2} \left[R^{b+n-2} \int_0^{\frac{1}{R}} \tau^{n-1} g_{ij}(x, \tau) d\tau \right. \\
 & \quad \left. + C \int_{\frac{1}{R}}^\infty t^{-b-n+1} \left(\int_0^t \tau^{n-1} g_{ij}(x, \tau) d\tau \right) dt \right] \\
 & \leq C \text{HL}(D_{ij}f)(x).
 \end{aligned}$$

Thus

$$\left\| \sup_{R>0} \int_1^\infty g_{ij}\left(x, \frac{t}{R}\right) |B(t)| dt \right\|_p \leq C \|f\|_{L_2^p} \leq C e^{|\tau|^2} \|f\|_{L_2^p}.$$

□

Using the same method, we have

Lemma 2'. *Let $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 1$. If $f \in L_1^p(\mathbb{R}^n)$, then $\|M_1^{\alpha,b} f\|_p \leq C e^{|\tau|^2} \|f\|_{L_1^p}$, where C is independent of τ and f .*

Lemma 3. *Let $0 \leq \lambda \leq 2$, $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 2$. If $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, then*

$$\|M_\lambda^{\alpha,b}\|_p \leq C e^{|\tau|^2} \|f\|_{\mathcal{L}_\lambda^p},$$

where C is independent of τ and f .

PROOF: Suppose $\{r_k\}$ is a sequence consisting of all positive rational numbers. Let $A_k = \{r_1, r_2, \dots, r_k\}$. Define

$$(F_\lambda^{\alpha,b,k} f)(x) = \sup_{r_j \in A_k} r_j^\lambda |(S_{r_j}^{\alpha,b} f)(x) - f(x)|.$$

Then

$$(F_\lambda^{\alpha,b,k} f)(x) \leq (F_\lambda^{\alpha,b,k+1} f)(x)$$

and

$$(M_\lambda^{\alpha,b} f)(x) = \lim_{k \rightarrow \infty} (F_\lambda^{\alpha,b,k} f)(x).$$

For any fixed $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, there exists $g \in L^p(\mathbb{R}^n)$ such that $f = j_\lambda g$. Fix k and let $S_j (1 \leq j \leq k)$ be a set such that for $x \in S_j$,

$$(F_\lambda^{\alpha,b,k} f)(x) = r_j^\lambda |(S_{r_j}^{\alpha,b} f)(x) - f(x)|$$

and

$$(F_\lambda^{\alpha,b,k} f)(x) > r_i^\lambda |(S_{r_i}^{\alpha,b} f)(x) - f(x)|, \quad i < j.$$

It is easy to see that the sets $\{S_j\}$ do not intersect each other. Let $\Omega = \{z \in \mathbb{C}, 0 \leq \text{Re } z \leq 1\}$ and define

$$\begin{aligned} \phi_j(x) &= \text{sign}\{(S_{r_j}^{\alpha,b} f)(x) - f(x)\}, \\ (T_z g)(x) &= \sum_{r_j \in A_k} r_j^{2z} \chi_{S_j}(x) \{S_{r_j}^{\alpha,b}(j_{2z} g)(x) - (j_{2z} g)(x)\} \phi_j. \end{aligned}$$

Now we assert that $\{T_z\}$ is an admissible family of operators in the sense of E.M. Stein (see [7]). Indeed, $\{j_{2z}g\}$ is an admissible family of operators in the sense of E.M. Stein, so we only set up the following inequality:

$$(5) \quad \|S_R^{\alpha,b} f\|_p \leq C \|f\|_p, \quad f \in L^p(\mathbb{R}^n), \quad p > 1.$$

By (4), we must prove that

$$(6) \quad \left\| \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy \right\|_p \leq \|f\|_p.$$

In fact,

$$\begin{aligned} & \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n dy \\ &= \int_0^{1/R} \left(\int_{\Sigma_{n-1}} f(x-ty') dy' \right) R^{\alpha,b}(Rt) R dt \\ & \quad + \int_{1/R}^\infty \left(\int_{\Sigma_{n-1}} f(x-ty') dy' \right) R^{\alpha,b}(Rt) R dt \\ &= I_1 + I_2. \end{aligned}$$

We have

$$\begin{aligned} |I_1| &\leq \int_0^{1/R} \left(\int_{\Sigma_{n-1}} f(x-ty') dy' \right) (Rt)^{n-1} R dt \\ &= \left(\frac{1}{R}\right)^{-n} \int_{|y| < 1/R} f(x-y) dy \leq C \text{HL}(f)(x) \end{aligned}$$

and

$$\begin{aligned} |I_2| &\leq \left| \int_{1/R}^\infty (Rt)^{-b-1} R \left(\int_{\Sigma_{n-1}} f(x-ty') dy' \right) dt \right| \\ &= \left| \int_{1/R}^\infty (Rt)^{-b-1} R t^{-n+1} d \left(\int_0^t \tau^{n-1} \left(\int_{\Sigma_{n-1}} f(x-\tau y') dy' \right) d\tau \right) \right| \\ &\leq \left| R^n \int_0^{1/R} \tau^{n-1} \left(\int_{\Sigma_{n-1}} f(x-\tau y') dy' \right) d\tau \right| \\ & \quad + \left| C \int_{1/R}^\infty R^{-b} t^{-n-b-1} \int_0^t \tau^{n-1} \left(\int_{\Sigma_{n-1}} f(x-\tau y') dy' \right) d\tau \right| \\ &\leq C \text{HL}(f)(x) + \text{HL}(f)(x) C \int_{1/R}^\infty R^{-b} t^{-b-1} dt \leq C \text{HL}(f)(x). \end{aligned}$$

Thus (6) is true. So (5) is also true, therefore $\{T_z\}$ is an admissible family of operators in the sense of E.M. Stein.

Now we write

$$J_\lambda g = G_\lambda * g, \quad \hat{G}_\lambda(x) = (1 + 4\pi^2|x|^2)^{-\frac{\lambda}{2}}.$$

A multiplier theorem (see [3, p.96]) implies that

$$\|J_{i\eta}g\|_p \leq P(\eta)\|g\|_p,$$

where $P(x)$ is a polynomial of degree $k > n/2$. Note that $\text{Re } \alpha > (n - 1)/2$ and we have

$$\begin{aligned} \|T_{i\eta}g\|_p &\leq \|F_0^{\alpha,b,k}(J_{2i\eta}g)\|_p \leq \|M_0^{\alpha,b}(J_{2i\eta}g)\|_p \\ &\leq Ce^{|\tau|^2}\|J_{2i\eta}g\|_p \leq Ce^{|\tau|^2}P(2\eta)\|g\|_p. \end{aligned}$$

By Lemma 2, we get

$$\begin{aligned} \|T_{1+i\eta}g\|_p &< \|F_2^{\alpha,b,k}(J_{2+2i\eta}g)\|_p \leq \|M_2^{\alpha,b}(J_{2+2i\eta}g)\|_p \\ &\leq Ce^{|\tau|^2}\|J_{2+2i\eta}g\|_{L_2^p} \leq Ce^{|\tau|^2}\|J_{2i\eta}g\|_p \leq Ce^{|\tau|^2}P(2\eta)\|g\|_p. \end{aligned}$$

Using Stein's interpolation theorem of analytic operators, we obtain

$$\|F_\lambda^{\alpha,b,k}f\|_p \leq \|T_{\frac{\lambda}{2}}g\|_p \leq Ce^{|\tau|^2}\|g\|_p \leq Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p}.$$

Finally, by Lebesgue's monotone convergence theorem, we get

$$\|M_\lambda^{\alpha,b}f\|_p \leq Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p}.$$

□

If we define T_z in the proof of Lemma 3 as

$$(T_zg)(x) = \sum_{j \in A_k} r_j^z \chi_{S_j}(x) \{S_{r_j}^{\alpha,b}(J_zg)(x) - (J_zg)(x)\} \phi_j$$

and using the same method of proof of 2', we have

Lemma 3'. *Let $0 \leq \lambda \leq 1$, $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 1$. If $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, then*

$$\|M_\lambda^{\alpha,b}f\|_p \leq Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p},$$

where C is independent of τ and f .

Define

$$(N_\lambda^{\alpha,b}f)(x) = \sup_{R>0} R^\lambda |(S_R^{\alpha+1,b}f)(x) - (S_R^{\alpha,b}f)(x)|.$$

Lemma 4. Let $\alpha = \sigma + i\tau$, $\sigma > 0$, $0 \leq \lambda \leq \tau$, $b > \lambda$. If $f \in \mathcal{L}_\lambda^2(\mathbb{R}^n)$, then

$$\|N_\lambda^{\alpha,b} f\|_2 \leq \|f\|_{\mathcal{L}_\lambda^2}.$$

PROOF: Let $\beta \in \mathbb{C}$, $\operatorname{Re} \beta > \frac{1}{2}$, $\delta > -\frac{1}{2}$, $0 < \lambda < 2$. Then

$$\begin{aligned} & (S_R^{\beta+\delta+1,b} f)(x) - (S_R^{\beta+\delta,b} f)(x) \\ &= \int_0^R t^{n-1} \left(\int_{\Sigma_{n-1}} \hat{f}(x - ty') dy' \right) \left[\left(1 - \frac{t^b}{R^b} \right)^{\beta+\delta+1} - \left(1 - \frac{t^b}{R^b} \right)^{\beta+\delta} \right] dt \\ &= \int_0^R t^{n-1} \left(1 - \frac{t^b}{R^b} \right)^\beta \left(\int_{\Sigma_{n-1}} \hat{f}(x - ty') dy' \right) \\ &\quad \times \left[\left(1 - \frac{t^b}{R^b} \right)^{1+\delta} - \left(1 - \frac{t^b}{R^b} \right)^\delta \right] dt \\ &= \int_0^R \left(1 - \frac{t^b}{R^b} \right)^\beta d \left(\int_0^t \tau^{n-1} \int_{\Sigma_{n-1}} \hat{f}(x - \tau y') dy' \right. \\ &\quad \left. \times \left[\left(1 - \frac{\tau^b}{R^b} \right)^{1+\delta} - \left(1 - \frac{\tau^b}{R^b} \right)^\delta \right] d\tau \right) \\ &= \beta \int_0^R \left(1 - \frac{t^b}{R^b} \right)^{\beta-1} b \frac{t^{b-1}}{R^b} \left\{ (S_t^{\delta+1,b} f)(x) - (S_t^{\delta,b} f)(x) \right\} dt \\ &= b\beta \frac{1}{R^{b(\beta-1)+b}} \int_0^R (R^b - t^b)^{\beta-1} t^{b-1} \left\{ (S_t^{\delta+1,b} f)(x) - (S_t^{\delta,b} f)(x) \right\} dt. \end{aligned}$$

Writing $(G_\lambda^{\delta,b} f)(x) = \left\{ \int_0^\infty t^{2\lambda-1} |(S_t^{\delta+1,b} f)(x) - (S_t^{\delta,b} f)(x)|^2 dt \right\}^{\frac{1}{2}}$, we have

$$\begin{aligned} & R^\lambda |(S_R^{\beta+\delta+1,b} f)(x) - (S_R^{\beta+\delta,b} f)(x)| \\ &\leq b|\beta| \frac{1}{R^{b\operatorname{Re} \beta - \lambda}} \left\{ \int_0^R (R^b - t^b)^{2\operatorname{Re} \beta - 2} t^{2b-2\lambda-1} dt \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \int_0^R t^{2\lambda-1} |(S_t^{\delta+1,b} f)(x) - (S_t^{\delta,b} f)(x)|^2 dt \right\}^{\frac{1}{2}} \\ &\leq b|\beta| \left\{ \int_0^1 (1-t)^{2\operatorname{Re} \beta - 2} t^{2b-2\lambda-1} dt \right\}^{\frac{1}{2}} (G_\lambda^{\delta,b} f)(x) \leq C(G_\lambda^{\delta,b} f)(x). \end{aligned}$$

Choose $\delta = \frac{\sigma-1}{2} > -\frac{1}{2}$, $\beta = \alpha - \delta = \frac{\sigma+1}{2} + i\tau$. Then

$$\|N_\lambda^{\alpha,b} f\|_2^2 \leq C \|G_\lambda^{\frac{\sigma-1}{2},b} f\|_2^2.$$

Further,

$$\begin{aligned} & \|G_\lambda^{\frac{\sigma-1}{2},b} f\|_2^2 \\ & \leq \int_{\mathbb{R}^n} \left(\int_0^\infty t^{2\lambda-1} |(S_t^{\frac{\sigma+1}{2},b} f)(x) - (S_t^{\frac{\sigma-1}{2},b} f)(x)|^2 dt \right) dx \\ & = \int_0^\infty t^{2\lambda-1} \int_{|y|<t} \left| \left(1 - \frac{|y|^b}{t^b}\right)^{\frac{\sigma+1}{2}} - \left(1 - \frac{|y|^b}{t^b}\right)^{\frac{\sigma-1}{2}} \right|^2 \\ & \quad \times |\hat{f}(y)|^2 dy dt \\ & = \int_0^\infty t^{2\lambda-1} \int_{|y|<t} \left(1 - \frac{|y|^b}{t^b}\right)^{\sigma-1} \frac{|y|^{2b}}{t^{2b}} |\hat{f}(y)|^2 dy dt \\ & = \int_{\mathbb{R}^n} |y|^{2b} |\hat{f}(y)|^2 \left[\int_{|y|}^\infty t^{2\lambda-2b-1} \left(1 - \frac{|y|^b}{t^b}\right)^{\sigma-1} dt \right] dy. \end{aligned}$$

Setting $\frac{|y|}{t} = t$, we get

$$\int_{|y|}^\infty t^{2\lambda-2b-1} \left(1 - \frac{|y|^b}{t^b}\right)^{\sigma-1} dt = \int_0^1 |y|^{2\lambda-2b-1} t^{-2\lambda+2b+1} (1-t^b)^{\sigma-1} \frac{|y|}{t^2} dt.$$

Therefore

$$\|G_\lambda^{\frac{\sigma-1}{2},b} f\|_2^2 = \int_{\mathbb{R}^n} |y|^{2\lambda} |\hat{f}(y)|^2 dy \int_0^1 t^{2b-2\lambda-1} (1-t^b)^{\sigma-1} dt \leq C \|f\|_{\mathcal{L}_\lambda^2}^2. \quad \square$$

PROOF OF THEOREM 2: Let $f = \mathcal{J}_\lambda g$, $g \in L^p(\mathbb{R}^n)$. For $\sigma > 0$, we choose $k \in \mathbb{N}$ such that $\sigma + k > \frac{n-1}{2}$. By Lemma 3, Lemma 4 and inequality

$$(M_\lambda^{\alpha,b} f)(x) \leq \sum_{j=0}^{k-1} (N_\lambda^{\alpha+j,b} f)(x) + (M_\lambda^{\alpha+k,b} f)(x),$$

we have

$$(7) \quad \|M_\lambda^{\alpha,b} f\|_2 \leq C e^{|\tau|^2} \|f\|_{\mathcal{L}_\lambda^2} = C e^{|\tau|^2} \|g\|_2.$$

Let $p_1 > 1, \sigma > \frac{n-1}{2}$. Then by Lemma 3

$$\|M_\lambda^{\alpha,b} f\|_{p_1} \leq C e^{|\tau|^2} \|f\|_{\mathcal{L}^{p_1}} = C e^{|\tau|^2} \|g\|_{p_1}.$$

For $p_1 < p < 2$, there exists $0 < t < 1$ such that $\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_1}$. Let $u_0 > 0, u_1 > \frac{n-1}{2}, \delta(z) = u_0(1-z) + u_1 z, 0 \leq \operatorname{Re} z \leq 1$. Then

$$\delta(t) = u_0(1-t) + u_1 t = u_0 + (u_1 - u_0)t = u_0 + (u_1 - u_0) \frac{\frac{2}{p} - 1}{\frac{2}{p_1} - 1}.$$

Thus

$$\delta(t) \longrightarrow \frac{n-1}{2} \cdot \frac{\frac{2}{p} - 1}{\frac{2}{p_1} - 1} > \frac{n-1}{2} \left(\frac{2}{p} - 1\right) \text{ as } u_0 \searrow 0, u_1 \searrow \frac{n-1}{2},$$

and

$$\delta(t) \longrightarrow \frac{n-1}{2} \left(\frac{2}{p} - 1\right) \text{ as } p_1 \searrow 1.$$

So for $\sigma > \frac{n-1}{2} \left(\frac{2}{p} - 1\right)$, there exist u_0, u_1, p_1 such that $\delta(t) = \sigma$, where $t = \frac{1/p-1/2}{1/p_1-1/2}$. Thus for given $1 < p < 2$ and $\sigma > \frac{n-1}{2} \left(\frac{2}{p} - 1\right)$, we can find u_0, u_1 and p_1 such that $1 < p_1 < p < 2, \delta(t) = \sigma$.

Fix such u_0, u_1, p_1 . Let $\{R_j\}$ be a sequence consisting of all positive rational numbers. Denote $A_k = \{R_1, \dots, R_k\}$, and

$$(F_\lambda^{\alpha,b,k} f)(x) = \sup_{R \in A_k} \{R^\lambda (S_R^{\alpha,b} f)(x) - f(x)\}.$$

Then we have $(F_\lambda^{\alpha,b,k} f)(x) \leq (F_\lambda^{\alpha,b,k+1} f)(x)$, and,

$$(M_\lambda^{\alpha,b} f)(x) = \lim_{k \rightarrow \infty} (F_\lambda^{\alpha,b,k} f)(x).$$

For $1 \leq j \leq k$, let

$$E_j = \left\{ x \in \mathbb{R}^n : \sup_{R \in A_k} \left[R^\lambda |(S_R^{\alpha,b} f)(x) - f(x)| \right] = R_j^\lambda |(S_{R_j}^{\alpha,b} f)(x) - f(x)| \right\},$$

and $F_1 = E_1, F_j = E_j - \bigcup_{i=1}^{j-1} E_i, j = 2, 3, \dots, k$. Define

$$(T_z g)(x) = \sum_{j=1}^k R_j^\lambda \chi_{F_j}(x) \{S_{R_j}^{\delta(z),b} (J_\lambda g)(x) - (J_\lambda g)(x)\} \phi_j(x),$$

where

$$\phi_j(x) = \text{sign}\{S_{R_j}^{\sigma,b}(j_\lambda g)(x) - (j_\lambda g)(x)\}.$$

It is easy to verify that $\{T_z\}$ is an admissible family of linear operators. Since

$$\begin{aligned} \|T_{i\tau}g\|_2 &\leq \|F_\lambda^{\delta(i\tau),b,k} f\|_2 \leq \|M_\lambda^{\delta(i\tau),b}\|_2 \leq Ce^{(u_1-u_0)^2\tau^2} \|g\|_2, \\ \|T_{1+i\tau}g\|_{p_1} &\leq \|F_\lambda^{\delta(1+i\tau),b,k} f\|_{p_1} \leq Ce^{(u_1-u_0)^2\tau^2} \|g\|_{p_1}, \end{aligned}$$

by Stein's interpolation theorem of analytic operators, we have

$$\|F_\lambda^{\sigma,b,k} f\|_p = \|F_\lambda^{\delta(t),b,k} f\|_p = \|T_tg\|_p \leq C\|g\|_p = C\|f\|_{\mathcal{L}_\lambda^p}.$$

Hence, by the monotone convergence theorem we obtain

$$\|M_\lambda^{\sigma,b} f\|_p \leq C\|f\|_{\mathcal{L}_\lambda^p}, 1 < p \leq 2.$$

Finally, it should be pointed out that the proof in the case $2 < p < \infty$ is similar to the above. □

Applying the same method and using Lemma 2', we have

Theorem 2'. *Let $0 \leq \lambda \leq 1$, $1 < p < \infty$, $b > 1$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}|\frac{2}{p} - 1|$. If $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, then*

$$\|M_\lambda^{\sigma,b} f\|_p \leq C\|f\|_{\mathcal{L}_\lambda^p},$$

where C is independent of f .

To prove Theorem 1, we first need to establish a weak type estimation of the maximal operator $M_1^{\frac{n-1}{2},b}$ on any block.

Lemma 5. *Let $a(x)$ be a $(q, 1)$ -block, $b > 1$. Then*

$$(8) \quad |\{x : (M_1^{\frac{n-1}{2},b} a)(x) > \lambda\}| \leq C\lambda^{-1},$$

where C is independent of λ and $a(x)$.

PROOF: We have

$$\begin{aligned} &R \left\{ (S_R^{\frac{n-1}{2},b} a)(x) - a(x) \right\} \\ &= R \left\{ \left(\frac{b}{2}\right)^{\frac{n-1}{2}} (S_R^{\frac{n-1}{2}} a)(x) + \sum_{k=1}^m C_k (S_R^{\frac{n-1}{2}+k} a)(x) \right. \\ &\quad \left. + \int_{\mathbb{R}^n} a(x+y) \frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n dy - a(x) \right\}. \end{aligned}$$

Denote

$$v_0 = \int_{\mathbb{R}^n} \frac{R^{\frac{n-1}{2},b}(|y|)}{|y|^{n-1}} dy = 1 - \sum_{k=1}^m C_k - \left(\frac{b}{2}\right)^{\frac{n-1}{2}},$$

$$(\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) = \sup_{R>0} R \left| \int_{\mathbb{R}^n} [a(x+y) - a(x)] \frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n dy \right|.$$

By Lemma 4 in [2], we must prove that

$$|\{x : (\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) > \lambda\}| \leq C\lambda^{-1}.$$

In fact, setting $g(x, t) = \int_{\Sigma_{n-1}} [a(x - ty') - a(x)] dy'$, we have

$$\begin{aligned} & R \left| \int_{\mathbb{R}^n} [a(x+y) - a(x)] \frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n dy \right| \\ &= R \left| \int_0^\infty \left\{ \int_{\Sigma_{n-1}} [a(x - \frac{t}{R}y') - a(x)] dy' \right\} R^{\frac{n-1}{2},b}(t) dt \right| \\ &= R \left| \int_0^\infty g(x, \frac{t}{R}) R^{\frac{n-1}{2},b}(t) dt \right| \\ &= R \left| \int_0^\infty g(x, \frac{t}{R}) d \left(\int_\tau^\infty R^{\frac{n-1}{2},b}(\tau) d\tau \right) \right| \\ &= R \int_0^\infty \frac{d}{dt} g(x, \frac{t}{R}) A(t) dt, \end{aligned}$$

where $A(t) = \int_t^\infty R^{\frac{n-1}{2},b}(\tau) d\tau$.

For $t > 1$,

$$|A(t)| \leq \int_t^\infty |R^{\frac{n-1}{2},b}(\tau)| d\tau \leq C \int_t^\infty t^{-b-1} d\tau = Ct^{-b} < C\frac{1}{t}$$

and for $0 < t < 1$,

$$|A(t)| \leq \left| \int_1^\infty R^{\frac{n-1}{2},b}(\tau) \right| + \int_t^1 |R^{\frac{n-1}{2},b}(\tau)| d\tau \leq C \leq \frac{C}{t}.$$

Hence

$$\begin{aligned}
 & R \left| \int_{\mathbb{R}^n} [a(x+y) - a(x)] \frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n dy \right| \\
 &= C \int_0^\infty \left\{ \int_{\Sigma_{n-1}} |D_x a(x + \frac{t}{R}y')| dy' \right\} |A(t)| dt \\
 &= C \int_0^\infty \left\{ \int_{\Sigma_{n-1}} |D_x a(x + ty')| dy' \right\} |A(Rt)| R dt \\
 &\leq C \int_0^\infty \int_{\Sigma_{n-1}} |D_x a(x + ty')| dy' \frac{dt}{t} \\
 &= C \int_{\mathbb{R}^n} |D_x a(x+y)| \frac{dy}{|y|^n} = C \int_{\mathbb{R}^n} \frac{|Da(u)|}{|u-x|^n} dy.
 \end{aligned}$$

Therefore

$$(\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) \leq C \int_Q \frac{|Da(u)|}{|x-u|^n} du,$$

where $\text{supp } a(x) \subset Q$.

Let $\tilde{Q} = 2Q$. Then for $x \notin \tilde{Q}$,

$$\begin{aligned}
 (\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) &\leq \frac{C}{|x|^n} \int_Q |Da(u)| du \leq \frac{C}{|x|^n} \|Da(u)\|_q |Q|^{1-\frac{1}{q}} \\
 &\leq \frac{C}{|x|^n} \|a\|_{\mathcal{L}_1^q} |Q|^{1-\frac{1}{q}} \leq \frac{C}{|x|^n}.
 \end{aligned}$$

So

$$\left| \left\{ x \notin \tilde{Q} : (\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) > \lambda, \lambda \leq \frac{1}{|Q|} \right\} \right| \leq C\lambda^{-1}.$$

Clearly,

$$|\{x \in \tilde{Q} : (\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) > \lambda, \lambda \leq \frac{1}{|Q|}\}| \leq |\tilde{Q}| \leq C\lambda^{-1}.$$

By Theorem 2', we have

$$\|\mathcal{M}_1^{\frac{n-1}{2},b} a\|_q \leq C \|a\|_{\mathcal{L}_1^q}, \quad 1 < q < \infty.$$

Thus

$$\begin{aligned}
 & \left| \left\{ x \in \mathbb{R}^n : (\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) > \lambda, \lambda > \frac{1}{|Q|} \right\} \right| \\
 & \leq C(\lambda^{-1} \|a\|_{\mathcal{L}_\lambda^q})^q \leq C \left(\frac{|Q|^{\frac{1}{q}-1}}{\lambda} \right)^q \leq C\lambda^{-1}.
 \end{aligned}$$

Therefore (8) holds. □

PROOF OF THEOREM 1: Suppose $f(x) \in B_q^1(\mathbb{R}^n)$. Then

$$f(x) = \sum_k m_k b_k(x) = \sum_{k=1}^N m_k b_k(x) + \sum_{k=N+1}^{\infty} m_k b_k(x) = g(x) + h(x),$$

where b_k is a $(q, 1)$ -block and $N\{m_k\} < \infty$.

To complete the proof of Theorem 1, we must prove for all $\lambda > 0$,

$$\left| \left\{ x : \limsup_{R \rightarrow \infty} |R\{(S_R^{\frac{n-1}{2}, b} f)(x) - f(x)\}| > \lambda \right\} \right| = 0.$$

Since $g \in L_1^q(\mathbb{R}^n)$, by theorem in [9] we have

$$(S_R^{\frac{n-1}{2}, b} g)(x) - g(x) = o\left(\frac{1}{R}\right), \text{ a.e.}$$

So

$$\left| \left\{ x : \limsup_{R \rightarrow \infty} |R\{(S_R^{\frac{n-1}{2}, b} g)(x) - g(x)\}| > \lambda/2 \right\} \right| = 0.$$

Thus by Lemma 5 and Lemma 1.3 in [10], we get

$$\begin{aligned} & |\{x : \limsup_{R \rightarrow \infty} |R\{(S_R^{\frac{n-1}{2}, b} f)(x) - f(x)\}| > \lambda\}| \\ & \leq |\{x : (M_1^{\frac{n-1}{2}, b} h)(x) > \lambda/2\}| \\ & \leq C\lambda^{-1} \sum_{k=N+1}^{\infty} |m_k| \left(1 + \log \frac{\sum_l |m_l|}{|m_k|} \right) \longrightarrow 0, \quad N \rightarrow \infty. \end{aligned}$$

□

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