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# Asymptotic stability for a nonlinear evolution equation

ZHANG HONGWEI, CHEN GUOWANG

*Abstract.* We establish the asymptotic stability of solutions of the mixed problem for the nonlinear evolution equation  $(|u_t|^{r-2}u_t)_t - \Delta u_{tt} - \Delta u - \delta \Delta u_t = f(u)$ .

*Keywords:* nonlinear evolution equation, mixed problem, asymptotic stability of solutions

*Classification:* 35L35, 35L25

## 1. Introduction

This paper deals with asymptotic stability, as time tends to infinity, of solutions of the following mixed problem

$$(1.1) \quad (|u_t|^{r-2}u_t)_t - \Delta u_{tt} - \Delta u - \delta \Delta u_t = f(u), \quad x \in \Omega, \quad t > 0,$$

$$(1.2) \quad u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0,$$

$$(1.3) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

where  $\Omega \subset \mathbb{R}^n$  ( $n \geq 1$  is a natural number) is a bounded open set with smooth boundary  $\partial\Omega$ ,  $r \geq 2$  and  $\delta > 0$  are real number. Problems related to the equation

$$(1.4) \quad f(u_t)u_{tt} - \Delta u_{tt} - \Delta u = 0$$

are interesting not only from the point of view of PDE general theory, but also due to its applications in mechanics. For instance, when the material density,  $f(u_t)$ , is equal to 1, Equation (1.4) describes the extensional vibrations of thin rods, see Love [1] for the physical details. When the material density  $f(u_t)$  is not constant, we are dealing with a thin rod which possesses a rigid surface and whose interior is somehow permissive to slight deformations such that the material density varies according to the velocity, see [2], [3]. J. Ferreira and M.A. Rojas-Medar [2] have studied the existence of global weak solutions to the problem (1.1)–(1.3) with

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$\delta = 0$  in noncylindrical domain. Cavalcanti et al. [3] studied the existence and uniform decay of global weak solution to the following problem

$$(|u_t|^{r-2}u_t)_t - \Delta u_{tt} - \Delta u - \delta \Delta u_t + \int_0^t g(t-z)\Delta u(z) dz = 0$$

with initial and boundary condition, where  $r > 2$  and  $\delta > 0$  are constants,  $g$  represents the kernel of the memory term. However, no asymptotic stability result was presented in [2], [3] for the problem (1.1)–(1.3). In this paper, we study the asymptotic stability of solutions of the problem (1.1)–(1.3). Throughout this paper, we use the following notations.  $(\cdot, \cdot)$  denotes the inner product of  $L^2(\Omega)$ .  $\|\cdot\|$ ,  $\|\cdot\|_r$  and  $\|\cdot\|_0$  denote the norms of the spaces  $L^2(\Omega)$ ,  $L^r(\Omega)$  and  $H_0^1(\Omega)$  respectively.

### 2. Main theorem

We assume that the function  $f(s)$  satisfies the following condition

(H)  $|f(s)| \leq a|s|^{p-1}$ ,  $0 \leq F(s) \leq a|s|^p$ ,

where  $F(s) = \int_0^s f(\rho)d\rho$  for  $2 < p \leq \infty$  if  $n = 1, 2$  or for  $2 < p \leq \frac{2n}{n-2}$  if  $n \geq 3$ , and  $a$  is a positive constant. Furthermore, let  $2 \leq r \leq p$ .

Now, we define the energy associated with Equation (1.1) by

$$E(t) = \frac{r-1}{r}\|u_t\|_r^r + \frac{1}{2}\|\nabla u_t(t)\|^2 + J(u(t)), \quad t \in \mathbb{R}^+ = [0, +\infty),$$

where

$$J(u) = J(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2 - \int_{\Omega} F(u(t)) dx.$$

We see that the energy has the so-called energy identity

(2.1) 
$$E(t) + \delta \int_0^t \|\nabla u_t(s)\|^2 ds = E(0),$$

where  $E(0) = \frac{r-1}{r}\|u_1\|_r^r + \frac{1}{2}\|\nabla u_1\|^2 + J(u_0)$  is the initial energy. Obviously,  $E(t)$  is a non-increasing function in time.

**Lemma 2.1.** *Let  $u_0 \in H_0^1(\Omega)$  and  $u_1 \in H_0^1(\Omega)$ . Then under the assumption (H), the problem (1.1)–(1.3) possesses at least one weak solution  $u : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$  with*

$$u \in L^\infty(0, \infty; H_0^1(\Omega)), \quad u_t \in L^\infty(0, \infty; H_0^1(\Omega)), \quad u_{tt} \in L^2(0, \infty; H_0^1(\Omega)),$$

and for all  $\eta \in C_0^\infty(0, T; H_0^1)$  we have

$$\begin{aligned} & \left[ (|u_t(s)|^{r-2} u_t(s), \eta(s)) + (\nabla u_t(s), \nabla \eta(s)) \right] \Big|_{s=0}^{s=t} \\ &= \int_0^t \left[ (|u_t(s)|^{r-2} u_t(s), \eta_t(s)) + (\nabla u_t(s), \nabla \eta_t(s)) - (\nabla u(s), \nabla \eta(s)) \right. \\ & \quad \left. - \delta(\nabla u_t(s), \nabla \eta(s)) + (f(u(s)), \eta(s)) \right] ds. \end{aligned}$$

The proof of Lemma 2.1 is omitted, since the proof of Lemma 2.1 is analogous to Theorem 3.1 in [2].

In order to get the asymptotic stability of the solution of the problem (1.1)–(1.3), we introduce the set

$$\Sigma = \{(\lambda, E(0)) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 \leq \lambda < \lambda_1, 0 \leq \frac{1}{2}\lambda^2 - aC_0^p \lambda^p < E(0) < E_1\},$$

where

$$\lambda_1 = \left( \frac{1}{paC_0^p} \right)^{\frac{1}{p-2}}, \quad E_1 = \lambda_1^2 \left( \frac{1}{2} - \frac{1}{p} \right)$$

and  $C_0$  is the embedding constant (when  $H_0^1$  is embedded into  $L^p$ ).

Then our main theorem reads as follows:

**Main theorem.** *Under the assumptions of Lemma 2.1, if  $(\|\nabla u_0\|, E(0)) \in \Sigma$  and  $u$  is a solution of the problem (1.1)–(1.3), then*

$$(2.2) \quad \lim_{t \rightarrow \infty} E(t) = 0.$$

We divide the proof into several steps.

**Lemma 2.2.** *Let  $u$  be a weak solution of the problem (1.1)–(1.3). If  $(\|\nabla u_0\|, E(0)) \in \Sigma$ , then for all  $t \in \mathbb{R}^+$ ,*

- (i)  $(\|\nabla u(t)\|, E(t)) \in \Sigma$ ;
- (ii)  $E(t) \geq \frac{r-1}{r} \|u_t\|_r^r + \frac{1}{2} \|\nabla u_t\|^2$ ;
- (iii)  $\frac{1}{2} \|\nabla u\|^2 - \frac{1}{2} (f(u), u) \geq \frac{1}{4} \|\nabla u\|^2$ .

PROOF: By the definition of  $E(t)$ , (H) and embedding theorem, we have

$$(2.3) \quad E(t) \geq \frac{r-1}{r} \|u_t\|_r^r + \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 - aC_0^p \|\nabla u\|_p \geq G(\|\nabla u\|),$$

where  $G(\lambda) = \frac{1}{2}\lambda^2 - aC_0^p\lambda^p$ . It is easy to see that  $G(\lambda)$  attains its maximum  $E_1$  for  $\lambda = \lambda_1$ ,  $G(\lambda)$  is strictly decreasing for  $\lambda \geq \lambda_1$  and  $G(\lambda) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ . Since  $E(t) \leq E(0) < E_1$  for  $t \in \mathbb{R}^+$  by (2.1), we have  $\|\nabla u\| < \lambda_1$  for  $t \in \mathbb{R}^+$ . From (2.3) and  $G(\|\nabla u\|) \geq 0$  for  $0 \leq \|\nabla u\| < \lambda_1$ , we get  $E(t) \geq G(\|\nabla u\|) \geq 0$ , so (i) holds.

To obtain (ii), it remains to note that  $G(\|\nabla u\|) \geq 0$  whenever  $0 \leq \|\nabla u\| < \lambda_1$  and to use (2.3) again, then (ii) follows at once.

By (H) and embedding theorem, we obtain

$$\frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}(f(u), u) \geq \frac{1}{4}\|\nabla u\|^2 + \frac{1}{2}\left(\frac{1}{2}\|\nabla u\|^2 - aC_0^p\|\nabla u\|^p\right).$$

Hence (iii) holds since  $0 \leq \|\nabla u(t)\| < \lambda_1$  for  $t \in \mathbb{R}^+$  and  $G(\|\nabla u\|) \geq 0$  for  $0 \leq \|\nabla u\| < \lambda_1$ . The lemma is proved.  $\square$

**Lemma 2.3.** *Let  $(\|\nabla u_0\|, E(0)) \in \Sigma$  and  $E(t) \geq \beta$ , where  $\beta > 0$ . Then there exists  $\alpha = \alpha(\beta) > 0$  such that*

$$(2.4) \quad \frac{r-1}{r}\|u_t\|_r^r + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}(f(u), u) \geq \alpha, \quad \text{for } t \in \mathbb{R}^+.$$

PROOF: By the definition of  $E(t)$ , (H) and  $E(t) \geq \beta$ , we have

$$(2.5) \quad \frac{r-1}{r}\|u_t\|_r^r + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 \geq \beta, \quad t \in \mathbb{R}^+.$$

Now suppose that (2.4) does not hold. For Lemma 2.1(iii), there is a sequence  $\{t_n\} \subset \mathbb{R}^+$  such that

$$\begin{aligned} & \frac{r-1}{r}\|u_t(t_n)\|_r^r + \frac{1}{2}\|\nabla u_t(t_n)\|^2 + \frac{1}{2}\|\nabla u(t_n)\|^2 - \frac{1}{2}(f(u(t_n)), u(t_n)) \\ & \geq \frac{r-1}{r}\|u_t(t_n)\|_r^r + \frac{1}{2}\|\nabla u_t(t_n)\|^2 + \frac{1}{4}\|\nabla u(t_n)\|^2 \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Then we get

$$\frac{r-1}{r}\|u_t(t_n)\|_r^r + \frac{1}{2}\|\nabla u_t(t_n)\|^2 \rightarrow 0, \quad \|\nabla u(t_n)\|^2 \rightarrow 0, \quad n \rightarrow \infty.$$

This is contradiction with (2.5). The lemma is proved.  $\square$

PROOF OF MAIN THEOREM: Suppose that (2.2) fails. Then there exists  $\beta > 0$  such that  $E(t) \geq \beta$  for all  $t \in \mathbb{R}^+$  since (2.1) and  $E(t) \geq 0$  by Lemma 2.2 (i).

Multiplying both sides of (1.1) by  $u$ , integrating over  $[T, t]$  ( $0 < T \leq t < \infty$ ) and integrating by parts with respect to  $t$ , we obtain

$$\begin{aligned}
 (2.6) \quad & \left[ (|u_t(s)|^{r-2} u_t(s), u(s)) + (\nabla u_t(s), \nabla u(s)) \right] \Big|_{s=T}^t \\
 & = \int_T^t \left\{ \frac{3r-2}{r} \|u_t(s)\|_r^r + 2 \|\nabla u_t(s)\|^2 - 2 \left[ \frac{r-1}{r} \|u_t(s)\|_r^r \right. \right. \\
 & \quad \left. \left. + \frac{1}{2} \|\nabla u_t(s)\|^2 + \frac{1}{2} \|\nabla u(s)\|^2 - \frac{1}{2} (f(u(s)), u(s)) \right] - \delta(\nabla u(s), \nabla u_t(s)) \right\} ds \\
 & = \int_T^t (I_1 + I_2 + I_3) ds.
 \end{aligned}$$

Using  $H_0^1 \hookrightarrow L^r$ ,  $E(t) \leq E(0) < \infty$ , Hölder inequality and  $\|\nabla u_t\|^2 \in L^1(0, \infty)$ , we have

$$\begin{aligned}
 (2.7) \quad & \int_T^t I_1 ds \leq C_1 \int_T^t (\|\nabla u_t(s)\|^r + \|\nabla u_t(s)\|^2) ds \\
 & \leq C_2 (E^{\frac{r-1}{r}}(0) + E^{\frac{1}{2}}(0)) \int_T^t \|\nabla u_t(s)\| ds \\
 & \leq C_3 \left( \int_T^t \|\nabla u_t(s)\|^2 ds \right)^{\frac{1}{2}} \left( \int_T^t ds \right)^{\frac{1}{2}} \\
 & \leq C_4 \left( \int_T^t ds \right)^{\frac{1}{2}}.
 \end{aligned}$$

Here and in the following  $C_i$  ( $i = 1, 2, \dots$ ) denotes positive constants which do not depend on  $t$  and  $T$ . By virtue of Lemma 2.3, we have

$$(2.8) \quad \int_T^t I_2 ds \leq -2\alpha \int_T^t ds.$$

Furthermore, by use of  $\|\nabla u\| \leq \lambda_1$ ,  $E(t) \geq 0$ , Lemma 2.2, Hölder inequality and  $\|\nabla u_t\|^2 \in L^1(0, \infty)$ , we have

$$\begin{aligned}
 (2.9) \quad & \int_T^t I_3 \leq \delta \left( \int_T^t \|\nabla u_t(s)\|^2 ds \right)^{\frac{1}{2}} \left( \int_T^t \|\nabla u(s)\|^2 ds \right)^{\frac{1}{2}} \\
 & \leq \lambda_1 \delta \left( \int_T^\infty \|\nabla u_t(s)\|^2 ds \right)^{\frac{1}{2}} \left( \int_T^t ds \right)^{\frac{1}{2}} \leq C_5 \left( \int_T^t ds \right)^{\frac{1}{2}}.
 \end{aligned}$$

Then from (2.6)–(2.9) we know

$$\begin{aligned}
 (2.10) \quad & \left[ (u_t(s)|^{r-2} u_t(s), u(s)) + (\nabla u_t(s), \nabla u(s)) \right] \Big|_{s=T}^t \\
 & \leq C_6 \left( \int_T^t ds \right)^{\frac{1}{2}} - 2\alpha \int_T^t ds.
 \end{aligned}$$

On the other hand, from Young inequality,  $H_0^1 \hookrightarrow L^r$ ,  $\|\nabla u\| \leq \lambda_1 < \infty$ ,  $E(t) < E(0) < \infty$  and Lemma 2.2(i), we get

$$\begin{aligned} & \left| (|u_t(t)|^{r-2}u_t(t), u(t)) + (\nabla u_t(t), \nabla u(t)) \right| \\ & \leq C_7 \left( \|u_t\|_r^r + \|\nabla u\|^r + \|\nabla u_t\|^2 + \|\nabla u\|^2 \right) \leq C_8 < \infty. \end{aligned}$$

In turn, we reach a contradiction with (2.10) for fixing  $T$  when  $t \rightarrow \infty$ . Hence we derive  $\lim_{t \rightarrow \infty} E(t) = 0$ . This completes the proof.  $\square$

**Remark 1.** If we take  $f(s) = |s|^{p-2}s$  in (1.1), then  $F(s) = \frac{1}{p}|s|^p$  and  $\frac{1}{p}sf(s) = F(s)$ , so (H) holds. By straightforward calculation we get

$$\lambda_1 = C_0^{-\frac{p}{p-2}}, \quad E_1 = \left( \frac{1}{2} - \frac{1}{p} \right) \left( \frac{1}{C_0^p} \right)^{\frac{2}{p-2}}.$$

It is easy to see that  $E_1$  is exactly the potential well depth corresponding to the problem (1.1)–(1.3) obtained by Payne and Sattinger [10], that is

$$E_1 = \inf_{u \in H_0^1 \setminus \{0\}} \sup_{\lambda \in \mathbb{R}} J(\lambda u),$$

where  $J(u) = \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|_p^p$ .

**Remark 2.** If the initial point  $(\|u_0\|, E(0))$  lies in set

$$\begin{aligned} \Sigma_0 = \left\{ (\lambda, E(0)) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 \leq \lambda < \lambda_2 = \left( \frac{1}{2pcC_0^p} \right)^{\frac{1}{p-2}}, \right. \\ \left. 0 \leq \frac{1}{4}\lambda^2 - aC_0^p\lambda^p < E(0) < E_2 = \frac{1}{2}\lambda_1^2 \left( \frac{1}{2} - \frac{1}{p} \right) \right\}, \end{aligned}$$

which is smaller than  $\Sigma$ , we can prove (2.2) and moreover,

$$\lim_{t \rightarrow \infty} \|\nabla u(t)\|^2 = 0.$$

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