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On the complexity of some σ-ideals of σ-P-porous sets

Luděk Zajíček, Miroslav Zelený

Abstract. Let $\mathbf{P}$ be a porosity-like relation on a separable locally compact metric space E. We show that the σ-ideal of compact σ-$\mathbf{P}$-porous subsets of E (under some general conditions on $\mathbf{P}$ and E) forms a $\mathbf{\Pi}^1_1$-complete set in the hyperspace of all compact subsets of E, in particular it is coanalytic and non-Borel. Our general results are applicable to most interesting types of porosity. It is shown in the cases of the σ-ideals of σ-porous sets, σ-$\langle g\rangle$-porous sets, σ-strongly porous sets, σ-symmetrically porous sets and σ-strongly symmetrically porous sets. We prove a similar result also for σ-very porous sets assuming that each singleton of E is very porous set.

Keywords: σ-porous sets, σ-ideal, coanalytic sets, Hausdorff metric

Classification: 54H05, 28A05

1. Introduction

The aim of this paper is to investigate descriptive complexity of σ-ideals of compact sets which are related to the notion of σ-porosity. As for history and applications of the notions of porosity and σ-porosity we refer to [Za$_2$].

It is proved in [ZP] that compact σ-porous sets form a $\mathbf{\Pi}^1_1$-complete set (in particular, a coanalytic non-Borel set) in the "hyperspace" of all compact sets (equipped with the Hausdorff metric), if the underlying space is nonempty compact and has no isolated point. This result was firstly proved by G. Debs and D. Preiss ([DP]), but the proof was not published. The proof of [ZP] is based on a direct method of construction of non-σ-porous sets. This method is applicable to a number of problems but it is rather complicated and uses some special properties of the ordinary (i.e. Denjoy-Dolzhenko) porosity. This method could be probably adapted also to some other types of porosity, but then it would become very technical.

In this paper we prove that this theorem is true also for compact σ-$\langle g\rangle$-porous sets, σ-strongly porous sets, σ-symmetrically porous sets, σ-strongly symmetrically porous sets and σ-very porous sets. In fact, we prove a general theorem (Theorem 6.1) which is applicable to most interesting types of porosity.

Moreover, our proof is considerably simpler than that in [ZP]. In the proof we need no explicit construction of special non-σ-porous sets; we use a non-constructive existence proof based on a result of descriptive set theory (namely

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Kunen-Martin theorem on analytic well-founded relations on a Polish space, see Theorem 5.1 below). To be able to use this theorem, we need also a construction (of "complicated σ-porous sets"), but this construction is much simpler than constructions of [ZP]. However, note that some new results of the present paper can be obtained easily using results of [ZP]. Namely, in [ZP] there was proved the following theorem:

Let E be a compact metric space and $\mathcal{A}$ be an analytic subset of the space of all compact subsets of E. If $\mathcal{A}$ contains all countable compact subsets of E, then there is $K \in \mathcal{A}$, which is not σ-porous.

Thus there is no Borel σ-ideal of compact sets of E, which contains all singletons and is contained in the σ-ideal of σ-porous sets. Therefore non-Borelness of several considered σ-ideals (e.g. σ-strongly porous sets, σ-symmetrically porous sets, σ-strongly symmetrically porous sets and σ-very porous sets) can be inferred in rather general spaces via this theorem. But there are σ-$\langle g\rangle$-porous sets which are not σ-porous and therefore we cannot use the above theorem here.

Our non-constructive method (which is a form of the well-known "overspill method", cf. [K_1, p. 290]) is a simplification of the one used in an unpublished manuscript [Ze] where it was proved that, in a compact metric space, each non-σ-porous Suslin set contains a closed non-σ-porous set. (Note that this theorem was proved in [ZP] by a constructive method in topologically complete metric spaces, but also for ordinary porosity only.) We are now working on generalization of this theorem to other interesting types of porosity, using a version of the descriptive method used here. The "inscribing problem" appears to be much more difficult than the "complexity problem" treated in the present article, and we still succeeded only partially. We formulate here some lemmas in generality, which is not needed (but does not complicate the proofs), since some of them are applicable to the "inscribing problem" (and possibly to other problems). Note that in both [ZP] and the present article the general theory of σ-ideals ([KLW]) is essentially used.

Further note that we use new (more general) versions of the notion of Foran's system and corresponding Foran's lemma. In applications to $\langle g\rangle$-porosity, strong porosity and strong symmetrical porosity it would be sufficient to use the version of Foran's lemma from [Za_2]. This version is sufficient also for applications to the ordinary porosity. Indeed, in this case we could use the result of [Za_1] which says that for each $0 < c < 1$, a set is σ-porous if and only if it is σ-c-porous. Thus, investigating σ-porous sets, we can deal with c-porosity only (with fixed c); this advantage is used e.g. in [Za_3] and [ZP]. On the other hand, the new version of Foran's lemma is necessary for us in the case of symmetric porosity. This is caused by the fact (cf. [EH, Example 2]) that there exists a σ-symmetrically porous set which is σ-c-symmetrically porous for no $c > 0$.

Let us recall several notions of descriptive set theory used in this paper. Following [K_1] a *Polish space* is a separable completely metrizable topological space.

We will use the well-known facts that a separable locally compact metric space is Polish and that a countable product of Polish spaces is Polish. Let X be a metric space. A set $A \subset X$ is called *analytic* if there is a Polish space Y and a continuous mapping $f : Y \to X$ with $f(Y) = A$. A subset C of a Polish space X is called *coanalytic* ($\mathbf{\Pi^1_1}$) if $X \setminus C$ is analytic. A set $C \subset X$ is $\mathbf{\Pi^1_1}$-complete if C is $\mathbf{\Pi^1_1}$ and for every Polish space Y and every $\mathbf{\Pi^1_1}$ set $B \subset Y$ there exists a Borel mapping $f : Y \to X$ with $B = f^{-1}(C)$ (cf. [K_1, Definition 26.7 and the following remark] and [K_2]). Recall that no $\mathbf{\Pi^1_1}$-complete set is analytic or Borel.

2. Point-set relations and a new version of Foran's lemma

The standard method [Za_1] of the proof that a concrete ("small") set is not σ-porous was formalized in [F]. A very general version of Foran's lemma [Za_2, Lemma 4.3] works with point-set relations ("abstract porosities") which have three basic properties of the notion of porosity. In the present article we found it useful to work with quite general point-set relations and "abstract porosities" are called *porosity-like* relations here.

The basic notion in Foran's lemma is this of Foran's system (of closed sets) in a topologically complete metric space. By a proof which is similar to the proof of Baire's theorem it is proved that no member of Foran's system is σ-porous.

The notion of (a new form of) Foran's system is quite fundamental for our proof.

Let (X, ρ) be a metric space. Then the open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. Let $A \subset X$, $A \neq \emptyset$, and $\varepsilon > 0$. Then the symbol $B(A, \varepsilon)$ stands for the set $\{y \in X;\ \rho(y, A) < \varepsilon\}$.

We say that $\mathbf{R}$ is a *point-set relation on* X if it is a relation between points of X and subsets of X. Thus a point-set relation $\mathbf{R}$ is a subset of $X \times 2^X$. The symbol $\mathbf{R}(x, A)$, where $x \in X$ and $A \subset X$, means that $(x, A) \in \mathbf{R}$, i.e. $\mathbf{R}$ holds for the pair (x, A).

We consider the following properties of a point-set relation $\mathbf{R}$ on X.

(A1) If $A \subset B \subset X$, $x \in X$ and $\mathbf{R}(x, B)$, then $\mathbf{R}(x, A)$.
(A2) $\mathbf{R}(x, A)$ iff there is $r > 0$ such that $\mathbf{R}(x, A \cap B(x, r))$.
(A3) $\mathbf{R}(x, A)$ iff $\mathbf{R}(x, \overline{A})$.

We say that a point-set relation $\mathbf{P}$ on X is a *porosity-like relation* if $\mathbf{P}$ satisfies the "axioms" **(A1)**–**(A3)**.

Let $\mathbf{P}$ be a porosity-like relation on X. We say that $A \subset X$ is

- $\mathbf{P}$-*porous at* $x \in X$ if $\mathbf{P}(x, A)$,
- $\mathbf{P}$-*porous* if $\mathbf{P}(x, A)$ for every $x \in A$,
- σ-$\mathbf{P}$-*porous* if A is a countable union of $\mathbf{P}$-porous sets.

The following notions and notation will be useful.

Notation 2.1. Let (X, ρ) be a metric space and $\mathbf{R}$ be a point-set relation on X.

- If $A \subset X$ and $B \subset X$, then
$$\mathbf{R}(A, B) \overset{\text{def}}{\iff} \forall\, a \in A: \ \mathbf{R}(a, B).$$
- Let moreover I be a nonempty index set and $\mathbf{R}_\iota$, $\iota \in I$, be point-set relations on X. Then the point-set relations $(\neg\mathbf{R})$, $\bigcup_{\iota \in I} \mathbf{R}_\iota$, $\bigcap_{\iota \in I} \mathbf{R}_\iota$ on X are defined in the natural way: $(\neg\mathbf{R})(x, A) \overset{\text{def}}{\iff} \neg(\mathbf{R}(x, A))$, $\left(\bigcup_{\iota \in I} \mathbf{R}_\iota\right)(x, A) \overset{\text{def}}{\iff} \exists\, \iota \in I: \ \mathbf{R}_\iota(x, A)$ and $\left(\bigcap_{\iota \in I} \mathbf{R}_\iota\right)(x, A) \overset{\text{def}}{\iff} \forall\, \iota \in I:$ $\mathbf{R}_\iota(x, A)$.
- We say that a point-set relation $\mathbf{R}_1$ is stronger than a point-set relation $\mathbf{R}_2$ if $\mathbf{R}_1 \subset \mathbf{R}_2$; i.e. $\mathbf{R}_1(x, A)$ implies $\mathbf{R}_2(x, A)$.

Now we can formulate and prove a new version of Foran's lemma.

Definition 2.2. Let X be a metric space. Let $\mathcal{O}$ be the system of all open subsets of X. Let $\mathbf{R}$ be a point-set relation on X and $\mathcal{F}$ be a nonempty system of nonempty closed subsets of X. We say that $\mathcal{F}$ is a *Foran-$\mathbf{R}$-system*, if for each $F \in \mathcal{F}$, $B \in \mathcal{O}$ with $F \cap B \neq \emptyset$ there exist $F^\star \in \mathcal{F}$ and $B^\star \in \mathcal{O}$ such that

- $F^\star \subset F \cap B$,
- $F^\star \cap B^\star \neq \emptyset$,
- $(\neg\mathbf{R})(F^\star \cap B^\star, F \cap B)$.

Let $\mathcal{T}$ be a set of point-set relations on X. We say that $\mathcal{F}$ is a *Foran-$\mathcal{T}$-system*, if $\mathcal{F}$ is a Foran-$\mathbf{T}$-system for every $\mathbf{T} \in \mathcal{T}$.

Remark 2.3. We will frequently use in the sequel the obvious fact that if all $\mathbf{T} \in \mathcal{T}$ satisfy **(A1)** and $\mathcal{B}$ is an arbitrary basis of open sets, then we can equivalently write $\mathcal{B}$ instead of $\mathcal{O}$ in the above definition.

Lemma 2.4. *Let $\mathcal{T}$ be a nonempty countable set of point-set relations on a complete metric space X. Suppose that*

- $\mathbf{P} = \bigcup \mathcal{T}$ *is a porosity-like relation,*
- *each* $\mathbf{T} \in \mathcal{T}$ *satisfies* **(A1)** *and* **(A3)**.

Let $\mathcal{F}$ be a Foran-$\mathcal{T}$-system. Then no set of $\mathcal{F}$ is σ-$\mathbf{P}$-porous.

The following proof is a slight modification of the proof of [Za$_2$, Lemma 4.3].

Proof: Suppose on the contrary that $F \in \mathcal{F}$ is σ-$\mathbf{P}$-porous. Then $F = \bigcup_{m=1}^{\infty} C_m$, where each C_m is $\mathbf{P}$-porous. For every $\mathbf{T} \in \mathcal{T}$ and $m \in \mathbb{N}$ we put
$$C(\mathbf{T}, m) = \{x \in C_m; \ \mathbf{T}(x, C_m)\}.$$

We have $C_m = \bigcup\{C(\mathbf{T}, m); \ \mathbf{T} \in \mathcal{T}\}$ and $\mathbf{T}(x, C(\mathbf{T}, m))$ for every $x \in C(\mathbf{T}, m)$ since $\mathbf{T}$ satisfies **(A1)**. We order the $C(\mathbf{T}, m)$'s into a sequence $\{A_n\}_{n=1}^{\infty}$. Clearly

$F = \bigcup_{n=1}^{\infty} A_n$. We will define inductively a sequence $\{F_n\}_{n=0}^{\infty}$ of elements of $\mathcal{F}$ and a sequence of open balls $\{B(x_n, r_n)\}_{n=0}^{\infty}$ such that $F_0 = F$, $\lim r_n = 0$ and for every $n \in \mathbb{N}$ we have

- $F_{n-1} \cap \overline{B(x_{n-1}, r_{n-1})} \supseteq F_n \cap \overline{B(x_n, r_n)}$,
- $F_{n-1} \cap B(x_{n-1}, r_{n-1}) \neq \emptyset$,
- $F_n \cap \overline{B(x_n, r_n)} \cap A_n = \emptyset$.

Put $F_0 = F$, choose $x_0 \in F$ and put $r_0 = 1$. If F_{n-1} and $B(x_{n-1}, r_{n-1})$ are defined for some $n \geq 1$, then we define F_n and $B(x_n, r_n)$ distinguishing two cases:

a) If $A_n \cap F_{n-1} \cap B(x_{n-1}, r_{n-1})$ is not dense in $F_{n-1} \cap B(x_{n-1}, r_{n-1})$, then we put $F_n = F_{n-1}$ and choose $x_n \in F_{n-1} \cap B(x_{n-1}, r_{n-1})$ and $r_n \in (0, 1/n)$ such that

- $F_{n-1} \cap \overline{B(x_n, r_n)} \cap A_n = \emptyset$,
- $B(x_n, r_n) \subset B(x_{n-1}, r_{n-1})$.

b) Suppose that $A_n \cap F_{n-1} \cap B(x_{n-1}, r_{n-1})$ is dense in $F_{n-1} \cap B(x_{n-1}, r_{n-1})$. There is some $\mathbf{T} \in \mathcal{T}$ such that $\mathbf{T}(x, A_n)$ for every $x \in A_n$. Using the definition of Foran-$\mathcal{T}$-system (for $F = F_{n-1}$, $B = B(x_{n-1}, r_{n-1})$ and $\mathbf{T}$) we find $F^\star \in \mathcal{F}$ and an open set $B^\star$ such that

- $F^\star \subset F_{n-1} \cap B(x_{n-1}, r_{n-1})$,
- $F^\star \cap B^\star \neq \emptyset$,
- $(\neg\mathbf{T})(F^\star \cap B^\star, F_{n-1} \cap B(x_{n-1}, r_{n-1}))$.

Since $F_{n-1} \cap B(x_{n-1}, r_{n-1}) \subset \overline{A_n}$ we have $(\neg\mathbf{T})(F^\star \cap B^\star, \overline{A_n})$. This implies $(\neg\mathbf{T})(F^\star \cap B^\star, A_n)$ by **(A3)**. Then we see that $F^\star \cap B^\star \cap A_n = \emptyset$, since $\mathbf{T}(x, A_n)$ for every $x \in A_n$. We put $F_n = F^\star$. Now we choose $x_n \in F_n$ and $r_n \in (0, 1/n)$ such that $\overline{B(x_n, r_n)} \subset B^\star$. This finishes the construction of the desired sequences.

Clearly $\bigcap_{n=1}^{\infty} \left(F_n \cap \overline{B(x_n, r_n)}\right) = \{a\}$, $a \in F$ and $a \notin \bigcup_{n=1}^{\infty} A_n = F$, a contradiction. $\square$

Remark 2.5. (i) In the above proof we proved a little bit more than Lemma 2.4 states. The assumption that $\mathbf{P}$ is a porosity-like relation is used only because of terminological reasons (otherwise the notion of σ-$\mathbf{P}$-porosity would not be well-defined). If we do not assume that $\mathbf{P}$ is a porosity-like relation then the conclusion of Lemma 2.4 can be formulated as follows: *Then no set of $\mathcal{F}$ can be written as a countable union of sets A_n's such that* $\mathbf{P}(A_n, A_n)$.

(ii) In the subsequent paper on "inscribing problem" (in contrast to the present article) we will apply Lemma 2.4 to relations which do not satisfy **(A2)**.

(iii) It is easy to see that the proof of Lemma 2.4 works if we write $F^\star \cap B^\star \subset F \cap B$ instead of $F^\star \subset F \cap B$ in Definition 2.2. Moreover, if $\mathbf{T}$ satisfies **(A2)**, then $(\neg\mathbf{T})(F^\star \cap B^\star, F \cap B)$ is equivalent to $(\neg\mathbf{T})(F^\star \cap B^\star, F)$. Thus in the case when $\mathcal{T}$ consists from one porosity-like relation, Lemma 2.4 coincides with [Za$_2$, Lemma 4.3].

It will be useful to introduce other properties **(A4)**, **(A5)**, **(D1)** and **(D2)** a point-set relation $\mathbf{R}$ on a metric space X can have, and the notion of a stable relation.

The first property is satisfied by most natural porosity-like relations.

(A4)
- $\mathbf{R}(x,\emptyset)$ for every $x \in X$.
- $(\neg\mathbf{R})(x,X)$ for every $x \in X$.
- $\mathbf{R}(x,\{x\})$ for every $x \in X$ which is not isolated.

The next property is satisfied by most interesting relations if X is locally compact.

(A5) For every $\varepsilon > 0$ and every nonempty compact $K \subset X$ there exists a countable system $\mathcal{S}$ of open sets such that
- $\bigcup \mathcal{S} \subset B(K,\varepsilon)$,
- $\mathcal{S}$ is locally finite in $X \setminus K$ (i.e. for every $y \in X \setminus K$ there is $r > 0$ such that $B(y,r)$ intersects at most finitely many elements of $\mathcal{S}$),
- if $J \subset X$ intersects each element of $\mathcal{S}$, then $(\neg\mathbf{R})(K, K \cup J)$.

Remark 2.6. It is easy to see that if a point-set relation $\mathbf{R}_1$ is stronger then $\mathbf{R}_2$ (i.e. $\mathbf{R}_1 \subset \mathbf{R}_2$) and $\mathbf{R}_2$ satisfies **(A5)**, then $\mathbf{R}_1$ satisfies **(A5)** as well.

The other property is a "descriptive" one.

(D1) For each compact set $K \subset X$, the set $\{x \in K;\ \mathbf{R}(x,K)\}$ is a $\boldsymbol{G_\delta}$ set.

To formulate two other properties, denote by $\mathcal{C}_b(X)$ the hyperspace of all nonempty bounded closed subsets of a metric space X equipped with the Hausdorff metric and let $\mathcal{K}^*(X)$ be the subspace of $\mathcal{C}_b(X)$ formed by all nonempty compact sets.

The second descriptive property reads as follows.

(D2) The set

$$\{(L,K) \in \mathcal{K}^*(X)\times\mathcal{K}^*(X);\ \exists\, O \subset X \text{ open} : L\cap O \neq \emptyset,\ (\neg\mathbf{R})(L\cap O, K\cap G)\}$$

is analytic for every $G \subset X$ open.

Note that c-porosity in separable complete metric space and c-symmetric porosity in $\mathbb{R}$ (in contrast to ordinary porosity and symmetric porosity) have properties **(D1)** and **(D2)** but it is not quite easy to verify it. To formulate a condition implying **(D1)** and **(D2)**, the verification of which is almost immediate (see Lemma 3.7), we need the natural notion of a stable point-set relation. Note that no interesting porosity-like relation is stable, but they are generated in simple ways by stable relations.

Definition 2.7. We say that a point-set relation $\mathbf{R}$ on a metric space X is *stable*, if the set

$$\{(x,F) \in X \times \mathcal{C}_b(X);\ \mathbf{R}(x,F)\}$$

is open.

We will need also the natural notion of the kernel of a set with respect to a porosity-like relation.

Definition 2.8. Let $\mathbf{Q}$ be a porosity-like relation on a metric space X. If $A \subset X$, then we define

$$\ker_{\mathbf{Q}}(A) = A \setminus \bigcup \{O;\ O \subset X \ \text{ is open, } \ A \cap O \ \text{ is } \sigma\text{-}\mathbf{Q}\text{-porous}\}.$$

We will need the following lemma.

Lemma 2.9. *Let $\mathbf{Q}$ be a porosity-like relation on a metric space X.*

(i) *If $A \subset X$ is closed, then $\ker_{\mathbf{Q}}(A)$ is closed.*
(ii) *The set $\ker_{\mathbf{Q}}(A) \cap G$ is non-σ-$\mathbf{Q}$-porous for every open set G intersecting $\ker_{\mathbf{Q}}(A)$.*

Proof: The statement (i) is obvious. The assertion (ii) easily follows from the fact that $M \subset X$ is σ-$\mathbf{Q}$-porous iff for each $x \in M$ there exists a neighbourhood U_x of x such that $M \cap U_x$ is σ-$\mathbf{Q}$-porous. This fact is easy to see if X is separable; for the non-separable case (which we need not in the present article) see [Za$_4$, Lemma 3]. $\square$

The following simple but useful lemma can be considered as a partial converse of Foran's lemma. Related results were already applied in [Za$_3$] and in [ZP].

Lemma 2.10. *Let X be a separable metric space, $\mathbf{P}$ be a porosity-like relation on X and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi \in \Xi} \mathbf{V}^{\xi}$, where each point-set relation $\mathbf{V}^{\xi}$ satisfies* **(A1)**, **(A3)** *and* **(D1)**. *Let $F_0 \subset X$ be a compact non-σ-$\mathbf{P}$-porous set. Then there exists a countable Foran-$\{\mathbf{V}^{\xi};\ \xi \in \Xi\}$-system $\mathcal{F}$ such that each element of $\mathcal{F}$ is compact and is contained in F_0.*

Proof: Fix a countable open basis $\mathcal{B}$ of X. To construct $\mathcal{F}$, we will need the following assertion.

(*) Let $K \in \mathcal{K}^*(X)$, $\ker_{\mathbf{P}}(K) = K$, $\xi \in \Xi$, $B \in \mathcal{B}$ and $K \cap B \neq \emptyset$. Then there exists a set $L = L(K, B, \xi) \in \mathcal{K}^*(X)$ such that $\ker_{\mathbf{P}}(L) = L$, $L \subset K \cap B$ and $(\neg \mathbf{V}^{\xi})(L, K \cap B)$.

To prove (*), choose $x \in K \cap B$, a neighbourhood U of x with $\overline{U} \subset B$ and put $K_1 := K \cap \overline{U}$. Since $x \in \ker_{\mathbf{P}}(K)$, K_1 is not σ-$\mathbf{P}$-porous. Since $\mathbf{V}^{\xi}$ satisfies **(A1)** and **(D1)**, $\{x \in K_1;\ (\neg \mathbf{V}^{\xi})(x, K_1)\}$ is an F_σ non-σ-$\mathbf{P}$-porous set. Therefore there exists a non-σ-$\mathbf{P}$-porous compact set $K_2 \subset K_1$ such that $(\neg \mathbf{V}^{\xi})(K_2, K_1)$. By **(A1)** we have $(\neg \mathbf{V}^{\xi})(K_2, K \cap B)$. By Lemma 2.9 we can put $L := \ker_{\mathbf{P}}(K_2)$.

Now we will define by induction countable systems $\mathcal{F}_n$, $n = 0, 1, \dots$, of nonempty compact sets. Put $\mathcal{F}_0 := \{\ker_{\mathbf{P}}(F_0)\}$. Further suppose that $n \in \mathbb{N}$ and we have defined $\mathcal{F}_{n-1}$. Then we put

$$\mathcal{F}_n := \{L(F, B, \xi);\ F \in \mathcal{F}_{n-1}, B \in \mathcal{B}, F \cap B \neq \emptyset, \xi \in \Xi\}.$$

It is easy to see that

$$\mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n$$

forms a countable Foran-$\{\mathbf{V}^{\xi};\ \xi \in \Xi\}$-system of compact sets, which are contained in F_0. $\square$

3. The hyperspace of compact sets and related lemmas

Let (E, ρ) be a metric space. Recall that $\mathcal{C}_b(E)$ denotes the space of all nonempty bounded closed subsets of E with the Hausdorff metric

$$h(F, C) = \sup\{\max\{\rho(x, F), \rho(y, C)\};\ x \in C, y \in F\}$$

and $\mathcal{K}^*(E)$ is its subspace of all nonempty compact subsets of E.

It will be useful to work with the metric space $\mathcal{K}(E)$ of *all* compact subsets of E. The space $\mathcal{K}(E)$ is also equipped with Hausdorff metric; the empty set is considered as an isolated point of $\mathcal{K}(E)$. (To have a fixed metric on $\mathcal{K}(E)$, we choose $a \in E$ and the distance of $\emptyset$ and a nonempty compact subset $K \subset E$ define as $h(\emptyset, K) := \operatorname{dist}(a, K) + \operatorname{diam}(K) + 1$.)

Remark 3.1. It is well-known (cf. [M]) that the topology on $\mathcal{K}(E)$ can be characterized by the fact that the sets

$$\{K \in \mathcal{K}(E);\ K \cap O \neq \emptyset\}, \qquad \{K \in \mathcal{K}(E);\ K \subset O\},$$

where $O \subset E$ is open, form an open subbasis of $\mathcal{K}(E)$.

Also the following facts are well-known.

Lemma 3.2. *Let E be a metric space.*

(i) *If E is compact (complete, separable, locally compact), then $\mathcal{K}(E)$ has the same property.*

(ii) *If $K_1 \subset K_2 \subset \dots$, $K = \overline{\bigcup_{i=1}^{\infty} K_i}$ and $K \in \mathcal{K}(E)$, $K_i \in \mathcal{K}(E)$, then $K_i \to K$ in $\mathcal{K}(E)$.*

(iii) *The set $\{(F, x) \in \mathcal{K}(E) \times E;\ x \in F\}$ is closed in $\mathcal{K}(E) \times E$.*

(iv) *The set $\{(L, K) \in \mathcal{K}(E) \times \mathcal{K}(E); L \subset K\}$ is closed in $\mathcal{K}(E) \times \mathcal{K}(E)$.*

(v) *If $G \subset E$ is a $\boldsymbol{G_\delta}$ set, then $\{K \in \mathcal{K}(E); K \subset G\}$ is a $\boldsymbol{G_\delta}$ set in $\mathcal{K}(E)$.*

We will need also the following facts; we omit their easy and straightforward proofs.

Lemma 3.3. (i) *Let E be a compact metric space. Let $K \in \mathcal{K}(E)$ and $\mathcal{S}$ be a system of open sets, which is locally finite in $E \setminus K$. Suppose that for every $S \in \mathcal{S}$ a set $K_S \in \mathcal{K}(E)$ with $K_S \subset S$ is given. Then $K \cup \bigcup\{K_S;\ S \in \mathcal{S}\} \in \mathcal{K}(E)$.*

(ii) *Let E be a metric space. Let $K \in \mathcal{K}(E)$. Then there exists a system $\mathcal{S}$ of open subsets of E such that $\mathcal{S}$ is locally finite in $E \setminus K$ and for every $x \in K$ and $r > 0$ there exists $S \in \mathcal{S}$ with $S \subset B(x, r)$.*

Lemma 3.4. *Let E be a separable metric space and $O \subset E$ be open. Then the mapping $F_O : \mathcal{K}(E) \to \mathcal{K}(E)$ defined by $F_O(K) := \overline{K \cap O}$ is Borel measurable.*

PROOF: Fix $O \subset E$ open. The space $\mathcal{K}(E)$ is separable by Lemma 3.2(i). Thus it is sufficient to show that $F_O^{-1}(V)$ is Borel for every V from some fixed subbasis of $\mathcal{K}(E)$. This implies that it is sufficient to prove that the sets

$$\mathcal{X} = \{K \in \mathcal{K}(E);\ \overline{K \cap O} \cap G \neq \emptyset\}, \qquad \mathcal{Y} = \{K \in \mathcal{K}(E);\ \overline{K \cap O} \subset G\}$$

are Borel for every G open in E. The set $\mathcal{X}$ is open since

$$\mathcal{X} = \{K \in \mathcal{K}(E);\ K \cap O \cap G \neq \emptyset\}.$$

We have also

$$\begin{aligned}\mathcal{K}(E) \setminus \mathcal{Y} &= \{K \in \mathcal{K}(E);\ \overline{K \cap O} \cap (E \setminus G) \neq \emptyset\} \\ &= \bigcap_{n=1}^{+\infty} \{K \in \mathcal{K}(E);\ K \cap O \cap B(E \setminus G, 1/n) \neq \emptyset\}\end{aligned}$$

and therefore $\mathcal{Y}$ is Borel. □

Definition 3.5. Let E be a metric space. A set $\mathcal{I} \subset \mathcal{K}(E)$ is called a *σ-ideal of compact sets* if the following conditions hold:

- $\mathcal{I}$ is *hereditary*, i.e. $K, L \in \mathcal{K}(E)$, $K \in \mathcal{I}$, $L \subset K$, then $L \in \mathcal{I}$;
- if $K, K_1, K_2, \dots \in \mathcal{K}(E)$, $K_n \in \mathcal{I}$ for all $n \in \mathbb{N}$ and $K = \bigcup_{n=1}^{+\infty} K_n$, then $K \in \mathcal{I}$.

The theory of σ-ideals of compact sets was developed by Kechris, Louveau and Woodin in [KLW]. Their results were applied in the theory of trigonometric series (cf. [DSR], [KL]) and in other fields. We will need the so called dichotomy theorem.

Theorem 3.6 (Kechris-Louveau-Woodin, [K_1, Theorem 33.3])**.** *Let E be a Polish space and $\mathcal{I} \subset \mathcal{K}(E)$ be a $\boldsymbol{\Pi}^1_1$ σ-ideal of compact sets. Then $\mathcal{I}$ is either $\boldsymbol{\Pi}^1_1$-complete or a $\boldsymbol{G_\delta}$ subset of $\mathcal{K}(E)$.*

Lemma 3.7. *Let E be a separable complete metric space. Let $\mathbf{R}_k$, $k \in \mathbb{N}$, be stable point-set relations on E with* **(A1)** *and* **(A3)***. Then the relations $\mathbf{V}_1 := \bigcap_{k=1}^{\infty} \mathbf{R}_k$ and $\mathbf{V}_2 := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathbf{R}_k$ have the properties* **(A1)**, **(A3)**, **(D1)** *and* **(D2)**.

Proof: We start with $\mathbf{V}_1$. It is easy to see that $\mathbf{V}_1$ has **(A1)** and **(A3)**.

(D1) Fix $K \in \mathcal{K}(E)$. We have

$$C := \{x \in K;\ \mathbf{V}_1(x, K)\} = \bigcap_{k=1}^{\infty} \{x \in K;\ \mathbf{R}_k(x, K)\}.$$

Stability of $\mathbf{R}_k$'s gives that C is $\boldsymbol{G_\delta}$.

(D2) Let $\mathcal{B}$ be a countable open basis of E. Fix $G \subset E$ open. Denote

$$\mathcal{A} = \{(L, K) \in \mathcal{K}(E) \times \mathcal{K}(E);\ \exists\, O \subset E \text{ open}:\ L \cap O \neq \emptyset,\ (\neg \mathbf{V}_1)(L \cap O, K \cap G)\}.$$

We have

$$(L, K) \in \mathcal{A} \Leftrightarrow \exists\, O_1 \in \mathcal{B} : \emptyset \neq L \cap O_1 \subset \bigcup_{k=1}^{\infty} \{x \in E;\ (\neg \mathbf{R}_k)(x, K \cap G)\}.$$

Stability and **(A3)** of $\mathbf{R}_k$ give that the set $\{x \in E;\ (\neg \mathbf{R}_k)(x, K \cap G)\}$ is closed for every $k \in \mathbb{N}$ and therefore Baire Category Theorem and **(A3)** of $\mathbf{R}_k$ imply

$$\begin{aligned}(L, K) \in \mathcal{A} &\Leftrightarrow \exists\, O_2 \in \mathcal{B}\, \exists\, k \in \mathbb{N}:\ \emptyset \neq L \cap O_2 \subset \{x \in E;\ (\neg \mathbf{R}_k)(x, K \cap G)\} \\ &\Leftrightarrow \exists\, O_3 \in \mathcal{B}\, \exists\, k \in \mathbb{N} : L \cap O_3 \neq \emptyset,\ \overline{L \cap O_3} \subset \{x \in E;\ (\neg \mathbf{R}_k)(x, \overline{K \cap G})\}.\end{aligned}$$

The set

$$\mathcal{C}(k) := \{(K_1, K_2) \in \mathcal{K}(E) \times \mathcal{K}(E);\ K_1 \not\subset \{x \in E;\ (\neg \mathbf{R}_k)(x, K_2)\}\}$$

is open by stability of $\mathbf{R}_k$. Therefore Lemma 3.4 and the equivalence

$$(L, K) \in \mathcal{A} \Leftrightarrow \exists\, O \in \mathcal{B}\, \exists\, k \in \mathbb{N}:\ L \cap O \neq \emptyset, (F_O(L), F_G(K)) \notin \mathcal{C}(k)$$

imply that $\mathcal{A}$ is Borel and therefore analytic.

To prove the desired properties of $\mathbf{V}_2$, it is sufficient to apply just the proved result to the point-set relations $\mathbf{R}_n^* := \bigcup_{k=n}^{\infty} \mathbf{R}_k$ which are clearly stable and have properties **(A1)** and **(A3)**. □

Notation 3.8. Let E be a metric space and Ξ be a nonempty countable set. Let $\mathbf{V}^\xi$, $\xi \in \Xi$, be point-set relations on E. Let $\mathfrak{M}$ be the space $[\mathcal{K}^*(E)]^{\mathbb{N}}$ equipped with the product topology. Let $\mathcal{O}$ be the set of all open subsets of E. We define a relation $\prec^*$ between elements of $\mathfrak{M}$ by

$$\begin{aligned}\mathcal{X} \prec^* \mathcal{Z} \overset{\text{def}}{\Longleftrightarrow} &\forall \xi \in \Xi \, \forall n \in \mathbb{N} \, \forall B \in \mathcal{O}, \mathcal{Z}(n) \cap B \neq \emptyset \, \exists m \in \mathbb{N} \, \exists B^\star \in \mathcal{O} : \\ &\mathcal{X}(m) \cap B^\star \neq \emptyset, \ \mathcal{X}(m) \subset \mathcal{Z}(n) \cap B, \ (\neg \mathbf{V}^\xi)(\mathcal{X}(m) \cap B^\star, \mathcal{Z}(n) \cap B).\end{aligned}$$

Remark 3.9. The relation $\prec^*$ depends on the considered relations $\mathbf{V}^\xi$, $\xi \in \Xi$. However, we will use the simple symbol $\prec^*$ instead of a symbol indicating the relationship to $\mathbf{V}^\xi$'s since this simplification will lead to no confusion.

Remark 3.10. It is easy to see that if all $\mathbf{V}^\xi$ satisfy **(A1)** and $\mathcal{B}$ is an arbitrary basis of open sets, then we can equivalently write $\mathcal{B}$ instead of $\mathcal{O}$ in Notation 3.8.

Remark 3.11. Let $\mathcal{X} \in \mathfrak{M}$. Then the set $\{\mathcal{X}(n);\ n \in \mathbb{N}\}$ forms a Foran-$\{\mathbf{V}^\xi;\ \xi \in \Xi\}$-system if and only if $\mathcal{X} \prec^* \mathcal{X}$.

Lemma 3.12. *Let E be a complete metric space and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi \in \Xi} \mathbf{V}^\xi$ be a porosity-like relation on E, where each point-set relation $\mathbf{V}^\xi$ satisfies* **(A1)** *and* **(A3)**. *Let $\mathcal{F}_n \in \mathfrak{M}$, $n \in \mathbb{N}$, and $\mathcal{F}_{n+1} \prec^* \mathcal{F}_n$ for every $n \in \mathbb{N}$. Then $\mathcal{F}_1$ contains only non-σ-$\mathbf{P}$-porous sets.*

Proof: We put $\mathcal{U} = \bigcup_{n=1}^\infty \mathcal{F}_n$ and $\mathcal{T} = \{\mathbf{V}^\xi;\ \xi \in \Xi\}$. It is easy to check that the system $\mathcal{U}$ forms a Foran-$\mathcal{T}$-system. The conclusion follows from Lemma 2.4. □

Lemma 3.13. *Let E be a separable complete metric space and Ξ be a nonempty countable set. Let $\mathbf{V}^\xi$, $\xi \in \Xi$, be point-set relations on E satisfying* **(A1)** *and* **(D2)**. *Then the relation $\prec^*$ is analytic on $\mathfrak{M}$, i.e. the set*

$$\{(\mathcal{X}, \mathcal{Z}) \in \mathfrak{M} \times \mathfrak{M};\ \mathcal{X} \prec^* \mathcal{Z}\}$$

is analytic.

Proof: We fix a countable open basis $\mathcal{B}$ of E. For $\xi \in \Xi$, $B \in \mathcal{B}$ we define

$$\begin{aligned}\mathcal{A}(\xi, B) = \{&(L, K) \in \mathcal{K}^*(E) \times \mathcal{K}^*(E);\ (K \cap B = \emptyset) \vee \\ &((L \subset K \cap B) \ \& \ (\exists B^\star \in \mathcal{B} :\ L \cap B^\star \neq \emptyset,\ (\neg \mathbf{V}^\xi)(L \cap B^\star, K \cap B)))\}.\end{aligned}$$

By Remark 3.10 we have

$$\mathcal{X} \prec^* \mathcal{Z} \Leftrightarrow \forall \xi \in \Xi \, \forall n \in \mathbb{N} \, \forall B \in \mathcal{B} \, \exists m \in \mathbb{N} :\ (\mathcal{X}(m), \mathcal{Z}(n)) \in \mathcal{A}(\xi, B).$$

The sets Ξ and $\mathcal{B}$ are countable and therefore it is sufficient to prove that $\mathcal{A}(\xi, B)$ is analytic for every $\xi \in \Xi$ and $B \in \mathcal{B}$. To this end we fix ξ, B and define

$$\mathcal{A}_1 = \{(L,K) \in \mathcal{K}^*(E) \times \mathcal{K}^*(E);\ K \cap B = \emptyset\},$$
$$\mathcal{A}_2 = \{(L,K) \in \mathcal{K}^*(E) \times \mathcal{K}^*(E);\ L \subset K\},$$
$$\mathcal{A}_3 = \{(L,K) \in \mathcal{K}^*(E) \times \mathcal{K}^*(E);\ L \subset B\},$$
$$\mathcal{A}_4 = \{(L,K) \in \mathcal{K}^*(E) \times \mathcal{K}^*(E);\ \exists B^\star \in \mathcal{B} : L \cap B^\star \neq \emptyset,\ (\neg \mathbf{V}^\xi)(L \cap B^\star, K \cap B)\}.$$

We have $\mathcal{A}(\xi, B) = \mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4)$. The sets $\mathcal{A}_1$ and $\mathcal{A}_3$ are open and the set $\mathcal{A}_2$ is closed by Lemma 3.2(iv). Since $\mathbf{V}^\xi$ satisfies **(D2)**, we have that $\mathcal{A}_4$ is analytic. □

Lemma 3.14. *Let E be a separable locally compact metric space and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi \in \Xi} \mathbf{V}^\xi$ be a porosity-like relation on E, where each $\mathbf{V}^\xi$ satisfies* **(A1)**, **(A3)**, **(D1)** *and* **(D2)**. *Then the σ-ideal $\mathcal{I}$ of all compact σ-$\mathbf{P}$-porous subsets of E forms a $\mathbf{\Pi}^1_1$ subset of $\mathcal{K}(E)$.*

Proof: Recall that $\mathfrak{M} = [\mathcal{K}^*(E)]^{\mathbb{N}}$. We define $\mathfrak{D} \subset \mathcal{K}(E) \times \mathfrak{M}$ by

$$(K, \mathcal{X}) \in \mathfrak{D} \overset{\text{def}}{\Longleftrightarrow} (\mathcal{X} \prec^* \mathcal{X})\ \&\ (\forall k \in \mathbb{N} :\ \mathcal{X}(k) \subset K).$$

According to Lemmas 3.2(iv) and 3.13 the set $\mathfrak{D}$ is analytic.

Using Lemma 2.10, Remark 3.11 and Lemma 2.4 we have that $K \in \mathcal{K}(E)$ is non-σ-$\mathbf{P}$-porous set if and only if there is $\mathcal{X}$ with $(K, \mathcal{X}) \in \mathfrak{D}$. The space $\mathfrak{M}$ is Polish and $\mathcal{K}(E) \setminus \mathcal{I}$ is a projection of $\mathfrak{D}$. Thus $\mathcal{K}(E) \setminus \mathcal{I}$ is analytic and we are done. □

4. Definition of a rank and the basic construction

Definition 4.1. Let E be a metric space, Ξ be a nonempty countable set and $\mathbf{V}^\xi$, $\xi \in \Xi$, be point-set relations on E. We will define systems $\mathcal{C}_\alpha$, $\alpha \leq \omega_1$, of nonempty compact subsets of E inductively. We put $\mathcal{C}_0 = \mathcal{K}^*(E)$. The system $\mathcal{C}_\alpha$, $0 < \alpha \leq \omega_1$, is defined by

$$\begin{aligned} K \in \mathcal{C}_\alpha \overset{\text{def}}{\Longleftrightarrow} (K \in \mathcal{K}^*(E)\ \&\ (\forall \beta < \alpha \, \forall \xi \in \Xi \\ \forall B \subset E \text{ open}, K \cap B \neq \emptyset \, \exists L \in \mathcal{C}_\beta :\ L \subset K \cap B, (\neg \mathbf{V}^\xi)(L, K \cap B))). \end{aligned}$$

Using the previous definition we define a rank function on $\mathcal{K}^*(E)$.

Definition 4.2. For any $K \in \mathcal{K}^*(E)$ set $\mathrm{rk}(K) = \sup\{\alpha;\ K \in \mathcal{C}_\alpha\}$.

Remark 4.3. (1) Clearly $\mathcal{C}_\gamma \subset \mathcal{C}_\alpha$ for $\alpha \le \gamma \le \omega_1$.
(2) It is easy to see that $\mathrm{rk}(K) \ge \alpha$ iff $K \in \mathcal{C}_\alpha$ and thus $\mathrm{rk}(K) = \max\{\alpha;\ K \in \mathcal{C}_\alpha\}$.
(3) Working with $\mathcal{C}_\alpha$'s and rk we will have in mind that these objects depend on $\mathbf{V}^\xi$'s. At each moment it will be clear which $\mathbf{V}^\xi$'s are considered.

The following lemma follows directly from the definition of rk, from the property **(A1)** of $\mathbf{V}^\xi$'s and from Remark 4.3(2).

Lemma 4.4. *Let E be a metric space, Ξ be a nonempty countable set, $\mathbf{V}^\xi$, $\xi \in \Xi$, be point-set relations on E with* **(A1)** *and let $\alpha < \omega_1$. Then the following assertions hold.*

(i) *Let $\mathcal{K} \subset \mathcal{K}^*(E)$, $\mathcal{K} \neq \emptyset$ and $\mathrm{rk}(K) \ge \alpha$ for each $K \in \mathcal{K}$. Then $\mathrm{rk}\left(\overline{\bigcup \mathcal{K}}\right) \ge \alpha$.*
(ii) *Let $K \in \mathcal{K}^*(E)$, $\mathrm{rk}(K) \ge \alpha$, $H \subset E$ be open and $K \cap H \neq \emptyset$. Then $\mathrm{rk}\left(\overline{K \cap H}\right) \ge \alpha$.*

Lemma 4.5. *Let $\alpha < \omega_1$. Let E be a locally compact metric space, D be a dense subset of E, Ξ be a nonempty countable set and $\mathbf{V}^\xi$, $\xi \in \Xi$, be point-set relations on E with* **(A1)**, **(A2)** *and* **(A5)**.

Let $H \subset E$ be a nonempty open subset and let $\mathcal{G} \subset \mathcal{K}(E)$ be a $\boldsymbol{G_\delta}$ σ-ideal of compact sets such that $\{\{x\}; x \in D\} \subset \mathcal{G}$. Then there exists $K \in \mathcal{G}$ such that $\emptyset \neq K \subset H$ and $\mathrm{rk}(K) \ge \alpha$.

To prove Lemma 4.5 we will need the next lemma.

Lemma 4.6. *Let E be a locally compact metric space, D be a dense subset of E, Ξ be a nonempty countable set and $\mathbf{V}^\xi$, $\xi \in \Xi$, be point-set relations on E with* **(A1)** *and* **(A5)**. *Let $\alpha < \omega_1$ and suppose that Lemma 4.5 holds for α. Let $H \subset E$ be a nonempty open set, $\mathcal{G} \subset \mathcal{K}(E)$ be a $\boldsymbol{G_\delta}$ σ-ideal of compact sets such that $\{\{x\}; x \in D\} \subset \mathcal{G}$, $\varepsilon > 0$, $\xi \in \Xi$ and $K \in \mathcal{G}$, $\emptyset \neq K \subset H$. Then there exists a nonempty compact set $L \in \mathcal{G}$, $K \subset L \subset H$, such that*

- $\mathrm{rk}(L) \ge \alpha$,
- $L \subset B(K, \varepsilon)$,
- $(\neg \mathbf{V}^\xi)(K, L)$.

Proof: We may and do assume that $B(K,\varepsilon) \subset H$ and that $\overline{H}$ is compact. Using **(A5)** of $\mathbf{V}^\xi$ we find a countable system $\mathcal{S}$ of open sets such that

(i) $\bigcup \mathcal{S} \subset B(K, \varepsilon)$,
(ii) $\mathcal{S}$ is locally finite in $E \setminus K$,
(iii) if $J \subset E$ intersects each element of $\mathcal{S}$, then $(\neg \mathbf{V}^\xi)(K, K \cup J)$.

Adding countably many appropriate open sets into $\mathcal{S}$, if necessary, we may assume that if J intersects each element of $\mathcal{S}$, then $K \subset \overline{J}$ (see Lemma 3.3(ii)).

We find for every $S \in \mathcal{S}$ a nonempty compact set $K_S \in \mathcal{G}$ such that $K_S \subset S$ and $\mathrm{rk}(K_S) \ge \alpha$. We put $L = K \cup \bigcup\{K_S;\ S \in \mathcal{S}\}$. Lemma 3.3(i) shows that $L \in \mathcal{K}^*(E)$. Then we have $L \in \mathcal{G}$ and also $L = \overline{\bigcup\{K_S;\ S \in \mathcal{S}\}}$. Using Lemma 4.4(i) we obtain that $\mathrm{rk}(L) \geq \alpha$.

The second condition is obviously satisfied. Using (iii) we obtain $(\neg\mathbf{V}^{\xi})(K, L)$ and therefore the last condition is also satisfied. □

Proof of Lemma 4.5: Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence containing each element of Ξ infinitely many times. We will proceed by transfinite induction.

If $\alpha = 0$, it is sufficient to choose $x \in H \cap D$ and put $K := \{x\}$. Now suppose that $\alpha > 0$ and the assertion holds for every $\beta < \alpha$. Let $\{\alpha_j\}_{j=1}^{\infty}$ be a nondecreasing sequence of ordinal numbers such that $\alpha_j < \alpha$, $j \in \mathbb{N}$, and $\lim(\alpha_j + 1) = \alpha$. The set $\mathcal{K}(H)$ is open in $\mathcal{K}(E)$ and therefore we can find a complete metric $\tilde{h}$ on $\mathcal{G} \cap \mathcal{K}(H)$ which is (on this set) equivalent to the Hausdorff metric h.

Now we will construct a sequence $\{K_j\}_{j=1}^{\infty}$ of nonempty compact sets such that the following conditions are satisfied for every $j \in \mathbb{N}$:

- $K_j \in \mathcal{G} \cap \mathcal{K}(H)$,
- $\mathrm{rk}(K_j) \geq \alpha_j$,
- $K_j \subset K_{j+1}$,
- $\tilde{h}(K_j, K_{j+1}) < 2^{-j}$,

(v) $(\neg\mathbf{V}^{\xi_j})(K_j, K_{j+1})$.

According to the induction hypothesis we find $K_1 \in \mathcal{G}$ with $\emptyset \neq K_1 \subset H$ and $\mathrm{rk}(K_1) \geq \alpha_1$. Now assume that we have defined $K_1, \dots, K_m$. We find $\varepsilon > 0$ so small that $\tilde{h}(K_m, K_m \cup T) < 2^{-m}$ whenever $T \subset B(K_m, \varepsilon)$ and $T \in \mathcal{G} \cap \mathcal{K}(H)$. Using Lemma 4.6 for $\alpha := \alpha_{m+1}$, $H := H$, $\varepsilon := \varepsilon$, $\xi := \xi_m$ and for $K := K_m$ we obtain $K_{m+1} \in \mathcal{G} \cap \mathcal{K}(H)$ with the desired properties.

Now we put $K = \overline{\bigcup_{j=1}^{\infty} K_j}$. Since $\{K_j\}_{j=1}^{\infty}$ forms a Cauchy sequence in $(\mathcal{G} \cap \mathcal{K}(H), \tilde{h})$, the sequence $\{K_j\}_{j=1}^{\infty}$ converges to some $K^{\star} \in \mathcal{G} \cap \mathcal{K}(H)$ with respect to $\tilde{h}$ and consequently also with respect to h. Lemma 3.2(ii) gives $K^{\star} = K$. Thus we have $K \in \mathcal{G} \cap \mathcal{K}(H)$.

To verify $\mathrm{rk}(K) \geq \alpha$, we will prove $\mathrm{rk}(K) \in \mathcal{C}_{\alpha}$ using Definition 4.1. To this end consider arbitrary $\beta < \alpha$, $\xi \in \Xi$ and an open set $B \subset E$ intersecting K. We can clearly find $j_0 \in \mathbb{N}$ such that $\mathrm{rk}(K_j) \geq \beta$ and $K_j \cap B \neq \emptyset$ for each $j \geq j_0$. Choose $j \geq j_0$ such that $\xi_j = \xi$. Then find an open set $H \subset E$ with $H \cap K_j \neq \emptyset$, $\overline{H} \subset B$ and put $L := \overline{H \cap K_j}$. Then clearly $L \subset K \cap B$ and (v) implies $(\neg\mathbf{V}^{\xi})(L, K)$. Since $\mathbf{V}^{\xi}$ satisfies **(A2)**, we have $(\neg\mathbf{V}^{\xi})(L, K \cap B)$. By Lemma 4.4(ii) we obtain $\mathrm{rk}(L) \geq \beta$, i.e. $L \in \mathcal{C}_{\beta}$. Thus we have proved $K \in \mathcal{C}_{\alpha}$ which completes the proof. □

5. Application of a version of the overspill method

Recall that a (binary) relation $\prec$ on a set X is said to be well-founded, if there is no sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X with $x_{n+1} \prec x_n$ for every $n \in \mathbb{N}$. For a well-founded relation $\prec$ on X there exists (see [K$_1$, Appendix B]) a unique function (called the rank function of $\prec$) which assigns to each $x \in X$ an ordinal number $\rho(x, \prec)$ such that

$$\rho(x, \prec) = \sup\{\rho(y, \prec) + 1; y \prec x\}.$$

The classical Kunen-Martin theorem reads as follows.

Theorem 5.1 ([K$_1$, Theorem 31.1]). *Let X be a Polish space and $\prec$ be a well-founded analytic relation on X. Then* $\sup\{\rho(x, \prec);\ x \in X\} < \omega_1$.

We will use this deep theorem via the following lemma.

Lemma 5.2. *Let E be a separable locally compact metric space and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi \in \Xi} \mathbf{V}^\xi$ be a porosity-like relation on E, where each $\mathbf{V}^\xi$ satisfies* **(A1)**, **(A3)** *and* **(D2)**. *Let $\mathcal{G} \subset \mathcal{K}(E)$ be an analytic subset of $\mathcal{K}(E)$ with* $\sup\{\mathrm{rk}(K);\ K \in \mathcal{G}\} = \omega_1$. *Then $\mathcal{G}$ contains a non-σ-$\mathbf{P}$-porous set.*

Proof: Without any loss of generality we may assume that $\mathcal{G}$ is hereditary. Indeed, if not then we replace $\mathcal{G}$ by its hereditary closure, i.e. by the set $\mathrm{her}(\mathcal{G}) = \{L \in \mathcal{K}(E);\ \exists\, K \in \mathcal{G}:\ L \subset K\}$. Since $\mathcal{G}$ is analytic, $\mathrm{her}(\mathcal{G})$ is also analytic since $\mathrm{her}(\mathcal{G})$ is a projection of the analytic set $\{(L, K) \in \mathcal{K}(E) \times \mathcal{K}(E);\ L \subset K, K \in \mathcal{G}\}$. (However note that in all applications we will work with hereditary systems.)

Using Notation 3.8 put

$$\mathfrak{C} = \{\mathcal{X} \in \mathfrak{M};\ \forall\, n \in \mathbb{N}:\ \mathcal{X}(n) \in \mathcal{G}\}$$

and define a relation $\prec^{**}$ on $\mathfrak{M}$ by

$$\mathcal{X} \prec^{**} \mathcal{Z} \overset{\mathrm{def}}{\Longleftrightarrow} (\mathcal{X} \in \mathfrak{C},\ \mathcal{Z} \in \mathfrak{C},\ \mathcal{X} \prec^{*} \mathcal{Z}).$$

It is easy to see that $\mathfrak{C}$ is an analytic subset of $\mathfrak{M}$. Since $\prec^*$ is analytic (Lemma 3.13), we have that $\prec^{**}$ is also analytic.

Assume for a while that $\prec^{**}$ is well-founded. Then $\rho(\mathcal{X}, \prec^{**})$ is well defined for all $\mathcal{X} \in \mathfrak{M}$. To obtain a contradiction, we will need the following claim.

Claim. Let $\alpha < \omega_1$. If $\mathcal{X} \in \mathfrak{C}$ with $\mathcal{X}(m) \in \mathcal{C}_\alpha$ for every $m \in \mathbb{N}$, then $\rho(\mathcal{X}, \prec^{**}) \geq \alpha$.

Proof of Claim: We will proceed by transfinite induction. The case $\alpha = 0$ is trivial. Now suppose that $\alpha > 0$, Claim holds for each $\tilde{\alpha} < \alpha$ and $\mathcal{X} \in \mathfrak{C}$ satisfies $\{\mathcal{X}(m);\ m \in \mathbb{N}\} \subset \mathcal{C}_\alpha$. Fix a countable open basis $\mathcal{B}$ of E. Let $\beta < \alpha$. By Definition 4.1, for each $m \in \mathbb{N}$, $\xi \in \Xi$ and $B \in \mathcal{B}$ with $\mathcal{X}(m) \cap B \neq \emptyset$, we can choose $K(m, B, \xi) \in \mathcal{C}_\beta$ with $K(m, B, \xi) \subset \mathcal{X}(m) \cap B$ and $(\neg \mathbf{V}^\xi)(K(m, B, \xi), \mathcal{X}(m) \cap B)$. Since $\mathcal{G}$ is hereditary we have $K(m, B, \xi) \in \mathcal{G}$. Order all $K(m, B, \xi)$'s into a sequence $\mathcal{S}$. Obviously $\mathcal{S} \prec^{**} \mathcal{X}$. We obtain $\rho(\mathcal{S}, \prec^{**}) \geq \beta$ by induction hypothesis and therefore $\rho(\mathcal{X}, \prec^{**}) \geq \beta + 1$. This implies $\rho(\mathcal{X}, \prec^{**}) \geq \alpha$. □

According to Claim we see that for $K \in \mathcal{G}$ with $\mathrm{rk}(K) \geq \alpha$ we have the inequality $\rho(\{K\}_{j=1}^{\infty}, \prec^{**}) \geq \alpha$. Consequently $\sup\{\rho(\mathcal{X}, \prec^{**});\ \mathcal{X} \in \mathfrak{C}\} = \sup\{\rho(\mathcal{X}, \prec^{**});\ \mathcal{X} \in \mathfrak{M}\} = \omega_1$. Since $\mathfrak{M}$ is a Polish space we obtain a contradiction with Theorem 5.1. Thus $\prec^{**}$ is not well-founded and there exists a sequence $\{\mathcal{X}_n\}_{n=1}^{\infty}$ of elements of $\mathfrak{C}$ such that $\mathcal{X}_{n+1} \prec^{**} \mathcal{X}_n$ for every $n \in \mathbb{N}$. This implies $\mathcal{X}_{n+1} \prec^{*} \mathcal{X}_n$ for every $n \in \mathbb{N}$. Therefore Lemma 3.12 implies that each element $K \in \mathcal{X}_1 \neq \emptyset$ is non σ-$\mathbf{P}$-porous. Since $\mathcal{X}_1 \subset \mathcal{G}$, the lemma is proved. □

6. Main abstract result

The following abstract theorem is an easy consequence of previous results.

Theorem 6.1. *Let E be a nonempty separable locally compact metric space without isolated points and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi \in \Xi} \mathbf{V}^{\xi}$ be a porosity-like relation on E, where each point-set relation $\mathbf{V}^{\xi}$ satisfies* **(A1)**, **(A2)**, **(A3)**, **(A4)**, **(A5)**, **(D1)** *and* **(D2)**.

Then the σ-ideal $\mathcal{I} \subset \mathcal{K}(E)$ of all compact σ-$\mathbf{P}$-porous sets is $\mathbf{\Pi}^1_1$-complete. In particular, $\mathcal{I}$ is coanalytic non-Borel.

PROOF: According to Lemma 3.14 we have that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$. According to the dichotomy theorem (Theorem 3.6) we have that $\mathcal{I}$ is either $\mathbf{\Pi}^1_1$-complete or $\boldsymbol{G_\delta}$. Assume that $\mathcal{I}$ is $\boldsymbol{G_\delta}$ to the contradiction. The σ-ideal $\mathcal{I}$ contains all singletons since E has no isolated point and **(A4)** holds for $\mathbf{V}^{\xi}$'s. Applying Lemma 4.5 to $\mathcal{G} := \mathcal{I}$, $D := E$ we obtain that $\sup\{\mathrm{rk}(K);\ K \in \mathcal{I}\} = \omega_1$. Lemma 5.2 gives $\mathcal{I} \setminus \mathcal{I} \neq \emptyset$, a contradiction. □

In the sequel we will always use the above theorem together with Lemma 3.7, which gives simple sufficient conditions for **(D1)** and **(D2)**.

7. Applications to concrete porosities

Now we will apply our abstract Theorem 6.1 to σ-ideals of compact σ-$\langle g \rangle$-porous sets, σ-porous sets, σ-strongly porous sets, σ-symmetrically porous sets and σ-strongly symmetrically porous sets.

First of all we recall definitions of the mentioned porosities.

We denote

$$G := \{g : [0, +\infty) \to [0, +\infty);\ g(0) = 0,\ g(x) > x \text{ for every } x > 0, \\ g \text{ is nondecreasing and continuous}\}.$$

The symbol g_α, $\alpha \in \mathbb{R}$, stands for the function $x \mapsto \alpha x$, $x \in [0, +\infty]$.

Let X be a metric space, $A \subset X$, $x \in X$, $g \in G$ and $c > 0$.

We say that

- A is *$\langle g\rangle$-porous at x* if there exists a sequence of balls $\{B(x_n, r_n)\}_{n=1}^{\infty}$ such that $x \in B(x_n, g(r_n))$, $\lim x_n = x$, $B(x_n, r_n) \cap A = \emptyset$,
- A is c-porous at x if A is $\langle g_\alpha\rangle$-porous at x for $\alpha = 1/c$,
- A is (ordinary) *porous at x* if A is d-porous at x for some $d > 0$,
- A is *strongly porous at x* if A is d-porous at x for every $0 < d < 1$.

The point-set relations which correspond to $\langle g\rangle$-porosity, (ordinary) porosity and strong porosity are denoted by $\mathbf{P_g}$, $\mathbf{P_{or}}$ and $\mathbf{P_{st}}$, respectively.

Let $A \subset \mathbb{R}$, $x \in \mathbb{R}$ and $c > 0$. We say that

- A is *c-symmetrically porous at x*, if there exists a sequence $\{B(x_n, r_n)\}_{n=1}^{\infty}$ of balls in $\mathbb{R}$ such that $\lim x_n = x$, $x \in B(x_n, r_n/c)$ and $(B(x_n, r_n) \cup B(x + (x - x_n), r_n)) \cap A = \emptyset$,
- A is *symmetrically porous at x*, if it is d-symmetrically porous at x for some $d > 0$,
- A is *strongly symmetrically porous at x* if it is d-symmetrically porous at x for every $0 < d < 1$.

The point-set relations which correspond to symmetrical porosity and strong symmetrical porosity are denoted by $\mathbf{P_{sy}}$ and $\mathbf{P_{ssy}}$, respectively.

It is easy to see that $\mathbf{P_g}$, $\mathbf{P_{or}}$, $\mathbf{P_{st}}$, $\mathbf{P_{sy}}$ and $\mathbf{P_{ssy}}$ are porosity-like relations.

Our aim is to prove the following theorems.

Theorem 7.1. *Let (E, ρ) be a nonempty separable locally compact metric space without isolated points. Let $\mathbf{Q}$ be $\mathbf{P_g}$ (for some $g \in G$) or $\mathbf{P_{or}}$ or $\mathbf{P_{st}}$ on E. Then the σ-ideal $\mathcal{I}$ of all compact σ-$\mathbf{Q}$-porous sets is $\mathbf{\Pi^1_1}$-complete, in particular coanalytic non-Borel.*

Theorem 7.2. *The σ-ideal of all compact σ-symmetrically porous (σ-strongly symmetrically porous, respectively) subsets of $\mathbb{R}$ is $\mathbf{\Pi^1_1}$-complete, in particular coanalytic non-Borel.*

To verify the property **(A5)** in concrete cases, we need the following lemma.

Lemma 7.3. *Let $g \in G$. Let (E, ρ) be a locally compact metric space, $K \in \mathcal{K}^*(E)$ and let $\varepsilon > 0$. Then there exists a system $\mathcal{S}$ of open sets such that*

(1) $\bigcup \mathcal{S} \subset B(K, \varepsilon)$,
(2) *$\mathcal{S}$ is locally finite in $E \setminus K$,*
(3) *if $J \subset E$ intersects each element of $\mathcal{S}$, then $K \cup J$ is $\langle g\rangle$-porous at no point of K.*

Proof: We may assume that the set $\overline{B(K, \varepsilon)}$ is compact. Denote

$$H_n = \left\{ y \in E;\ \varepsilon 2^{-(n+1)} \le \rho(y, K) < \varepsilon 2^{-n} \right\}, \qquad n \in \mathbb{N} \cup \{0\}.$$

For each $n \in \mathbb{N}$ find $0 < \delta_n < \varepsilon 2^{-(n+1)}$ so small that $g(\delta_n) < \varepsilon 2^{-(n+1)}$ and choose a finite set $C_n \subset H_n$ forming a $\delta_n/2$-net of H_n (i.e. for every $x \in H_n$ there exists $c \in C_n$ with $\rho(x, c) < \delta_n/2$). Put

$$\mathcal{S}_n := \{B(c, \delta_n/2);\ c \in C_n\}, \qquad \mathcal{S} := \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

Using the triangle inequality we easily infer that $\bigcup \mathcal{S}_n \subset H_{n-1} \cup H_n \cup H_{n+1}$ for every $n \in \mathbb{N}$. This implies (1) and also (2). To check (3), suppose to the contrary that $J \subset E$ intersects each element of $\mathcal{S}$ but $K \cup J$ is $\langle g \rangle$-porous at a point $z \in K$. Then there exists $x \in B(z, \varepsilon/2)$ and $r > 0$ such that $B(x, r) \cap (K \cup J) = \emptyset$ and $z \in B(x, g(r))$. Obviously there exist $n \in \mathbb{N}$, $c \in C_n$ and $t \in J$ such that $x \in H_n$, $\rho(x, c) < \delta_n/2$ and $\rho(t, c) < \delta_n/2$. Since $t \notin B(x, r)$, we have $r < \delta_n$ and therefore $g(r) < \varepsilon 2^{-(n+1)}$. Since $x \in H_n$, we obtain $z \notin B(x, g(r))$, a contradiction. $\square$

Proof of Theorem 7.1: **The case $\mathbf{Q} = \mathbf{P_g}$.**

Define point-set relations $\mathbf{R}_k^g$, $k \in \mathbb{N}$, on E by

$$\mathbf{R}_k^g(x, A) \overset{\text{def}}{\iff} \exists\, y \in E\ \exists\, r > 0 :\ \rho(x, y) < 1/k,\ \rho(x, y) < g(r),\ B(y, r) \cap A = \emptyset.$$

Clearly $\mathbf{P_g} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathbf{R}_k^g$. We know that $\mathbf{P_g}$ satisfies **(A1)**, **(A2)** and **(A3)**; since E has no isolated points we see that it satisfies also **(A4)**. The condition **(A5)** holds for $\mathbf{P_g}$ by Lemma 7.3. It is also clear that $\mathbf{R}_k^g$ satisfies **(A1)** and **(A3)**.

Since g is assumed to be continuous it is easy to observe that the condition $B(y, r) \cap A = \emptyset$ in the definition of $\mathbf{R}_k^g$ can be replaced by the condition $\rho(B(y, r), A) > 0$. Fixing y, r and k we have that the set

$$\{(x, F) \in E \times \mathcal{C}_b(E);\ \rho(x, y) < 1/k,\ \rho(x, y) < g(r),\ \rho(B(y, r), F) > 0\}$$

is open in the space $E \times \mathcal{C}_b(E)$. This and the above observation imply that $\mathbf{R}_k^g$ is stable.

Lemma 3.7 shows that $\mathbf{P_g}$ satisfies **(D1)** and **(D2)**. Consequently Theorem 6.1 applied to $\Xi = \{\xi\}$ and $\mathbf{P} = \mathbf{V}^\xi = \mathbf{P_g}$ gives $\mathbf{\Pi_1^1}$-completeness of $\mathcal{I}$.

The case $\mathbf{Q} = \mathbf{P_{or}}$. Put $\Xi = \mathbb{N} \setminus \{1\}$ and $\mathbf{V}^\xi := \mathbf{P}_{g_\xi}$ for $\xi \in \Xi$. The relation $\mathbf{V}^\xi$ satisfies **(A1)–(A5)**, **(D1)** and **(D2)** according to the previous case. Clearly

$$\mathbf{P_{or}} = \bigcup_{\xi \in \Xi} \mathbf{V}^\xi.$$

Consequently Theorem 6.1 implies the desired result.

The case $\mathbf{Q} = \mathbf{P_{st}}$.

We define point-set relations $\mathbf{R}_k$ on E by

$$\mathbf{R}_k(x, A) \Leftrightarrow \exists\, y \in E\, \exists\, r > 0 :\ \rho(x, y) < 1/k,\ \rho(x, y) < (1+1/k)r,\ B(y, r) \cap A = \emptyset.$$

Clearly

$$\mathbf{P_{st}} = \bigcap_{k=1}^{\infty} \mathbf{R}_k.$$

Since $\mathbf{R}_k = \mathbf{R}_k^g$ for $g = g_{(1+1/k)}$, $\mathbf{R}_k$'s are stable and satisfy **(A1)** and **(A3)**. Therefore Lemma 3.7 implies that $\mathbf{P_{st}}$ satisfies **(D1)** and **(D2)**. The relation $\mathbf{P_{st}}$ clearly satisfies **(A1)**–**(A4)**. Since $\mathbf{P}_{g_2}$ satisfies **(A5)**, the stronger relation $\mathbf{P_{st}}$ satisfies **(A5)** as well (cf. Remark 2.6). Consequently Theorem 6.1 applied to $\Xi = \{\xi\}$ and $\mathbf{P} = \mathbf{V}^{\xi} = \mathbf{P_{st}}$ gives $\mathbf{\Pi^1_1}$-completeness of $\mathcal{I}$. □

Proof of Theorem 7.2: **The case $\mathbf{Q} = \mathbf{P_{sy}}$.** For $\alpha > 1$ and $k \in \mathbb{N}$, define point-set relations $\mathbf{R}_k^{\alpha}$ on $\mathbb{R}$ by

$$\begin{aligned}\mathbf{R}_k^{\alpha}(x, A) \overset{\text{def}}{\Longleftrightarrow} \exists\, y \in \mathbb{R}\, \exists\, r > 0 :\ & |x-y| < 1/k, |x-y| < \alpha r,\\ & ((y-r, y+r) \cup (2x-y-r, 2x-y+r)) \cap A = \emptyset.\end{aligned}$$

Put $\Xi = \mathbb{N} \setminus \{1\}$ and

$$\mathbf{V}^{\xi} := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathbf{R}_k^{\xi}$$

for $\xi \in \Xi$. Clearly

$$\mathbf{P_{sy}} = \bigcup_{\xi \in \Xi} \mathbf{V}^{\xi}.$$

Each $\mathbf{R}_k^{\xi}$ clearly satisfies **(A1)** and **(A3)**. It is easy to prove that the relations $\mathbf{R}_k^{\xi}$'s are stable (the proof is almost identical to the proof of the stability of relations $\mathbf{R}_k^g$ above). Therefore Lemma 3.7 implies that each relation $\mathbf{V}^{\xi}$ satisfies **(D1)** and **(D2)**. Each relation $\mathbf{V}^{\xi}$ clearly satisfies **(A1)**–**(A4)**. Since $\mathbf{P}_{g_{\xi}}$ satisfies **(A5)**, the stronger relation $\mathbf{V}^{\xi}$ satisfies **(A5)** as well. Consequently Theorem 6.1 gives $\mathbf{\Pi^1_1}$-completeness of $\mathcal{I}$.

The case $\mathbf{Q} = \mathbf{P_{ssy}}$. For $k \in \mathbb{N}$, put

$$\mathbf{R}_k^* := \mathbf{R}_k^{\alpha},$$

where $\alpha := 1 + 1/k$. It is easy to see that

$$\mathbf{P_{ssy}} = \bigcap_{k=1}^{\infty} \mathbf{R}_k.$$

The relations $\mathbf{R}_k^*$ are stable and satisfy **(A1)** and **(A3)** according to the previous case. Therefore Lemma 3.7 implies that $\mathbf{P_{ssy}}$ satisfies **(D1)** and **(D2)**. The relation $\mathbf{P_{ssy}}$ clearly satisfies **(A1)**–**(A4)**. Since $\mathbf{P_{sy}}$ satisfies **(A5)**, the stronger relation $\mathbf{P_{ssy}}$ satisfies **(A5)** as well. Consequently Theorem 6.1 applied to $\Xi = \{\xi\}$ and $\mathbf{P} = \mathbf{V}^\xi = \mathbf{P_{sy}}$ gives $\mathbf{\Pi_1^1}$-completeness of $\mathcal{I}$. □

8. The case of σ-ideals generated by closed sets

In this section we investigate the complexity of other σ-ideals which are related to some porosity-like relations. Namely, we deal with σ-very porous sets and with the σ-ideal of compact sets which can be covered by countably many closed $\mathbf{P}$-porous sets, where $\mathbf{P}$ is a porosity-like relation.

Note that in some articles (e.g. [Zam]) "σ-porous sets" are defined in such a way that they coincide with the sets which can be covered by countably many of (ordinary) porous closed sets.

Definition 8.1 ([KLW]). Let E be a metric space and let $\mathcal{I} \subset \mathcal{K}(E)$ be a σ-ideal of compact sets. We say that $\mathcal{H} \subset \mathcal{I}$ is a basis of $\mathcal{I}$ if for every $K \in \mathcal{I}$ there exist compact sets $K_n \in \mathcal{H}$, $n \in \mathbb{N}$, with $K \subset \bigcup_{n=1}^{\infty} K_n$.

Theorem 8.2 ([KLW]). *Let E be a compact metric space and $\mathcal{I} \subset \mathcal{K}(E)$ be a σ-ideal of compact sets which has a hereditary $\mathbf{\Pi_1^1}$ basis $\mathcal{H} \subset \mathcal{K}(E)$. Then $\mathcal{I}$ is $\mathbf{\Pi_1^1}$.*

Remark 8.3. The above theorem holds also for each separable complete metric space E. Indeed, since the considered notions are topological (cf. Remark 3.1) we may suppose that E is a $\boldsymbol{G_\delta}$ subset of a compact metric space X (cf. [K$_1$, Theorem 4.14]). By Lemma 3.2(v) $\mathcal{K}(E)$ is a $\boldsymbol{G_\delta}$ subset of $\mathcal{K}(X)$ and so $\mathcal{H}$ is a $\mathbf{\Pi_1^1}$ subset of $\mathcal{K}(X)$. Thus $\mathcal{I}$ is $\mathbf{\Pi_1^1}$ in $\mathcal{K}(X)$ by the above theorem; therefore $\mathcal{I}$ is $\mathbf{\Pi_1^1}$ in $\mathcal{K}(E)$ as well. (See [K$_1$, 14.4, 14.10].)

Lemma 8.4. *Let E be a separable complete metric space and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi\in\Xi} \mathbf{V}^\xi$ be a porosity-like relation on E, where each point-set relation $\mathbf{V}^\xi$ satisfies* **(D2)**. *Then the set*

$$\mathcal{H} = \{K \in \mathcal{K}(E);\ K \textit{ is } \mathbf{P}\textit{-porous}\}$$

is $\mathbf{\Pi_1^1}$.

Proof: Denote

$$\mathcal{L}(\xi) = \{(x, K) \in E \times \mathcal{K}(E);\ \mathbf{V}^\xi(x, K)\}, \quad \xi \in \Xi.$$

Thus for $F \in \mathcal{K}(E)$ we have

$$F \in \mathcal{H} \Leftrightarrow \forall x \in E:\ x \notin F\ \vee\ \exists \xi \in \Xi:\ (x, F) \in \mathcal{L}(\xi).$$

This implies that

$$F \in \mathcal{K}(E) \setminus \mathcal{H} \Leftrightarrow \exists x \in E : \ x \in F \ \& \ \forall \xi \in \Xi : \ (x, F) \notin \mathcal{L}(\xi).$$

It is sufficient to show that $\mathcal{L}(\xi)$ is $\mathbf{\Pi^1_1}$. Indeed, then the set $\{(x, K) \in E \times \mathcal{K}(E); \ x \in F \ \& \ \forall \xi \in \Xi : \ (x, F) \notin \mathcal{L}(\xi)\}$ is analytic and so $\mathcal{K}(E) \setminus \mathcal{H}$ is also analytic.

Fix $\xi \in \Xi$. We have

$$(x, F) \notin \mathcal{L}(\xi) \Leftrightarrow$$
$$(\{x\}, F) \in \{(L, K) \in \mathcal{K}(E) \times \mathcal{K}(E); \ \exists O \text{ open}, L \cap O \neq \emptyset : \ (\neg \mathbf{V}^\xi)(L \cap O, K)\}.$$

The mapping $(x, F) \mapsto (\{x\}, F)$ from $E \times \mathcal{K}(E)$ to $\mathcal{K}(E) \times \mathcal{K}(E)$ is continuous. Now using **(D2)** (for $G := E$) we obtain that $\mathcal{L}(\xi)$ is $\mathbf{\Pi^1_1}$ and we are done. □

Theorem 8.5. *Let E be a nonempty separable locally compact metric space without isolated points and Ξ be a nonempty countable set. Let $\mathbf{P} = \bigcup_{\xi \in \Xi} \mathbf{V}^\xi$ be a porosity-like relation on E, where each point-set relation $\mathbf{V}^\xi$ satisfies* **(A1)**, **(A2)**, **(A3)**, **(A4)**, **(A5)** *and* **(D2)**. *Then the σ-ideal $\mathcal{I}$ of all compact sets which are countable unions of compact $\mathbf{P}$-porous sets is $\mathbf{\Pi^1_1}$-complete.*

Proof: The σ-ideal $\mathcal{I}$ has a $\mathbf{\Pi^1_1}$ basis (Lemma 8.4) and therefore $\mathcal{I}$ is $\mathbf{\Pi^1_1}$ (Theorem 8.2 and Remark 8.3).

According to the dichotomy theorem (Theorem 3.6) we obtain that $\mathcal{I}$ is either $\mathbf{\Pi^1_1}$-complete or $\boldsymbol{G_\delta}$. Assume that $\mathcal{I}$ is $\boldsymbol{G_\delta}$ to the contradiction. The σ-ideal $\mathcal{I}$ contains all singletons by the condition **(A4)**. Let rk be the rank related to the relations $\mathbf{V}^\xi$, $\xi \in \Xi$. Using Lemma 4.5 we obtain that $\sup\{\mathrm{rk}(K); \ K \in \mathcal{I}\} = \omega_1$. Lemma 5.2 gives that $\mathcal{I}$ contains a non-σ-porous set. This is a contradiction since each element of $\mathcal{I}$ is σ-$\mathbf{P}$-porous. □

Definition 8.6. Let (X, ρ) be a metric space. We say that $A \subset X$ is *very porous at* $x \in X$ if there exist $c > 0$ and $r_0 > 0$ such that for every $r \in (0, r_0)$ there exists $y \in B(x, r)$ with $B(y, cr) \cap A = \emptyset$. The corresponding point-set relation is denoted by $\mathbf{P_v}$.

Remark 8.7. It is clear that $\mathbf{P_v}$ is a porosity-like relation. It satisfies also **(A4)** in all normed linear spaces, but not in general. In fact, there exists a nonempty metric space M without isolated points such that each singleton is not very porous. Indeed, put $M := E_{\{\frac{1}{n+2}\}}$, where $E_{\{\frac{1}{n+2}\}}$ is the symmetric perfect set with dissection ratios $\{\frac{1}{n+2}\}$ (see [KL, pp. 87–88] for the definition). Then it is not difficult to see that M is the desired space.

Theorem 8.8. *Let (E,ρ) be a nonempty separable locally compact metric space without isolated points. If* $\mathbf{P_v}$ *satisfies* **(A4)** *on E, then the σ-ideal $\mathcal{I}$ of all compact σ-$\mathbf{P_v}$-porous sets is $\mathbf{\Pi^1_1}$-complete in $\mathcal{K}(E)$, in particular coanalytic non-Borel.*

To prove the last theorem we will use the following general theorem and Remark 8.10. The definition of the Cantor-Bendixson rank $\mathrm{rk}_{\mathcal{H}}$ determined by a given hereditary system of compact sets $\mathcal{H}$ can be found in [KL, pp. 197–198].

Theorem 8.9 ([KL, Theorem 6, p. 202])**.** *Let E be a compact metric space, $\mathcal{H}$ be a Borel hereditary subset of $\mathcal{K}(E)$ consisting of nowhere dense sets and $\mathcal{I}\subset\mathcal{K}(E)$ be a σ-ideal with the basis $\mathcal{H}$. If every nonempty open subset of E contains $K\in\mathcal{I}$ with $\mathrm{rk}_{\mathcal{H}}(K)>1$, then $\mathcal{I}$ is $\mathbf{\Pi^1_1}$-complete.*

Remark 8.10. The above theorem holds also for each separable complete metric space E. We may suppose that E is a dense $\boldsymbol{G_\delta}$ subset of a compact metric space X (cf. Remark 8.3). Each nonempty open subset O of X intersects E and therefore there exists $K\in\mathcal{I}$ such that $\mathrm{rk}_{\mathcal{H}}(K)>1$ and $K\subset O$. Thus $\mathcal{I}$ is a $\mathbf{\Pi^1_1}$-complete subset of $\mathcal{K}(X)$, in particular $\mathcal{I}\neq\mathcal{K}(E)$. Therefore the set $\mathcal{I}$ is a $\mathbf{\Pi^1_1}$ subset of $\mathcal{K}(E)$. Suppose that Y is a Polish space and $B\subset Y$ is $\mathbf{\Pi^1_1}$. Then there exists a Borel mapping $f:Y\to\mathcal{K}(X)$ with $f^{-1}(\mathcal{I})=B$. Choose $K_0\in\mathcal{K}(E)\setminus\mathcal{I}$ and define a mapping $g:\mathcal{K}(X)\to\mathcal{K}(E)$ by

$$g(K)=\begin{cases}\varphi^{-1}(K) & \text{if } K\in\mathcal{K}(E);\\ \varphi^{-1}(K_0) & \text{if } K\in\mathcal{K}(X)\setminus\mathcal{K}(E).\end{cases}$$

The mapping g is clearly Borel and we have $g\circ f:Y\to\mathcal{K}(E)$ and $(g\circ f)^{-1}(\mathcal{I})=B$. This shows that $\mathcal{I}$ is a $\mathbf{\Pi^1_1}$-complete subset of $\mathcal{K}(E)$.

Proof of Theorem 8.8: Having in mind the last remark it is sufficient to check that $\mathcal{I}$ satisfies the assumptions of Theorem 8.9.

The symbol $\mathbb{Q}^+$ stands for the set of positive rationals. Let D be a countable dense subset of E. For $c>0$ and $r_0>0$ we define $\mathcal{H}(c,r_0)\subset\mathcal{K}(E)$ by

$$K\in\mathcal{H}(c,r_0)\overset{\text{def}}{\Longleftrightarrow}$$
$$\forall x\in D\,\forall r\in\mathbb{Q}^+\cap(0,r_0)\,\exists y\in D:\ B(y,cr)\cap K=\emptyset\ \&\ y\in B(x,r).$$

The set $\mathcal{H}(r,c_0)$ is obviously a Borel subset of $\mathcal{K}(E)$. Thus also the set

$$\mathcal{H}:=\bigcup\{\mathcal{H}(c,r_0);\ c\in\mathbb{Q}^+,r_0\in\mathbf{Q}^+\}$$

is Borel.

Claim. The set $\mathcal{H}$ forms a basis of $\mathcal{I}$.

Proof of Claim: It is obvious that $\mathcal{H} \subset \mathcal{I}$. Now take $K \in \mathcal{I}$. We can write $K = \bigcup_{n=1}^{\infty} A_n$, where A_n is $\mathbf{P_v}$-porous. We define

$$S_n(c, r_0) = \{x \in A_n; \forall\, r \in (0, r_0)\, \exists\, y \in B(x, r) :\ B(y, cr) \cap A_n = \emptyset\}, \quad c, r_0 \in \mathbb{Q}^+.$$

Clearly, $A_n = \bigcup\{S_n(c, r_0);\ c \in \mathbb{Q}^+ \cap (0,1), r_0 \in \mathbb{Q}^+\}$.

We show that $\overline{S_n(c, r_0)} \in \mathcal{H}(c/2, r_0)$ for $c \in \mathbb{Q}^+ \cap (0,1)$ and $r_0 \in \mathbb{Q}^+$. Take $x \in D$ and $r \in \mathbb{Q}^+ \cap (0, r_0)$. If $B(x, r/2) \cap \overline{S_n(c, r_0)} = \emptyset$, then we put $y = x$. If $B(x, r/2) \cap \overline{S_n(c, r_0)} \neq \emptyset$, then there exists $\tilde{x} \in B(x, r/2) \cap S_n(c, r_0)$ and $y \in B(\tilde{x}, r/2)$ with $B(y, cr/2) \cap A_n = \emptyset$. Thus $B(y, cr/2) \cap S_n(c, r_0) = \emptyset$ and also $B(y, cr/2) \cap \overline{S_n(c, r_0)} = \emptyset$. □

Thus $\mathcal{H}$ is a hereditary Borel basis of $\mathcal{I}$. It is clear that each $H \in \mathcal{H}$ is nowhere dense. If $V \subset E$ is a nonempty open set, then proceeding similarly as in the proof of Lemma 7.3 we find a compact set $L \subset V$ such that L is countable and there is $x_0 \in L$, which is not a point of ordinary porosity of L. Since $\mathbf{P_v}$ satisfies **(A4)**, each singleton is $\mathbf{P_v}$-porous and therefore L is a σ-$\mathbf{P_v}$-porous set. For every open neighbourhood O of x_0 we have that $\overline{L \cap O}$ is not in $\mathcal{H}$, therefore $\mathrm{rk}_{\mathcal{H}}(L) > 1$. Applying Theorem 8.9 we obtain that $\mathcal{I}$ is $\mathbf{\Pi^1_1}$-complete. □

References

[DP] Debs G., Preiss D., private communication.

[DSR] Debs G., Saint-Raymond J., *Ensemblesboréliens d'unicité au sens large*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 3, 217–239.

[EH] Evans M.J., Humke P.D., *Contrasting symmetric porosity and porosity*, J. Appl. Anal. **4** (1998), no. 1, 19–41.

[F] Foran J., *Continuous functions need not have σ-porous graphs*, Real Anal. Exchange **11** (1985–86), 194–203.

[K_1] Kechris A.S., *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[K_2] Kechris A.S., *On the concept of $\mathbf{\Pi^1_1}$-completeness*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1811–1814.

[KL] Kechris A.S., Louveau A., *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Math. Soc. Lecture Notes Series **128**, Cambridge University Press, Cambridge, 1989.

[KLW] Kechris A.S., Louveau A., Woodin W.H., *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. **301** (1987), 263–288.

[M] Michael E., *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[Za_1] Zajíček L., *Sets of σ-porosity and σ-porosity (q)*, Časopis Pěst. Mat. **101** (1976), 350–359.

[Za_2] Zajíček L., *Porosity and σ-porosity*, Real Anal. Exchange **13** (1987–88), 314–350.

[Za_3] Zajíček L., *Products of non-σ-porous sets and Foran systems*, Atti Sem. Mat. Fis. Univ. Modena **44** (1996), 497–505.

[Za_4] Zajíček L., *Smallness of sets of nondifferentiability of convex functions in non-separable Banach spaces Czechoslovak Math. J.* **41** (116) (1991), 288–296.

[Zam] Zamfirescu T., *Nearly all convex bodies are smooth and strictly convex*, Monatsh. Math. (1987), 57–62.

[Ze] Zelený M., unpublished manuscript.

[ZP] Zelený M., Pelant J., *The structure of the σ-ideal of σ-porous sets*, submitted, available electronically under `http://www.karlin.mff.cuni.cz/kma-preprints/`.

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