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On conditions for the boundedness of the Weyl fractional integral on weighted $L^p$ spaces

L. de Rosa, A. de la Torre

Abstract. In this paper we give a sufficient condition on the pair of weights $(w, v)$ for the boundedness of the Weyl fractional integral $I^+_\alpha$ from $L^p(v)$ into $L^p(w)$. Under some restrictions on $w$ and $v$, this condition is also necessary. Besides, it allows us to show that for any $p : 1 \leq p < \infty$ there exist non-trivial weights $w$ such that $I^+_\alpha$ is bounded from $L^p(w)$ into itself, even in the case $\alpha > 1$.

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1. Introduction and main results

Let $0 < \alpha < 1$. Given a locally integrable function $f$ on $\mathbb{R}$, its Weyl fractional integral is defined by

$$I^+_\alpha f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy.$$  

Similarly, the Riesz fractional integral is given by

$$I^-_\alpha f(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} dy.$$  

By a weight $w$ we mean a locally integrable, non-negative function defined on $\mathbb{R}$. For any Lebesgue measurable set $E \subseteq \mathbb{R}$ we denote the $w$-measure of $E$ by $w(E) = \int_E w(x) \, dx$, and the characteristic function of $E$ by $\chi_E$.

Throughout the paper, $C$ shall be a positive constant not necessarily the same at each occurrence.

Let $w$ and $v$ be two weights on $\mathbb{R}$ and $1 < p < \infty$. We consider the weighted norm inequality,

$$\left[ \int_{-\infty}^{+\infty} |I^+_\alpha f(x)|^p w(x) \, dx \right]^{1/p} \leq C \left[ \int_{-\infty}^{+\infty} |f(x)|^p v(x) \, dx \right]^{1/p},$$

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for every \( f \) in \( L^p(v) \). If we denote \( \sigma(x) = v(x)^{1-p'} \), where \( 1/p + 1/p' = 1 \), then (1.3) is equivalent to

\[
(1.4) \quad \left[ \int_{-\infty}^{\infty} |I_\alpha^+(f\sigma)(x)|^p w(x) \, dx \right]^{1/p} \leq C \left[ \int_{-\infty}^{\infty} |f(x)|^p \sigma(x) \, dx \right]^{1/p},
\]

for every \( f \) in \( L^p(\sigma) \).

The fractional maximal operator,

\[
M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(t)| \, dt
\]
satisfies the inequality \( M_\alpha^+ f(x) \leq I_\alpha^+ (|f|)(x) \). The boundedness of \( M_\alpha^+ \) from \( L^p(v) \) into \( L^p(w) \) implies that there exists a constant \( C > 0 \) such that for every \( a < b \),

\[
(1.5) \quad \left( \int_{-\infty}^{a} \frac{w(y)}{(b-y)^{(1-\alpha)p}} \, dy \right)^{1/p} \left( \int_{a}^{b} \sigma(y) \, dy \right)^{1/p'} \leq C,
\]

see proof of Theorem 3 in [4]. Then, this condition (1.5) is necessary for the inequality (1.4) to hold. The following theorem gives a sufficient condition for (1.4), which is also necessary in some cases.

**Theorem 1.1.** Let \( w \) and \( \sigma \) be two weights on \( \mathbb{R} \). Let \( 1 < p < \infty \) and \( 0 < \alpha < 1 \). Then (1.4) holds if \( I_\alpha^- w \) belongs to \( L^{p'}_{loc}(\sigma) \) and

\[
(1.6) \quad I_\alpha^- [(I_\alpha^- w)^{p'} \sigma](x) \leq C I_\alpha^- w(x) \quad \sigma\text{-a.e.}
\]

**Theorem 1.2.** Let \( 1 < p < \infty \) and \( 0 < \alpha < 1 \). If \( w \) and \( \sigma \) satisfy

\[
(1.7) \quad \sup_{r>0} \left[ \int_{2r}^{\infty} \frac{w([x-\rho, x-2\rho]) \, d\rho}{\rho^{1-\alpha}} \right]^{1/p} \left[ \int_{0}^{r} \frac{\sigma([x-\rho, x]) \, d\rho}{\rho^{1-\alpha}} \right]^{1/p'} \leq C,
\]

for all \( x \in \mathbb{R} \), then condition (1.6) is necessary for the inequality (1.4) to hold.

Let \( w \) be any weight and \( \sigma = v^{1-p'} = (I_\alpha^- w)^{-p'} w \). Clearly, the pair \((w, \sigma)\) satisfies condition (1.6). Therefore, the inequality

\[
\int_{-\infty}^{+\infty} |I_\alpha^+(f)(x)|^p w(x) \, dx \leq C \int_{-\infty}^{+\infty} |f(x)|^p w(x) |1-p I_\alpha^- (w)(x)|^p \, dx
\]
Weyl fractional integral

holds. If \( w \) is a power weight, for instance \( w(x) = x^\gamma \chi_{(0,\infty)}(x), \gamma > -1 \), it is easy to see that \( w(x)^{1-p}I^-_\alpha(w)(x)^p \approx Cx^{\gamma+\alpha p}\chi_{(0,\infty)}(x) \) and therefore

\[
\int_0^\infty |I^+_\alpha(f)(x)|^p x^\gamma \, dx \leq C \int_0^\infty |f(x)|^p x^{\gamma+\alpha p} \, dx.
\]

A similar result for \( I^-_\alpha \) was obtained by E. Hernández in [3]. Furthermore, if the weight \( w \) satisfies \( I^-_\alpha w(x) \leq Cw(x) \) almost everywhere, then \( I^+_\alpha \) maps \( L^p(w) \) boundedly into itself. It is easy to check that \( w(x) = e^x \) satisfies this condition.

Therefore, the class of weights \( w \) such that \( I^+_\alpha \) maps \( L^p(w) \) boundedly into itself, is not empty. This is in sharp contrast with the case of the two-sided operator \( I\alpha f(x) = \int_{-\infty}^{+\infty} \frac{f(y)}{|y-x|^{1-\alpha}} \, dy \), for which this class is trivial. Indeed, there does not exist a non-zero weight \( w \) satisfying the condition

\[
(A_{p,\alpha}) \quad w(I)^{1/p} w^{1-p'} (I)^{1/p'} \leq C |I|^{1-\alpha}
\]

for all intervals \( I \), which is necessary for the boundedness of \( I\alpha \).

**Remark 1.3.** We can consider the operators \( I^+_\alpha \) and \( I^-_\alpha \) defined as in (1.1) and (1.2) for every \( \alpha > 0 \). In the case \( \alpha \geq 1 \) the weights for these operators were studied by F.J. Martín Reyes and E. Sawyer in [5].

**Definition 1.4.** For fixed \( 1 \leq p < \infty \) and \( 0 < \alpha < 1 \), we say that the weight \( w \) belongs to the class \( F^+_{p,\alpha} \), respectively \( F^-_{p,\alpha} \), if the operator \( I^+_\alpha \), respectively \( I^-_\alpha \), maps \( L^p(w) \) boundedly into itself.

We have seen above that these classes are non-trivial, at least in the case \( 1 < p < \infty \), \( 0 < \alpha < 1 \). The following theorems give us a characterization of them.

**Theorem 1.5.** Let \( 0 < \alpha < 1 \). The following are equivalent:

1. \( w \in F^+_{1,\alpha} \).
2. There exists a constant \( C \) such that for any \( f \)

\[
\int_{-\infty}^{+\infty} M^+_\alpha(f)(x)w(x) \, dx \leq C \int_{-\infty}^{+\infty} |f(x)|w(x) \, dx.
\]

3. There exists a constant \( C \) such that \( I^-_\alpha w(x) \leq Cw(x) \) a.e.

Actually the result is true for pairs of weights.

**Theorem 1.6.** Let \( v \) and \( w \) be two weights and \( 0 < \alpha < 1 \). The following are equivalent:

1. There exists a constant \( C \) such that for any \( f \)

\[
\int_{-\infty}^{+\infty} |I^+_\alpha(f)(x)|w(x) \, dx \leq C \int_{-\infty}^{+\infty} |f(x)|v(x) \, dx.
\]
2. There exists a constant $C$ such that for any $f$
\[
\int_{-\infty}^{+\infty} M_{\alpha}^+(f)(x)w(x) \, dx \leq C \int_{-\infty}^{+\infty} |f(x)|v(x) \, dx.
\]

3. There exists a constant $C$ such that $I_{\alpha}^- w(x) \leq C v(x)$ a.e.

**Remark 1.7.** By a duality argument, parts (1) and (3) of the previous theorem are equivalent even in the case $\alpha \geq 1$.

**Theorem 1.8.** Let $1 < p < \infty$ and $\alpha > 0$. The following are equivalent:

1. $w \in F_{p,\alpha}^+$.
2. There exist two weights $w_0 \in F_{1,\alpha}^+$ and $w_1 \in F_{1,\alpha}^-$ such that $w = w_0 w_1^{1-p}$.

Clearly we obtain similar theorems for $I_{\alpha}^-$ reversing the orientation of the real line.

2. **Proof of Theorems 1.1 and 1.2**

Let $w$ and $\sigma$ be two weights on $\mathbb{R}$. If $I_{\alpha}^- w$ belongs to $L_{\text{loc}}^p(\sigma)$, we denote

\[
\nu = (I_{\alpha}^- w)^{p'} \sigma.
\]

Then, we can write condition (1.6) in the form

\[
I_{\alpha}^- \nu \leq C I_{\alpha}^- w \quad \sigma \text{-a.e.}
\]

The following three lemmas shall be needed in the proof of Theorem 1.1.

**Lemma 2.1.** Let $1 < p < \infty$ and $\nu$ be defined by (2.1).

(i) Suppose that

\[
\left[ \int_{-\infty}^{+\infty} |I_{\alpha}^- (g\nu)|^{p'} \sigma \right]^{1/p'} \leq C \left[ \int_{-\infty}^{+\infty} |g|^{p'} \nu \right]^{1/p'},
\]

for all $g \in L^p(\nu)$. Then, for any $r : 1 < r \leq p'$ the inequality

\[
\left[ \int_{-\infty}^{+\infty} \left| I_{\alpha}^- (g\nu) \right| \frac{r}{I_{\alpha}^- w} \frac{\nu}{I_{\alpha}^- w} \right]^{1/r} \leq C \left[ \int_{-\infty}^{+\infty} |g|^{r'} \nu \right]^{1/r},
\]

holds for all $g \in L^r(\nu)$. 

(ii) If (2.2) holds, then (2.4) holds for all $r : 1 < r \leq \infty$. (In the case $r = \infty$, inequality (2.4) is to be interpreted in the $L^\infty(d\nu)$ norm.)

PROOF: In order to prove (i) we will make use of the theory of interpolation in the setting of Lorentz spaces. We recall that for $0 < p < \infty, 0 < q \leq \infty$, the space $L^{p,q}(\nu)$ is defined as the set of all measurable functions $f$ for which

$$\|f\|_{p,q} = \|t^p f^*(t)\|_{L^q(dt/t)} < \infty$$

where $f^*$ is the decreasing rearrangement of $f$ with respect to the measure $\nu$. It is known that if $1 < p < \infty$ then the associate space of $L^{p,1}(\nu)$ is $L^{p',\infty}(\nu)$ and that if a quasilinear operator $T$ maps $L^{p,1}(\nu)$ boundedly into $L^p(\nu)$ and $L^q(\nu)$ into $L^q(\nu)$, where $1 < p < q \leq \infty$ then $T$ is a bounded operator on $L^s(\nu)$ for any $p < s < q$ (see [1]).

We define the operator $A$ by

$$A g = \frac{I_\alpha^-(g \nu)}{I_\alpha w}.$$  

Taking into account (2.3) we have that

$$\|Ag\|_{L^{p'}(\nu)} \leq C\|g\|_{L^{p'}(\nu)}.$$  

That is, the operator $A$ is bounded from $L^{p'}(\nu)$ to $L^{p'}(\nu)$. We shall show that for all $1 < r < p'$,

$$\|Ag\|_{L^{r}(\nu)} \leq C\|g\|_{L^{r,1}(\nu)}.$$  

The adjoint operator of $A$ is defined by

$$A^* f = I_\alpha^+[f(I_\alpha^- w)^{-1}] \nu,$$

and (2.7) can be rewritten as

$$\|I_\alpha^+[f(I_\alpha^- w)^{-r} \nu]\|_{L^{r',\infty}(\nu)} \leq C\|f\|_{L^{r'}((I_\alpha^- w)^{-r} \nu)}.$$  

This inequality is equivalent to

$$\|I_\alpha^+ g\|_{L^{r',\infty}(\nu)} \leq C\|(I_\alpha^- w)^r \nu^{-1} g\|_{L^{r'}((I_\alpha^- w)^{-r} \nu)} = C\|g\|_{L^{r'}((I_\alpha^- w)^{r'\nu^{1-r'}}).$$
This is the same as asserting that $I^+_\alpha$ is bounded from $L^{r'}((I^-\alpha w)^{r'}\nu^{1-r'})$ to $L^{r',\infty}(\nu)$. By Theorem 2 in [4] this is equivalent to the existence of a constant $C > 0$ such that for any interval $I$,

\[(2.9)\]
\[
\int_I \left| \frac{I^-\alpha (\chi I \nu)}{I^-\alpha w} \right|^r \nu \leq C\nu(I).
\]

Using (2.3) with $g = \chi_I$, we get

\[(2.10)\]
\[
\int_I |I^-\alpha (\chi I \nu)|^{p'}\sigma \leq C\nu(I).
\]

Applying Hölder’s inequality with exponents $p'/r$ and its conjugate, by (2.10) we have that

\[
\int_I \left[ \frac{I^-\alpha (\chi I \nu)}{I^-\alpha w} \right]^r \nu \leq \left[ \int_I \left[ \frac{I^-\alpha (\chi I \nu)}{I^-\alpha w} \right]^{p'/r} \nu \right]^{r/p'} \nu(I)^{1-r/p'}
\]
\[
= \left[ \int_I [I^-\alpha (\chi I \nu)]^{p'} \sigma \right]^{r/p'} \nu(I)^{1-r/p'}
\]
\[
\leq C\nu(I).
\]

Then (2.9) holds, and it implies (2.8). Therefore, by duality we have (2.7). Now, by (2.6) and an interpolation theorem for $L^{r,1}(\nu)$, we obtain (2.4) for all $1 < r < p'$.

(ii) By inequality (2.2), the operator $A$ defined in (2.5) is bounded on $L^{\infty}(\nu)$ that is,

\[(2.11)\]
\[
\|Ag\|_{L^{\infty}(\nu)} \leq C\|g\|_{L^{\infty}(\nu)}.
\]

On the other hand, (2.2) implies that

\[
\int_I [I^-\alpha (\chi I \nu)]^{p'} \sigma \leq C \int_I (I^-\alpha w)^{p'} \sigma = C\nu(I),
\]

for any interval $I$. Then (2.10) holds and, as in part (i), (2.7) holds for all $r \leq p'$. Now, interpolating (2.11) and (2.7) we have that (2.4) holds for all $1 < r < \infty$. The case $r = \infty$ is straightforward and left to the reader. \(\square\)

**Lemma 2.2.** Let $w$ and $\sigma$ be two weights defined on $\mathbb{R}$. Let $0 < \alpha < 1$. Then, for every positive integer $m$, the inequality

\[(2.12)\]
\[
I^-\alpha [(I^+_\alpha \sigma)^m w] \leq C \left\{ (I^-\alpha w)(I^+_\alpha \sigma)^m + I^-\alpha [(I^-\alpha w)(I^+_\alpha \sigma)^{m-1}\sigma] \right\}
\]
holds with a constant $C$ depending on $\alpha$ and $m$.

**Proof:** Taking into account that $m > 0$ we get,

$$I_{\alpha}^{-}[(I_{\alpha}^{+}\sigma)^m w](x) = \int_{-\infty}^{x} \frac{I_{\alpha}^{+}\sigma(y)^m}{(x-y)^{1-\alpha}} w(y) \, dy$$

$$= \int_{-\infty}^{x} \frac{1}{(x-y)^{1-\alpha}} \left( \int_{x}^{\infty} \frac{\sigma(z) \, dz}{(z-y)^{1-\alpha}} \right)^m w(y) \, dy$$

(2.13)

$$\leq 2^m \left[ \int_{-\infty}^{x} \frac{1}{(x-y)^{1-\alpha}} \left( \int_{x}^{\infty} \frac{\sigma(z) \, dz}{(z-y)^{1-\alpha}} \right)^m w(y) \, dy \right]$$

$$+ 2^m \left[ \int_{-\infty}^{x} \frac{1}{(x-y)^{1-\alpha}} \left( \int_{y}^{x} \frac{\sigma(z) \, dz}{(z-y)^{1-\alpha}} \right)^m w(y) \, dy \right]$$

$$= A_m + B_m.$$

We have the estimate

$$A_m \leq C \int_{-\infty}^{x} \frac{1}{(x-y)^{1-\alpha}} \left( \int_{x}^{\infty} \frac{\sigma(z) \, dz}{(z-x)^{1-\alpha}} \right)^m w(y) \, dy$$

$$= CI_{\alpha}^{+} \sigma(x)^m I_{\alpha}^{-} w(x).$$

Then, in order to prove (2.12), by (2.13), it will be enough to show that

(2.14) $$B_m \leq CI_{\alpha}^{-} [(I_{\alpha}^{-} w)(I_{\alpha}^{+}\sigma)^{m-1}\sigma](x).$$

We can write $B_m$ in the form

$$B_m = C \int_{-\infty}^{x} \frac{1}{t^{2-\alpha}} \int_{x-t}^{x} \left( \int_{y}^{x} \frac{\sigma(z) \, dz}{(z-y)^{1-\alpha}} \right)^m w(y) \, dy \, dt.$$ (2.15)

Applying Fubini’s Theorem we have that

If we prove that for every positive integer $m$, the inequality

(2.16) $$\int_{x-t}^{x} \left( \int_{y}^{x} \frac{\sigma(z) \, dz}{(z-y)^{1-\alpha}} \right)^m w(y) \, dy \leq C \int_{x-2^m t}^{x} I_{\alpha}^{-} w(y) I_{\alpha}^{+} \sigma(y)^{m-1}\sigma(y) \, dy,$$

holds with a constant $C$ depending on $m$ and $\alpha$ only, then by (2.15) and Fubini’s Theorem, we obtain (2.14). We shall show (2.16) by induction. If $m = 1$, changing the order of integration,

$$\int_{x-t}^{x} \left( \int_{y}^{x} \frac{\sigma(z) \, dz}{(z-y)^{1-\alpha}} \right) w(y) \, dy \, dt = \int_{x-t}^{x} \left( \int_{x-t}^{z} \frac{w(y) \, dy}{(z-y)^{1-\alpha}} \sigma(z) \, dz \right)$$

$$\leq \int_{x-2t}^{x} I_{\alpha}^{-} w(z) \sigma(z) \, dz.$$
That is, (2.16) holds in the case \( m = 1 \).
Let \( m > 1 \) and assume that (2.16) holds for \( m - 1 \). Integrating by parts, we observe that

\[
\left( \int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^m = m \int_y^x \frac{1}{(u-y)^{1-\alpha}} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} \sigma(u) \, du.
\]

Then applying Fubini’s Theorem,

\[
I_m = \int_{x-t}^x \left( \int_y^{x-x} \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^m w(y) \, dy
\]

(2.17)

\[
= m \int_{x-t}^x \int_y^{x-t} \frac{1}{(u-y)^{1-\alpha}} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} \sigma(u) \, du \, w(y) \, dy
\]

\[
= m \int_{x-t}^x \int_t^x \frac{1}{(u-t-y)^{1-\alpha}} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, \sigma(u) \, du.
\]

By (2.17), we can write \( I_m \) in the form

\[
I_m = C \int_{x-t}^x \int_{x-t}^x \int_{u-y}^{2(u-y)} \frac{ds}{s^{2-\alpha}} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, \sigma(u) \, du.
\]

Changing the order of integration and enlarging the domain we have that

\[
I_m = C \int_{x-t}^x \int_{x-t}^x \int_{u-x}^{u-x+t} \frac{1}{s^{2-\alpha}} \int_{u-s}^{u-s/2} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, ds \, \sigma(u) \, du
\]

\[
+ C \int_{x-t}^x \int_{x-t}^x \int_{u-x}^{u-x+t} \frac{1}{s^{2-\alpha}} \int_{u-s}^{u-s/2} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, ds \, \sigma(u) \, du
\]

\[
\leq C \int_{x-t}^x \int_{x-t}^x \int_{u-x}^{u-x+t} \frac{1}{s^{2-\alpha}} \int_{u-s}^{u-s/2} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, ds \, \sigma(u) \, du
\]

\[
+ C \int_{x-t}^x \int_{x-t}^x \int_{u-x}^{u-x+t} \frac{1}{s^{2-\alpha}} \int_{u-s}^{u-s/2} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, ds \, \sigma(u) \, du
\]

\[
= C \int_{x-t}^x \int_{x-t}^x \int_{u-x}^{2(u-x)} \frac{1}{s^{2-\alpha}} \int_{u-s}^{u-s/2} \left( \int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}} \right)^{m-1} w(y) \, dy \, ds \, \sigma(u) \, du.
\]

Using (2.16) in the case \( m - 1 \), we get

\[
I_m \leq C \int_{x-t}^x \int_{0}^{2(u-x)} \frac{1}{s^{2-\alpha}} \int_{u-s}^{u-s/2} I_+^\alpha w(y) I_+^\alpha \sigma(y)^{m-2} \sigma(y) \, dy \, ds \, \sigma(u) \, du.
\]
Applying Fubini’s Theorem, we obtain the estimate

\[ I_m \leq C \int_{x-t}^{x} \int_{u-2m(u-x+t)}^{u} I_{\alpha}^{-} w(y) I_{\alpha}^{+} \sigma(y)^{m-2} \sigma(y) \frac{ds}{s^{2-\alpha}} \sigma(u) \, du \, dy \]

\[ \leq C \int_{x-t}^{x} \int_{u-2m(u-x+t)}^{u} \frac{I_{\alpha}^{-} w(y) I_{\alpha}^{+} \sigma(y)^{m-2}}{(u-y)^{1-\alpha}} \sigma(y) \, dy \, \sigma(u) \, du. \]

Changing the order of integration again, we have that

\[ I_m \leq C \int_{x-t}^{x} \int_{u-2m(u-x+t)}^{u} \frac{\sigma(u)}{(u-y)^{1-\alpha}} \sigma(y) \, dy \, du \]

\[ + C \int_{x-t}^{x} I_{\alpha}^{-} w(y) I_{\alpha}^{+} \sigma(y)^{m-2} \int_{u-2m(u-x+t)}^{u} \frac{\sigma(u)}{(u-y)^{1-\alpha}} \sigma(y) \, dy \, du. \]

Enlarging the domain of integration in the first term on the right hand,

\[ I_m \leq C \int_{x-2m}^{x} I_{\alpha}^{-} w(y) I_{\alpha}^{+} \sigma(y)^{m-2} I_{\alpha}^{+} \sigma(y) \, \sigma(y) \, dy \]

\[ = C \int_{x-2m}^{x} I_{\alpha}^{-} w(y) I_{\alpha}^{+} \sigma(y)^{m-1} \, \sigma(y) \, dy. \]

This shows that (2.16) holds for every positive integer \( m \), and finishes the proof of this lemma.

The following two lemmas are simple variants of Lemma 4 and Lemma 5 in [6], therefore we omit their proofs.

**Lemma 2.3.** Let \( w \) and \( \sigma \) be two weights, \( 0 < \alpha < 1 \) and \( 1 < p < \infty \). We assume that \( m < p \leq m + 1 \), where \( m \) is a positive integer. Let \( \delta = (p - 1)/m \).

Then, the inequality

\[ (2.18) \quad I_{\alpha}^{-} [(I_{\alpha}^{+} \sigma)^{p-1} w] \leq C \left\{ (I_{\alpha}^{-} w)(I_{\alpha}^{+} \sigma)^{p-1} + (I_{\alpha}^{-} w)^{1-\delta} (I_{\alpha}^{-} w)(I_{\alpha}^{+} \sigma)^{m-1} \sigma^\delta \right\} \]

holds, with a constant \( C \) depending on \( \alpha, p \) and \( m \).

Let \( w \) and \( \sigma \) be two weights on \( \mathbb{R} \) and \( 1 < p < \infty \). We define the operator \( B_p \) in the form

\[ (2.19) \quad B_p(f) = I_{\alpha}^{-} [(I_{\alpha}^{+} (f \sigma))^{p-1} w], \]

for each \( f \in L^p(\sigma) \).
Lemma 2.4. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Suppose that for every $f \in L^p(\sigma)$, we have the inequality

\begin{equation}
\int_{-\infty}^{+\infty} [B_p(f)]^{p'} \sigma \leq C \|f\|_{L^p(\sigma)}^p.
\end{equation}

Then, (1.4) holds.

Proof of Theorem 1.1: Let $\nu$ be defined as in (2.1), that is $\nu = (I_{\alpha}^- w)^{p'} \sigma$. Condition (1.5) is $I_{\alpha}^- \nu \leq CI_{\alpha}^- w$, $\sigma - a.e$. Then, by Lemma 2.1(ii), we get (2.4) for every $r : 1 < r \leq \infty$. In the case $r = p'$ we have that the inequality

\begin{equation}
\|I_{\alpha}^- (g\nu)\|_{L^{p'}(\sigma)} \leq C \|g\|_{L^{p'}(\nu)}
\end{equation}

holds for every $g \in L^{p'}(\nu)$. By duality (2.21) is equivalent to

\begin{equation}
\|I_{\alpha}^+ (f\sigma)\|_{L^p(\nu)} \leq C \|f\|_{L^p(\sigma)},
\end{equation}

for every $f \in L^p(\sigma)$. We shall show that (2.22) implies (2.21). Thus, applying Lemma 2.4 we obtain (1.4).

Let $f \in L^p(\sigma)$, $f \geq 0$. We consider the operator $B_p$ defined in (2.19). First of all, we prove that (2.20) holds for all positive integers $p \geq 2$. By Lemma 2.2 with $m = p - 1$,

$$B_p(f) \leq C \left\{ (I_{\alpha}^- w)(I_{\alpha}^+ (f\sigma))^{p-1} + I_{\alpha}^- [(I_{\alpha}^- w)(I_{\alpha}^+ (f\sigma))]^{p-2} f \sigma \right\}.$$ 

Raising both sides of this inequality to the power $p'$ and integrating with respect to the weight $\sigma$,

$$\int_{-\infty}^{+\infty} B_p(f)^{p'} \sigma \leq C \int_{-\infty}^{+\infty} I_{\alpha}^-(w)^{p'} (I_{\alpha}^+(f\sigma))^{p} \sigma$$

$$+ C \int_{-\infty}^{+\infty} I_{\alpha}^- [(I_{\alpha}^- w)(I_{\alpha}^+ (f\sigma))]^{p-2} f \sigma]^{p'} \sigma.$$ 

By (2.22), the first term on the right hand is bounded by $C \|f\|_{L^p(\sigma)}^p$. To estimate the second term we consider the function

$$g = (I_{\alpha}^- w)^{1-p'} [I_{\alpha}^+ (f\sigma)]^{p-2} f.$$ 

Using (2.21), we have that

$$\int_{-\infty}^{+\infty} I_{\alpha}^- [(I_{\alpha}^- w)(I_{\alpha}^+ (f\sigma))]^{p-2} f \sigma]^{p'} \sigma = \|I_{\alpha}^- (g\nu)\|_{L^{p'}(\sigma)}^{p'}$$

$$\leq C \|g\|_{L^{p'}(\nu)}^{p'} = C \int_{-\infty}^{+\infty} (I_{\alpha}^- w)^{2-p'} (I_{\alpha}^+ (f\sigma))^{p-2} p' f \sigma.$$ 

This inequality gives (2.20) for \( p = 2 \). From now on, assume that \( p > 2 \). By Hölder’s inequality with exponents \( \frac{p-1}{p-2} \) and \( \frac{p}{p'} \) and using the identity \((2-p')p'-1\) = \( p' \), we obtain that the last expression is bounded by

\[
C \left[ \int_{-\infty}^{+\infty} I_{\alpha}^+(f\sigma)^p(I_{\alpha}^-w)^{p'}\sigma \right]^{\frac{p-1}{p-2}} \|f\|_{L^p(\sigma)}^{p'} \leq C \|f\|_{L^p(\sigma)}^p.
\]

In consequence, (2.20) holds for every positive integer \( p \).

Now, we suppose that \( p \) is not an integer and \( m < p < m + 1 \) with \( m \) a positive integer. By Lemma 2.3 we get

\[
B_p(f) \leq C \left\{ (I_{\alpha}^-w)(I_{\alpha}^+f\sigma)^{p-1} + (I_{\alpha}^-w)^{1-\delta}[(I_{\alpha}^-w)(I_{\alpha}^+f\sigma)^{m-1}f\sigma]^\delta \right\},
\]

where \( \delta = \frac{p-1}{m} \). Raising both sides of this inequality to the power \( p' \) and integrating against \( \sigma(x)dx \), we have

\[
\int_{-\infty}^{+\infty} B_p(f)^{p'}\sigma \leq C \int_{-\infty}^{+\infty} (I_{\alpha}^+(f\sigma))^{p'\nu}
+ C \int_{-\infty}^{+\infty} (I_{\alpha}^-w)^{p'(1-\delta)} \left\{ I_{\alpha}^-[(I_{\alpha}^-w)(I_{\alpha}^+f\sigma)^{m-1}f\sigma] \right\}^{p'\delta}\sigma.
\]

Using (2.22), the first term on the right hand is bounded by \( C\|f\|_{L^p(\sigma)}^{p'} \). Now, let \( r = p'\delta < p' \) and

\[
g = (I_{\alpha}^-w)^{1-p'}[(I_{\alpha}^+f\sigma)]^{m-1}f.
\]

Applying Lemma 2.1(i), more precisely using (2.4) we have that

\[
\int_{-\infty}^{+\infty} (I_{\alpha}^-w)^{p'(1-\delta)} \left\{ I_{\alpha}^-[(I_{\alpha}^-w)(I_{\alpha}^+f\sigma)^{m-1}f\sigma] \right\}^{p'\delta}\sigma
= \int_{-\infty}^{+\infty} \left\{ (I_{\alpha}^-w)(I_{\alpha}^-w)^{\frac{r}{p'(1-\delta)}} \right\}^{p'\delta}\sigma
\leq C \int_{-\infty}^{+\infty} g^{p'\delta}\sigma
= C \int_{-\infty}^{+\infty} (I_{\alpha}^-w)^{p'(1-\frac{1}{m})}(I_{\alpha}^+f\sigma)^{(1-\frac{1}{m})p} f \frac{p}{m}\sigma.
\]

If \( 1 < p < 2 \) then \( m = 1 \) and the proof is complete in this case. Suppose \( p > 2 \). Applying Hölder’s inequality with exponent \( m \) and its conjugate, and taking into account (2.22) we obtain that

\[
\int_{-\infty}^{+\infty} (I_{\alpha}^-w)^{p'(1-\frac{1}{m})}(I_{\alpha}^+f\sigma)^{(1-\frac{1}{m})p} f \frac{p}{m}\sigma
\leq \left[ \int_{-\infty}^{+\infty} (I_{\alpha}^-w)^{p'} I_{\alpha}^+(f\sigma)^{p'} \sigma \right]^{1-\frac{1}{m}} \|f\|_{L^p(\sigma)}^{\frac{p}{m}} \leq C \|f\|_{L^p(\sigma)}^p.
\]

Thus, (2.20) is proved for every \( 1 < p < \infty \). This completes the proof of Theorem 1.1. \( \square \)
Remark 2.5. We observe that applying Lemma 2.4 we have proved that (2.22) implies (1.4).

We observe that by duality (1.4) is equivalent to

\[(2.23) \quad \|I_\alpha^-(f w)\|_{L^{p'}(\sigma)} \leq C \|f\|_{L^{p'}(w)}.\]

**Proof of Theorem 1.2:** Let us assume that (1.4) and (1.7) hold. We can write

\[I_\alpha^- [(I_\alpha^- w)^{p'} \sigma](x) = C \int_0^\infty \frac{[(I_\alpha^- w)^{p'} \sigma][|x - r, x|]}{r^{1-\alpha}} \frac{dr}{r}.\]

For each \(r > 0\), let \(w = w_{1,r} + w_{2,r}\) where,

\[w_{1,r} = w\chi_{[x-2r,x]} \quad \text{and} \quad w_{2,r} = w - w_{1,r}.\]

Then,

\[I_\alpha^- w = I_\alpha^- (w_{1,r}) + I_\alpha^- (w_{2,r}),\]

and it follows that

\[I_\alpha^- [(I_\alpha^- w)^{p'} \sigma](x) \leq C \int_0^\infty \frac{[(I_\alpha^- w_{1,r})^{p'} \sigma][|x - r, x|]}{r^{1-\alpha}} \frac{dr}{r} + C \int_0^\infty \frac{[(I_\alpha^- w_{2,r})^{p'} \sigma][|x - r, x|]}{r^{1-\alpha}} \frac{dr}{r} = A + B.\]

By (2.23), we have the estimate

\[[(I_\alpha^- w_{1,r})^{p'} \sigma][|x - r, x|] = \int_{x-r}^x I_\alpha^- (\chi_{[x-2r,x]} w)(y)^{p'} \sigma(y) dy \leq C \int_{-\infty}^{+\infty} |\chi_{[x-2r,x]}(y)| p' w(y) dy = Cw([x-2r,x]).\]

Therefore,

\[A \leq C \int_0^\infty \frac{w([x-2r,x])}{r^{1-\alpha}} \frac{dr}{r} = CI_\alpha^- w(x).\]

On the other hand, taking into account the definition of \(w_{2,r}\), for each \(z \in [x-r, x]\) we have that

\[I_\alpha^- (w_{2,r})(z) = C \int_r^\infty \frac{w_{2,r}([z - \rho, z])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \leq C \int_r^\infty \frac{w([x-2\rho, x-2r])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} = C \int_{2r}^\infty \frac{w([x-\rho, x-2r])}{\rho^{1-\alpha}} \frac{d\rho}{\rho}.\]
Then,
\[
\int_{x-r}^{x} I_{\alpha}^{-} (w_{2r})(z) \, d\sigma(z) \, dz \leq C \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r]) \, d\rho}{\rho^{1-\alpha}} \right)^{p'/p} \int_{x-r}^{x} \sigma(z) \, dz.
\]
In consequence,
\[
B \leq C \int_{0}^{\infty} \frac{\sigma([x - r, x])}{r^{1-\alpha}} \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r]) \, d\rho}{\rho^{1-\alpha}} \right)^{p'/p} \, dr. \tag{2.24}
\]
Applying Fubini’s Theorem, we observe that
\[
g(r) = \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r]) \, d\rho}{\rho^{1-\alpha}} = \frac{1}{r^{2-\alpha}} \int_{x-r}^{x} w(z) \, dz \, d\rho = \int_{-\infty}^{x-2r} w(z) \, dz \, r^{2-\alpha} = C \int_{-\infty}^{x-2r} \frac{w(z)}{(x-z)^{1-\alpha}} \, dz.
\]
Thus, the derivative \(g'(r)\) is equal to \(-C \frac{w(x-2r)}{r^{1-\alpha}}\). Integrating by parts from (2.24) it follows that we can dominate \(B\) by
\[
C \int_{0}^{\infty} \int_{0}^{r} \frac{\sigma([x - \rho, x]) \, d\rho}{\rho^{1-\alpha}} \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r]) \, d\rho}{\rho^{1-\alpha}} \right)^{p'/p} \frac{w(x-2r)}{r^{1-\alpha}} \, dr \, dr
\]
since (1.7) implies that
\[
\sup_{r>0} \int_{0}^{r} \frac{\sigma([x - \rho, x]) \, d\rho}{\rho^{1-\alpha}} \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r]) \, d\rho}{\rho^{1-\alpha}} \right)^{p'/p} \leq C.
\]
Then,
\[
B \leq C \int_{-\infty}^{x} \frac{w(y)}{(x-y)^{1-\alpha}} \, dy = C I_{\alpha}^{-} w(x),
\]
and the proof of this theorem is complete. \(\square\)

In order to state the next proposition, we need to introduce the following definition.

**Definition 2.6.** Let \(\beta > 0\). We say that a weight \(w\) belongs to \(RD^{-}(\beta)\) if there exists a constant \(C > 0\), such that
\[
w([x - \rho, x]) \leq C \left( \frac{\rho}{r} \right)^{\beta} w([x - r, x]),
\]
for all \(x \in \mathbb{R}, r > 0\) and \(0 < \rho < r\).
Proposition 2.7. Let $1 < p < \infty$. Let $w$ and $\sigma$ be two weights on $\mathbb{R}$. If $\sigma \in RD^-(\beta)$ for some $\beta > 1 - \alpha$, then (1.5) implies condition (1.7).

Proof: We suppose that $w$ and $\sigma$ satisfy condition (1.5) and $\sigma \in RD^-(\beta)$ with $\beta > 1 - \alpha$. For each $r > 0$ we have that

$$\int_0^r \frac{\sigma([x - \rho, x])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \leq C \frac{\sigma([x - r, x])}{r^\beta} \int_0^r \rho^{\beta + \alpha - 2} \frac{d\rho}{\rho} = C \frac{\sigma([x - r, x])}{r^{1-\alpha}}.$$ 

Therefore,

$$(2.25) \quad \int_0^r \frac{\sigma([x - \rho, x])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \right)^{p'-1} \leq C \frac{1}{r^{1-\alpha}} \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - 2r])}{\rho^{1-\alpha}} \frac{\sigma([x - r, x])^{p/p'}}{\rho} \frac{d\rho}{\rho} \right)^{p'-1} \leq C \frac{1}{r^{1-\alpha}} \left( \int_{2r}^{\infty} \frac{w([x - \rho, x - r])}{\rho^{1-\alpha}} \frac{\sigma([x - r, x + \rho - 2r])^{p/p'}}{\rho} \frac{d\rho}{\rho} \right)^{p'-1}.

On the other hand (1.5) implies condition $A^+_{p,\alpha}$, that is, there exists a constant $C$ such that for every $a \in \mathbb{R}$ and $h > 0$

$$(w([a - h, a]))^{1/p} (\sigma([a, a + h]))^{1/p'} \leq Ch^{1-\alpha}.$$ 

Applying condition $A^+_{p,\alpha}$, it follows that (2.25) is bounded by

$$C \frac{1}{r^{1-\alpha}} \left( \int_{2r}^{\infty} \frac{(\rho - r)^{(1-\alpha)p}}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \right)^{p'-1} \leq C \frac{1}{r^{1-\alpha}} r^{(1-\alpha)(p-1)(p'-1)} = C.$$ 

Then, $w$ and $\sigma$ satisfy (1.7). □

Corollary 2.8. Let $\sigma$ belong to $RD^-(\beta)$ for some $\beta > 1 - \alpha$. Then (1.6) is a necessary and sufficient condition for the inequality (1.4) to hold.

Proof: It is an immediate consequence of Proposition 2.7 part (ii) and Theorem 1.2. □
Remark 2.9. As an application of these results we consider the existence of non-negative solution of the non-linear integral equation

\[ u = I^{-\alpha}_\sigma (u^q \sigma) + I^{-\alpha}_\sigma w \quad \sigma \text{-a.e.,} \]

where we suppose that \( I^{-\alpha}_\sigma w < \infty \) \( \sigma \text{-a.e.} \) and we have the following result:

Let \( 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1, A(p) = (p-1)p^{-q} \) and \( 0 < \alpha < 1 \). Let \( w \) and \( \sigma \) be two locally integrable weights.

(i) If \( I^{-\alpha}_\sigma w \) belongs to \( L^q_{\text{loc}}(\sigma) \) and the inequality

\[ I^{-\alpha}_\sigma [(I^{-\alpha}_\sigma w)^q \sigma](x) \leq A(p)I^{-\alpha}_\sigma w(x) \quad \sigma \text{-a.e.} \]

holds, then equation (2.26) has a non-negative solution in \( L^q_{\text{loc}}(\sigma) \).

(ii) Assume that there exists a constant \( C \) such that

\[ \int_0^r \frac{\sigma([x-r, x])}{\rho^{1-\alpha}} \, d\rho \leq C \sigma([x-r, x+r])^{1/q} \sigma([x-2r, x-r])^{1/p} r^{1-\alpha} \]

for all \( x \in \mathbb{R} \) and \( r > 0 \). If (2.26) has a non-negative solution in \( L^q_{\text{loc}}(\sigma) \), then \( I^{-\alpha}_\sigma w \) belongs to \( L^q_{\text{loc}}(\sigma) \) and there exists a constant \( A > 0 \) such that

\[ I^{-\alpha}_\sigma [(I^{-\alpha}_\sigma w)^q \sigma](x) \leq AI^{-\alpha}_\sigma w(x) \quad \sigma \text{-a.e.} \]

The proof is similar to the one in [6].

Definition 2.10. We say that a weight \( w \) belongs to \( D^- \) if there exists a constant \( C > 0 \), such that for all \( x \) belonging to \( \mathbb{R} \) and \( r > 0 \),

\[ w([x, x+r]) \leq Cw([x-r, x]). \]

Taking into account Definition 2.10 we state the next proposition.

Proposition 2.11. Let \( 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1 \) and \( 0 < \alpha < 1 \). If \( \sigma \) belongs to \( D^- \) with a constant \( C : 0 < C < (2^{1-\alpha} - 1)^{-1} \) then condition (2.28) holds.

Proof: Since \( \sigma \in D^- \) with constant \( C \) we have that

\[ (1+C)\sigma([x, x+r]) \leq C\sigma([x-r, x+r]). \]

Then,

\[ \sigma([x, x+r]) \leq \frac{C}{1+C}\sigma([x-r, x+r]), \]

\[ (2.30) \]
for every \(x \in \mathbb{R}\) and \(r > 0\). Let \(\beta > 1 - \alpha\) such that

\[
0 < C \leq \frac{1}{2^\beta - 1}. \tag{2.31}
\]

We shall show that \(\sigma \in RD^-(\beta)\) with constant \(\frac{1+C}{C} = A^{-1}\). Let \(x \in \mathbb{R}\) and \(r > 0\). Fixing \(\rho : 0 < \rho < r\), there exists a positive integer \(i\) such that, \(2^{-i}r \leq \rho < 2^{-i+1}r\). Then, using (2.30) we have that

\[
\sigma([x - \rho, x]) \leq \sigma([x - 2^{-i+1}r, x]) \leq A^{i-1} \sigma([x - r, x]) \leq A^{-1}(A2^\beta)^i \left(\frac{\rho}{r}\right)^\beta \sigma([x - r, x]). \tag{2.32}
\]

Taking into account (2.31) we have that \(A = \frac{C}{1+C} \leq \frac{1}{2^\beta}\). Then, by (2.32) we obtain that

\[
\sigma([x - \rho, x]) \leq A^{-1} \left(\frac{\rho}{r}\right)^\beta \sigma([x - r, x]).
\]

Since \(\beta + \alpha > 1\) we have the estimate

\[
\int_0^r \frac{\sigma([x - \rho, x])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \leq \frac{A^{-1}}{r^\beta} \sigma([x - r, x]) \int_0^r \rho^{\beta + \alpha - 2} d\rho \leq \frac{A^{-1}}{\beta + \alpha - 1} \frac{\sigma([x - r, x])}{r^{1-\alpha}}. \tag{2.33}
\]

From the hypothesis \(\sigma \in D^-\) it follows that \(\sigma([x - r, x]) \leq C\sigma([x - 2r, x - r])\). Then, applying (2.33), we have that

\[
\int_0^r \frac{\sigma([x - \rho, x])}{\rho^{1-\alpha}} \frac{d\rho}{\rho} \leq \frac{A^{-1}}{\beta + \alpha - 1} C^{1/p} \frac{\sigma([x - r, x])^{1/q}}{r^{1-\alpha}} \frac{\sigma([x - 2r, x - r])^{1/p}}{r^{1-\alpha}}.
\]

This shows that (2.28) holds and completes the proof of the proposition. \(\square\)

3. The case of equal weights

As we have observed in Section 1, the class of weights \(w\) such that \(I_\alpha^+\) maps \(L^p(w), 1 < p < \infty\), boundedly into itself, is non-empty. In fact, it is non-empty even in the case \(p = 1\).

Proof of Theorem 1.5: \((1) \Rightarrow (2)\): It follows from the inequality \(M_\alpha^+(f)(x) \leq I_\alpha^+(|f|)(x)\).

\((2) \Rightarrow (3)\): We assume that

\[
\int M_\alpha^+ f(x)w(x) \, dx \leq C \int |f(x)|w(x) \, dx.
\]
Let $x$ be a Lebesgue point for $w$, $h > 0$ and $I = (x, x + h)$. If $a = \text{essinf}_{y \in I} w(y)$ and $\varepsilon > 0$ we consider the set $E = \{x \in I : w(y) \leq a + \varepsilon\}$ and the function $f = |E|^{-1} \chi_E$. It is clear that for any $y < x$

$$M^+_\alpha f(y) \geq \frac{1}{(x + h - y)^{1-\alpha}} \int_y^{x+h} f(y) \, dy = \frac{1}{(x + h - y)^{1-\alpha}}.$$

Therefore,

$$\int_{-\infty}^{x} \frac{w(y)}{(x + h - y)^{1-\alpha}} \, dy \leq \frac{C}{|E|} \int \chi_E w \leq C(a + \varepsilon).$$

Thus,

$$\int_{-\infty}^{x} \frac{w(y)}{(x + h - y)^{1-\alpha}} \, dy \leq Ca \leq \frac{1}{h} \int_x^{x+h} w(y) \, dy.$$

When $h$ goes to zero the left hand side converges to $I^-_\alpha (f)(x)$ while the right hand side converges to $w(x)$.

$(3) \implies (1)$: Indeed,

$$\int_{-\infty}^{+\infty} |I^+_\alpha (f)| \, w \, dx \leq \int_{-\infty}^{+\infty} I^+_\alpha (|f|) w \, dx$$

$$= \int_{-\infty}^{+\infty} |f| I^-_\alpha (w) \, dx \leq C \int_{-\infty}^{+\infty} |f| w \, dx. \quad \square$$

**Proof of Theorem 1.6**: The proof is similar to the previous one and we omit it. \hfill \square

**Proof of Theorem 1.8**: $(1) \implies (2)$: By duality $w \in F^+_{p,\alpha}$ is equivalent to $w^{1-p'} \in F^-_{p',\alpha}$. It follows easily that the operators $M_1(g) = [w^{1/p} I^+_\alpha (w^{-1/p}|g|^{p'})]^{1/p'}$ and $M_2(g) = [w^{-1/p} I^-_\alpha (w^{1/p}|g|^p)]^{1/p}$ are bounded from $L^{pp'}(\mathbb{R})$ into itself. Applying the Rubio de Francia algorithm, see [2, Lemma 5.1, p.434], we can obtain a weight $v$ such that

$$M_1(v) \leq Cv \quad \text{and} \quad M_2(v) \leq Cv.$$

Then, $w_0 = w_1^{1/p} v^p$ belongs to $F^+_{1,\alpha}$ and $w_1 = w^{-1/p} v^{p'}$ belongs to $F^-_{1,\alpha}$. Clearly $w = w_0 w_1^{1-p}$.

$(2) \implies (1)$: We suppose that $w = w_0 w_1^{1-p}$, with $w_0 \in F^+_{1,\alpha}$ and $w_1 \in F^-_{1,\alpha}$. It follows easily from Hölder’s inequality that

$$|I^+_\alpha (f)(x)|^p \leq I^+_\alpha (|f|^{p} w_1^{1-p})(x) I^+_\alpha (w_1)(x)^{p-1}. $$
Therefore, by duality
\[
\int_{-\infty}^{+\infty} |I^+(f)(x)|^p w(x) \, dx \leq \int_{-\infty}^{+\infty} I^+_\alpha(|f|^p w_1^{1-p})(x)I^+_\alpha(w_1)(x)^{p-1} w(x) \, dx
\]
\[
= \int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} I^-_\alpha[I^+_\alpha(w_1)^{p-1} w](x) \, dx
\]
\[
\leq C \int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} I^-_\alpha[w^{p-1} w](x) \, dx
\]
\[
= C \int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} I^-_\alpha(w_0)(x) \, dx
\]
\[
\leq C \int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} w_0(x) \, dx
\]
\[
= C \int_{-\infty}^{+\infty} |f(x)|^p w(x) \, dx.
\]

In the rest of the paper we will make some remarks about the classes $F_{p,\alpha}^+$. 

**Proposition 3.1.** Let $w$ be a weight and $0 < \alpha < 1$. Then 

(a) $F_{1,\alpha}^+ \subset F_{p,\alpha}^+$ for $1 < p < \infty$; 

(b) if $w \in F_{1,\alpha}^+$ and $f$ is a non-negative increasing function then $f w \in F_{p,\alpha}^+$; 

(c) there exists a weight $u_0 \in F_{1,\alpha}^+$ for all $0 < \alpha < 1$, that is not essentially increasing; 

(d) for any $1 < p < \infty$ there exists a weight $u \in F_{p,\alpha}^+ \setminus F_{1,\alpha}^+$; 

(e) there exists an increasing weight $w$ such that $w / \notin F_{p,\alpha}^+$.  

**Proof:** 
(a) Theorem 1.5 states that $w \in F_{1,\alpha}^+$ is equivalent to $I^-_\alpha w \leq C w$. 

Therefore $(I^-_\alpha w)^{p'} w^{1-p'} \leq C w$ and the result follows from Theorem 1.1.

In order to prove (b) we observe that it is easy to check that if $w$ satisfies part (3) of Theorem 1.5 then so does $f w$ for any non-negative increasing $f$.

(c) Simple computations show that the function $u$ defined by 

\[
u(x) = \sum_{n=1}^{\infty} 2^n x(2-2n,2-2n+2)(x) + e^x x(1,\infty)(x),\]

satisfies $I^-_\alpha(u)(x) \leq C u(x)$ almost everywhere and it is clearly not increasing.

(d) Let $u_0$ be the function defined in part (c) and $u_1(x) = u_0(1-x)$. From the equality $I^+_\alpha(u_1)(x) = I^+_\alpha(u_0)(1-x)$ it follows that $u_1$ belongs to $F_{1,\alpha}^-$. By Theorem 1.8 we have that $w = u_0 u_1^{1-p}$ belongs to the class $F_{p,\alpha}^+$. 

□
We shall show that there does not exist a constant $C$ such that $I^-_\alpha(w)(x) \leq Cw(x)$ a.e. Let $x$ be such that $2^{-2n_0} < 1 - x \leq 2^{-2n_0+2}$ for some $n_0 > 1$. Then, $u(x) = 2^{n_0(1-p)+1}$, while for any $x \in [3/4,1)\)

$$I^-_\alpha(w)(x) \geq \int_0^{1/4} \frac{w(y)}{(x-y)^{1-\alpha}} dy = \sum_{n=2}^{\infty} 2^{n+1-np} \int_{2^{-n}}^{2^{-n+2}} \frac{1}{(x-y)^{1-\alpha}} dy \geq 3 \sum_{n=2}^{\infty} 2^{n+1-np} 2^{-2n} = A > 0.$$ In consequence the inequality $I^-_\alpha(w)(x) \leq Cw(x)$ a.e. would imply $0 < A < 2^{n_0(1-p)+1}$ for every $n_0 > 1$.

The function $w(x) = \chi_{[0,\infty)}(x)$ satisfies that $I^-_\alpha(w)(x) = \frac{2^n}{\alpha} \chi_{[0,\infty)}(x)$ and (e) follows.

**Proposition 3.2.** Let $w$ be a weight. Then,

(a) for any $0 < \gamma < 1$, there exists $u$ satisfying:

(i) $u \in F_{1,\alpha}^+$ for every $\alpha : \gamma < \alpha < 1$,

(ii) $u \notin F_{1,\alpha}^+$ for every $0 < \alpha \leq \gamma$;

(b) let $\alpha,\beta > 0$. If $w \in F_{1,\alpha}^+$ then $I^-_\beta(w) \in F_{1,\alpha}^+$;

(c) for every $1 \leq p < \infty$, if $0 < \alpha < \beta$ then $F_{p,\alpha}^+ \subset F_{p,\beta}^+$.

**Remark 3.3.** It follows from (a) of Proposition 3.1 and (c) of Proposition 3.2 that for any $0 < \alpha < 1 < \beta$ and $1 < p < \infty$ we have $F_{1,\alpha}^+ \subset F_{p,\beta}^+$. This inclusion provides easy examples of equal weights satisfying conditions (1.4) and (1.5) in [5, p.728].

**Proof:** In order to prove (a) we consider the sequence $a_n = 1 - \frac{1}{2^n}$, $n \geq 0$ and we define the function

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{(x-a_{n-1})^\gamma} \chi(a_{n-1},a_n)(x) + e^x \chi_{[1,\infty)}(x).$$

It is an easy but tedious computation to check that $I^-_\alpha(u)(x) \leq Cu(x)$ for any $\gamma < \alpha < 1$. On the other hand, for $0 < \alpha < \gamma$ and any positive integer $n_0$, if $1 < x < 1 + 2^{-n_0}$ we have

$$I^-_\alpha(u)(x) \geq \int_0^{1} \frac{u(y)}{(x-y)^{1-\alpha}} dy \geq \sum_{n=1}^{n_0} \int_{a_{n-1}}^{a_n} \frac{dy}{(y-a_{n-1})^\gamma (1 + 2^{-n_0} - y)^{1-\alpha}}.$$
A change of variables gives

\[ I_{\alpha}^{-}(u)(x) \geq C \sum_{n=1}^{n_0} 2^n(\gamma - \alpha). \]

Therefore, the inequality \( I_{\alpha}^{-}(u)(x) \leq C u(x) \) almost everywhere for \( 1 < x < 1 + 2^{-n_0} \) would imply \( \sum_{n=1}^{n_0} 2^n(\gamma - \alpha) \leq C e^2 \) for every \( n_0 > 1 \).

Part (b) is a consequence of the equality \( I_{\alpha}^{-} \circ I_{\beta}^{-}(w) = I_{\beta}^{-} \circ I_{\alpha}^{-}(w) \).

We shall prove part (c). Let us assume that \( w \in F_{p,\alpha}^{+} \). There exists a positive integer \( n > 1 \) such that \( \alpha < \beta < n\alpha \). Then, for any positive \( f \) we have

\[
I_{\beta}^{+}(f)(x) = \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\beta}} \, dy \\
\leq \int_{x+1}^{\infty} \frac{f(y)}{(y-x)^{1-n\alpha}} \, dy + \int_{x}^{x+1} \frac{f(y)}{(y-x)^{1-\alpha}} \, dy \\
\leq I_{\alpha}^{+} \circ I_{\alpha}^{+} \circ \cdots \circ I_{\alpha}^{+}(f)(x) + I_{\alpha}^{+}(f)(x),
\]

which implies that \( I_{\beta}^{+} \) is bounded from \( L^p(w) \) into itself. \( \square \)

**References**


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