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A sufficient condition for maximal resolvability of topological spaces

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Abstract. We show a new theorem which is a sufficient condition for maximal resolvability of a topological space. We also discuss some relationships between various theorems about maximal resolvability.

Keywords: maximally resolvable space, base at a point, π-base

Classification: 54A10, 54A25

At the beginning of XX century the problem of resolvability of a topological space became a matter of intense research and the subject of various publications. The first results were achieved by W. Sierpiński in [S]. He proved that if in a metric space $X$ each non-void open set contains at least $m \geq \aleph_0$ points, then $X$ is the union of $m$ disjoint sets every of which contains at least $m$ points of each non-void open set in $X$. In 1943 E. Hewitt considered the problem of determining the largest number of possible pairwise disjoint dense subsets in a topological space (including locally compact Hausdorff spaces and metric spaces). In 1964 J.G. Ceder generalized some of these results in the work [C]. In [CGF] W.W. Comfort and S. Garcia-Ferreira gave a brief introduction to the theory of spaces which are resolvable in the Hewitt sense. We will make use of some definitions and theorems introduced in [C] and [CGF]. The main aim of this paper is to give some theorems on resolvability in the language of bases at points.

Throughout the paper $X$ will denote a topological space which is dense-in-itself, i.e. no point of $X$ is isolated in $X$. Let $w(X)$ stand for the weight of $X$, that means

$$w(X) = \min\{|B| : B \text{ is a base in } X\}.$$  

A dispersion character of $X$ is a cardinal given by

$$\Delta(X) = \min\{|U| : U \text{ is a non-void open subset of } X\}.$$  

Let $\kappa$ be an arbitrary cardinal greater than 1. A space $X$ is called $\kappa$-resolvable if there is a family of $\kappa$-many pairwise disjoint and dense subsets of $X$. If $X$ is
κ-resolvable it becomes apparent that \( \kappa \leq \Delta(X) \). The space \( X \) is called maximally resolvable if it is \( \Delta(X) \)-resolvable. We say that a space \( X \) is cardinality-homogeneous (in short card-homogeneous) if every non-empty open subset \( V \) of \( X \) satisfies condition \(|V| = |X|\).

W.W. Comfort and S. Garcia-Ferreira in [CGF] presented the following theorem:

**Theorem 1.** If a topological space \( X \) is card-homogeneous satisfying condition \( w(X) \leq |X| \), then it is maximally resolvable.

We can formulate a similar theorem using bases at points.

**Theorem 2.** Let \( X \) be a card-homogeneous topological space and \( X_0 \) be a dense subset of \( X \). If for every \( x \in X_0 \) there exists a point base \( B(x) \) such that

\[ |B(x)| \leq |X|, \]

then \( X \) is maximally resolvable.

From [CGF, Lemma 3.5 and Remark 3.6] (see also [B, Proposition 2]), by the use the notions of \( \pi \)-base and \( \pi \)-weight of a topological space we obtain the following

**Theorem 3.** If \( X \) is a card-homogeneous topological space such that \( \pi w(X) \leq |X| \), then \( X \) is maximally resolvable.

**Remark 4.** The following implications take place: assumption of Th. 1 ⇒ assumption of Th. 2 ⇒ assumption of Th. 3.

(Hence Theorem 1 results from Theorem 2 and Theorem 2 results from Theorem 3.) Indeed, assume that \( w(X) \leq |X| \). Let \( B \) be a base of \( X \) such that \( |B| = w(X) \). For \( X_0 = X \) and a point \( x \in X \) we put

\[ B(x) = \{U \in B : x \in B\}. \]

Then \( B(x) \subset B \), so \(|B(x)| \leq |B| = w(X) \leq |X|\).

Now, assume that for every point \( x \) belonging to a dense \( X_0 \) subset of \( X \) there exists a point base \( B(x) \) such that \(|B(x)| \leq |X|\). Put \( B = \bigcup_{x \in X_0} B(x) \). Then \( B \) is a \( \pi \)-base of \( X \) such that \(|B| \leq |X|\), so \( \pi w(X) \leq |B| \leq |X| \). □

Observe that for metric spaces, the assumptions of Theorems 1, 2 and 3 are equivalent. We do not know whether this holds in general.

The main aim of next theorems is to omit the assumption that \( X \) is card-homogeneous. By a dispersion character of a space \( X \) at a point \( x \in X \) we mean a cardinal number

\[ \Delta(X, x) = \min\{|U| : U \text{ is an open neighbourhood of } x \text{ in } X\}. \]
Lemma 5. If $X$ is a dense-in-itself topological space of cardinality $\kappa$, then there exist pairwise disjoint, open and card-homogeneous sets $G_\alpha$, $\alpha < \kappa$, such that

$$X = \bigcup_{\alpha < \kappa} G_\alpha.$$ 

Proof: Consider a relation $\prec$ on $X$ defined as follows. Let $\Gamma = \{\Delta(X, x) : x \in X\}$. For each $\gamma \in \Gamma$ let $\prec_\gamma$ be a well ordering of the set $K_\gamma = \{x \in X : \Delta(X, x) = \gamma\}$. For any $x, y \in X$ we say that $x \prec y$ if either $\Delta(X, x) < \Delta(X, y)$ or $x, y$ are in $K_\gamma$ for some $\gamma \in \Gamma$ and $x \prec_\gamma y$. Then $X$ is well ordered by $\prec$. Thus $\prec$ is isomorphic to the set of ordinals less than an ordinal $\xi$ of cardinality $|\xi| = \kappa$, with the usual ordering (this set is usually identified with $\xi$). Hence we can arrange points of the set $X$ as $x_\alpha$, $\alpha < \xi$, and we have $x_\alpha \prec x_\beta$ iff $\alpha < \beta$.

For the point $x_0$, pick its neighbourhood $U$ such that $|U| = \Delta(X, x_0)$. Put $G_0 = U$. Thus $G_0$ is card-homogeneous since if $V \neq \emptyset$ is an open subset of $G_0$ then for any $y \in V$ we have $y = x_\beta$ for some $\beta \geq 0$, so

$$|V| \leq |G_0| = \Delta(X, x_0) \leq \Delta(X, x_\beta) \leq |V|.$$ 

Assume that $0 < \alpha < \xi$ and that the sets $G_\gamma$, $\gamma < \alpha$ have been chosen. If $X = \bigcup_{\gamma < \alpha} G_\gamma$, we put $G_\gamma = \emptyset$ for $\alpha \leq \gamma < \xi$. Otherwise, pick the smallest $\zeta < \xi$ such that $x_\zeta \in X \setminus \bigcup_{\gamma < \alpha} G_\gamma$. Take a neighbourhood $V$ of $x_\zeta$ such that $|V| = \Delta(X, x_\zeta)$ and put

$$G_\alpha = V \setminus \bigcup_{\gamma < \alpha} G_\gamma.$$ 

Then $|G_\alpha| = \Delta(X, x_\zeta)$ and similarly as for $G_0$ we show that $G_\alpha$ is card-homogeneous. This finishes the construction. Since $|\xi| = \kappa$, we can renumber sets $G_\alpha$ by indices $\alpha < \kappa$. Thus $X = \bigcup_{\gamma < \kappa} G_\gamma$. □

Theorem 6. Let $X$ be an arbitrary dense-in-itself topological space and let $X_0$ be a dense subset of $X$. If for every point $x \in X_0$ there exists a local base $B(x)$ at $x$ such that

$$|B(x)| \leq \Delta(X),$$ 

then $X$ is maximally resolvable.

Proof: By Lemma 5 there exists a disjoint family of open and card-homogeneous sets $G_\alpha$, $\alpha < |X|$, such that $X = \bigcup_{\alpha} G_\alpha$. We will use Theorem 2 for sets $G_\alpha$ and their dense subsets $G_\alpha \cap X_0$. Fix $\alpha < |X|$. For every point $x \in G_\alpha \cap X_0$ pick a local base $B(x)$ satisfying condition $|B(x)| \leq \Delta(X) \leq |G_\alpha|$. For any $\alpha$, each set
\(G_\alpha\) is \(|G_\alpha|\)-resolvable and according to the inequality \(\Delta(X) \leq |G_\alpha|\) it is \(\Delta(X)\)-resolvable. Hence we obtain \(\Delta(X)\)-many pairwise disjoint and dense sets \(S^{(\alpha)}_\gamma\), \(\gamma < \Delta(X)\). Denote \(X_\gamma = \bigcup_{\alpha} S^{(\alpha)}_\gamma\), \(\gamma < \Delta(X)\). Then, the sets \(X_\gamma\) are pairwise disjoint. We shall prove that they are dense in \(X\). For a fixed \(\alpha\), every set \(S^{(\alpha)}_\gamma\) is dense in \(G_\alpha\), so

\[
S^{(\alpha)}_\gamma \supset G_\alpha \Rightarrow \bigcup_{\alpha} S^{(\alpha)}_\gamma \supset \bigcup_{\alpha} G_\alpha.
\]

Hence

\[
X_\gamma = \bigcup_{\alpha} S^{(\alpha)}_\gamma = \bigcup_{\alpha} S^{(\alpha)}_\gamma \supset \bigcup_{\alpha} G_\alpha = X.
\]

The above inclusions imply that \(X_\gamma\) are dense in \(X\) and the space \(X\) is maximally resolvable.

In [C] J.G. Ceder obtained a similar theorem

**Theorem 7.** If \(X\) is dense-in-itself topological space satisfying condition \(w(X) \leq \Delta(X)\), then \(X\) is maximally resolvable.

A general sufficient condition for the maximal resolvability of a topological space is the following theorem proved by A. Bella in [B] (see also [CGF]).

**Theorem 8.** If \(X\) is dense-in-itself topological space satisfying condition \(\pi w(X) \leq \Delta(X)\), then \(X\) is maximally resolvable.

**Remark 9.** We claim that if a space \(X\) satisfies the assumption of Theorem 7, then it satisfies the assumptions of Theorems 6 and 8.

First we show that the assumption of Theorem 6 drives from inequality \(w(X) \leq \Delta(X)\). Put \(X_0 = X\). We can take a base \(\mathcal{B}\) such that \(|\mathcal{B}| = w(X)\) and for \(x \in X\) we put \(\mathcal{B}(x) = \{U \in \mathcal{B} : x \in U\}\). Then \(\mathcal{B}(x) \subset \mathcal{B}\), so \(|\mathcal{B}(x)| \leq |\mathcal{B}| = w(X) \leq \Delta(X)\).

The assumption of Theorem 8 drives immediately from inequality \(\pi w(X) \leq w(X) \leq \Delta(X)\).

The following examples witness that the above assumptions are not equivalent.

**Example 10.** We shall find a space which fulfils the assumption of Theorem 6 but does not these of Theorems 7 and 8.

Let \(X_1\) be a discrete topological space of cardinality \(c = |\mathbb{R}|\) and \(\mathbb{Q}_+\) be the set of nonnegative rationals. Put \(X = X_1 \times \mathbb{Q}_+\). In \(X\) we introduce a topology in the following way: if \(p = (x, 0)\), then a neighbourhood of \(p\) is of the form \(U(p, r) = \{x\} \times ([0, r) \cap \mathbb{Q}_+)\) where \(r > 0\); if \(p = (x, q)\), \(q \neq 0\) then a neighbourhood of \(p\) is of the form \(U(p, r) = \{x\} \times ((x - r, x + r) \cap \mathbb{Q}_+)\), where \(0 < r < |q|\). In this topology every open set is countable, so \(\Delta(X) = \aleph_0\). The space \(X\) has a countable base at every point \(p\), which can be taken as the family of open sets
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$U(p, \frac{1}{n})$, $n \in \mathbb{N}$. Hence $X$ fulfils the assumption of Theorem 6, so is maximally resolvable.

The space $X$ does not fulfil the assumption of Theorem 8 (so it does not fulfil the assumption of Theorem 7). Observe that $d(X) = \mathfrak{c}$, where $d(X)$ stands for the smallest cardinality of a dense subset of $X$. Indeed, $\{U(p, 1) : p \in X_1\}$ is a disjoint family of cardinality $\mathfrak{c}$, it consists of open sets, and each member of its family contains a point of a fixed dense set. As for the space $X$ the following inequalities $w(X) \geq \pi w(X) \geq d(X) > \Delta(X) = \aleph_0$ come true, the space $X$ neither has a countable $\pi$-base nor a countable base.

**Example 11.** This example introduces a space which fulfils the assumption of Theorem 8 but does not fulfil the assumption of Theorem 7 (does not have a countable base).

Let $X_1$ be a space of negative rationals equipped with the natural topology. Let $X_2$ be the space of positive real numbers equipped with Hashimoto topology generated by the ideal of nowhere dense sets. We put $X = X_1 \cup X_2$. As open sets in $X_1$ are countable, $\Delta(X) = \aleph_0$. Obviously $X_1$ has a countable base which is a countable $\pi$-base of this space. As every open set in $X_2$ contains an interval, an Euclidean base of $X_2$ is a $\pi$-base of $X_2$. Hence we obtain $\pi w(X) = \aleph_0$.

The space $X$ does not have a countable base, because $X_2$ does not have such a base. This follows from the fact that the number of open sets in $X_2$ is $\leq 2^{w(X_2)}$ but in $X_2$ we have $2^\mathfrak{c}$ open sets (every set of the form $(0, 1) \setminus A$, where $A$ is a subset of the Cantor set, is open in $X_2$).

**Example 12.** There exists a space $X$ which fulfils the assumption of Theorem 8 but does not fulfil the assumption of Theorem 6 (does not have a countable base in any point).

Let $X = (\mathbb{R}, T)$ be the space introduced in Example 11 (it does have a countable $\pi$-base). Let us fix a point $x \in \mathbb{R}$. Suppose that the family $\{B_\alpha : \alpha < \omega\}$ is a countable base at the point $x$. Each of the sets $B_\alpha$ is of the form $B_\alpha = U_\alpha \setminus I_\alpha$ ($U_\alpha$ is an interval, $I_\alpha$ is nowhere dense in natural topology) and $x \in B_\alpha$ for every $\alpha < \omega$. The set $I = \bigcup_{\alpha < \omega} I_\alpha$ is of first category. Let $C \subset \mathbb{R} \setminus I$ be a Cantor set such that $x \in C$. The set $U = (\mathbb{R} \setminus C) \cup \{x\}$ is an open neighbourhood of $x$ in $T$ and does not include any set $B_\alpha$.

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**References**


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