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Non-autonomous implicit integral equations
with discontinuous right-hand side

GIOVANNI ANELLO, PAOLO CUBIOTTI

Abstract. We deal with the implicit integral equation

\[ h(u(t)) = f(t, \int_I g(t, z) u(z) \, dz) \text{ for a.a. } t \in I, \]

where \( I := [0, 1] \) and where \( f : I \times [0, \lambda] \to \mathbb{R}, \ g : I \times I \to [0, +\infty[ \) and \( h : ]0, +\infty[ \to \mathbb{R}. \)

We prove an existence theorem for solutions \( u \in L^\infty(I) \) where the continuity of \( f \) with respect to the second variable is not assumed.

Keywords: implicit integral equations, discontinuity, lower semicontinuous multifunctions, operator inclusions, selections

Classification: 45P05, 47G10

1. Introduction

Let \( I := [0, 1] \) and \( J := [0, \lambda], \) with \( \lambda > 0. \) Let us first consider the implicit integral equation

\[ h(u(t)) = f\left( \int_I g(t, z) u(z) \, dz \right) \text{ for a.a. } t \in I, \]

where \( f : J \to \mathbb{R}, \ g : I \times I \to [0, +\infty[ \) and \( h : ]0, +\infty[ \to \mathbb{R}. \) Recently, in [4], an existence theorem for solutions \( u \in L^\infty(I) \) of equation (1) has been proved, where, unlike other recent results in the field, the continuity of the function \( f \) is not assumed. More precisely, \( f \) is assumed to be a.e. equal to a function \( f^* : J \to \mathbb{R} \) such that the set

\[ \{ x \in J : f^* \text{ is discontinuous at } x \} \]

has null Lebesgue measure. It is immediate to check that such a function \( f \) can be discontinuous at each point of the set \( J. \)

For the special case where \( h \) is the identity mapping, the latter result has been later extended to the non-autonomous version of problem (1), that is to the equation

\[ u(t) = f\left( t, \int_I g(t, z) u(z) \, dz \right) \text{ for a.a. } t \in I, \]
where \( f : I \times J \to \mathbb{R} \) (see Theorem 1 of [6]). For this latter problem, the above assumption (which specifies what kind of discontinuity is allowed for \( f \)) has the following form: there exists a function \( f^* : I \times J \to \mathbb{R} \) and a set \( E \subseteq J \), with null Lebesgue measure, such that \( f(\cdot, x) \) is measurable for each \( x \) in a countable dense subset of \( J \) and, for a.a. \( t \in I \), one has

\[
\{ x \in J : f^*(t, \cdot) \text{ is discontinuous at } x \} \cup \{ x \in J : f^*(t, x) \neq f(t, x) \} \subseteq E.
\]

(3) It was also proved that none of the two sets on the left hand side of (3) can depend on \( t \).

At this point, it is natural to consider the implicit non-autonomous integral equation

\[
h(u(t)) = f(t, \int_I g(t, z) u(z) \, dz) \quad \text{for a.a. } t \in I,
\]

(4) which contains equations (1) and (2) as special cases, and to ask whether it is possible to extend to this latter problem the existence results of [4] and [6]. Our effort in this paper goes exactly in such a direction. Indeed, our aim is to prove the following result (where \( m \) denotes the Lebesgue measure on the real line and “int” stands for “interior”).

**Theorem 1.** Let \( I := [0, 1] \) and \( J := [0, \lambda] \), with \( \lambda > 0 \). Let \( s \in ]1, +\infty[ \), \( A \subseteq ]0, +\infty[ \) an interval, \( h : A \to \mathbb{R} \) a continuous functions. Let \( f : I \times J \to \mathbb{R} \), \( g : I \times I \to ]0, +\infty[ \), \( \beta \in L^s(I) \), \( \phi_0 \in L^j(I) \), with \( j \geq s' \) and \( j > 1 \), \( \phi_1 \in L^{s'}(I) \), and let \( P \) be a countable dense subset of \( J \). Assume that:

(i) there exist a function \( f^* : I \times J \to \mathbb{R} \) and two sets \( E_1, E_2 \subseteq J \), with \( E_2 \) closed and \( m(E_1 \cup E_2) = 0 \), such that for each \( x \in P \) the function \( f^*(\cdot, x) \) is measurable and for a.a. \( t \in I \) one has

\[
\{ x \in J : f^*(t, x) \neq f(t, x) \} \subseteq E_1
\]

(5) and

\[
\{ x \in J : f^*(t, \cdot) \text{ is discontinuous at } x \} \subseteq E_2;
\]

(6)

(ii) \( \text{int } h^{-1}(z) = \emptyset \) for all \( z \in \text{int } h(A) \);

(iii) if one puts

\[
v(t) := \text{ess inf}_{x \in J} f(t, x), \quad z(t) := \text{ess sup}_{x \in J} f(t, x),
\]

then for a.a. \( t \in I \) one has

\[
[v(t), z(t)] \subseteq h(A) \quad \text{and} \quad \sup h^{-1}([v(t), z(t)]) \leq \beta(t);
\]

(7)
(iv) one has
\[ 0 < \| \phi_0 \|_{L^{s'}(I)} \leq \frac{\lambda}{\| \beta \|_{L^{s}(I)}} ; \]
(v) for each \( t \in I \), the function \( g(t, \cdot) \) is measurable;
(vi) for a.a. \( z \in I \), the function \( g(\cdot, z) \) is continuous in \( ]0, 1[ \) and
\[ g(t, z) \leq \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t, z) \leq \phi_1(z) \quad \text{for all} \quad t \in ]0, 1[ . \]

Then there exists a solution \( \hat{u} \in L^{s}(I) \) to equation (4).

Theorem 1 partially extends the main results of [4] and [6] to problem (4). Such an extension is not full since it is assumed, in addition, that the set \( E_2 \) is closed. The reader can easily check that such a function \( f \) can be discontinuous (with respect to the second variable) at each point \( x \in J \). In particular, our assumption is weaker than the usual Carathéodory condition assumed in the literature (in this connection, the reader can see for instance [3], [7], [8], [10] and the references therein; in particular, we refer to [10] and to the references therein for motivations for studying equation (4)). The proof of Theorem 1 will be given in Section 3, while in Section 2 we shall fix some notations and give some preliminary technical results.

2. Notations and preliminary results

As before, \( m \) denotes the usual Lebesgue measure over the real line \( \mathbb{R} \). Moreover, we denote by \( \mathcal{L}(A) \) (resp., \( \mathcal{B}(A) \)) the family of all Lebesgue (resp., Borel) measurable subsets of the set \( A \). In the sequel, the word “measurable” will stand for “Lebesgue measurable”. Also, we denote by \( \overline{A} \) and \( \overline{co} A \) the closure and the closed convex hull of the set \( A \), respectively.

If \( p \in [1, +\infty[ \), we denote by \( p' \) the conjugate exponent of \( p \). As usual, we denote by \( L^p(I) \) the space of all (equivalence classes of) measurable functions \( u : I \to \mathbb{R} \) such that
\[ \int_I |u(t)|^p \, dt < +\infty \quad \text{if} \quad p < +\infty, \]
\[ \text{ess sup}_{t \in I} |u(t)| < +\infty \quad \text{if} \quad p = +\infty, \]
with the usual norm
\[ \| u \|_{L^p(I)} := \left( \int_I |u(t)|^p \, dt \right)^{\frac{1}{p}} \quad \text{if} \quad p < +\infty, \]
\[ \| u \|_{L^{\infty}(I)} := \text{ess sup}_{t \in I} |u(t)| \quad \text{if} \quad p = +\infty. \]
Moreover, we denote by $C^0(I)$ the space of all continuous functions $v : I \to \mathbb{R}$.

From now on, we denote by $X$ the space $\{0,1\}^\mathbb{N}$ endowed with the product topology, and we put

$$D := \left\{ \{a_n\} \in X : a_n = 0 \text{ for infinitely many } n \right\} \cup \{1_n\}$$

($\{1_n\}$ denoting the sequence which has each term equal to 1),

$$C := \left\{ \{a_n\} \in X : \{a_{2n}\} \in D \text{ and } \{a_{2n-1}\} \in D \right\},$$

$$H := \{s \in [0,1] : s = \frac{p}{2^m}, \text{ with } p, m \in \mathbb{N} \text{ and } p \leq 2^m \} \cup \{0\},$$

$$\Omega := (I \setminus H) \times (J \setminus \lambda H).$$

Finally, let $\varphi : X \to I \times J$ be the function defined by putting, for each $\{a_n\} \in X$,

$$\varphi(\{a_n\}) = \left( \sum_{n=1}^{\infty} \frac{a_{2n}}{2^n}, \lambda \sum_{n=1}^{\infty} \frac{a_{2n-1}}{2^n} \right).$$

The following lemma follows easily by well-known facts and can be checked directly by the reader.

**Lemma 2.** The function $\varphi$ is continuous in $X$ and its restriction $\varphi|_C : C \to I \times J$ is a bijection. Moreover, the function $(\varphi|_C)^{-1} : I \times J \to C$ is continuous at each point $(t, x) \in \Omega$.

For the definitions and the basic facts about multifunctions, we refer the reader to [2], [14]. Here we only recall that if $Y$ and $S$ are nonempty sets and $F : Y \to 2^S$ is a multifunction, then a function $f : Y \to S$ is called a selection of $F$ if $f(x) \in F(x)$ for all $x \in Y$. The following result comes directly from the proof of Lemma 2 of [19] (for the definition and the basic properties of 0-dimensional spaces, the reader is referred to [9]).

**Lemma 3.** Let $Y$ and $S$ be two metric spaces, and assume that $Y$ is 0-dimensional. Let $G : Y \to 2^S$ be a multifunction with nonempty and complete values, and let $M \subseteq Y$ a given set. If $G$ is lower semicontinuous at each point of $Y \setminus M$, then there exists a selection $s : Y \to S$ of $G$ which is continuous at each point of $Y \setminus M$.

**Lemma 4.** Let $S$ be a metric space, let $V \subseteq I \times J$ and $B \subseteq I \times J$ be two given sets (with $B \neq \emptyset$), and $F : B \to 2^S$ be a multifunction with nonempty and complete values. Assume that $F$ is lower semicontinuous at each point of $B \setminus V$. 
Then there exists a selection \( g : B \to S \) of \( F \) which is continuous at each point of the set \((B \cap \Omega) \setminus V\).

**Proof:** Let us put for simplicity \( \varphi_C := \varphi|_C \), and let \( Y := \varphi_C^{-1}(B) \). Then the space \( Y \) is 0-dimensional. Let \( G : Y \to 2^S \) be the multifunction defined by putting, for each \( \{a_n\} \in Y \),

\[
G(\{a_n\}) = F(\varphi(\{a_n\})).
\]

Since \( \varphi \) is continuous in \( X \), \( G \) is lower semicontinuous at each point of \( Y \setminus \varphi^{-1}(V) \). By Lemma 3, there exists a selection \( s : Y \to S \) of \( G \) which is continuous at each point of \( Y \setminus \varphi^{-1}(V) \). For each \((t, x) \in B\), let us put

\[
g(t, x) := s(\varphi^{-1}(t, x)).
\]

At this point, it is immediate to check that \( g \) satisfies the conclusion. \( \square \)

The following lemma follows at once from the proof of Lemma 2.3 of [1].

**Lemma 5.** Let \( Y \) and \( S \) be metric spaces, with \( S \) separable, \( F : Y \to 2^S \) a multifunction with nonempty values, \( \{u_n\} \) a dense sequence in \( S \), and \( y_0 \in Y \). Let \( d \) denotes the distance in \( S \). Then one has:

(a) if \( F \) is lower semicontinuous at \( y_0 \), then for each \( u \in S \) the function \( y \in Y \to d(u, F(y)) \) is upper semicontinuous at \( y_0 \);

(b) if for each \( n \in \mathbb{N} \) the function \( y \in Y \to d(u_n, F(y)) \) is upper semicontinuous at \( y_0 \), then \( F \) is lower semicontinuous at \( y_0 \).

**Lemma 6.** Let \( T \in \mathcal{L}(I) \), let \( f : T \times J \to \mathbb{R} \) be a function and \( E \subseteq J \) a given set. Assume that:

(i) \( f \) is \( \mathcal{L}(T) \otimes \mathcal{B}(J) \)-measurable;

(ii) for each \( t \in T \) one has

\[
\{x \in J : f(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E;
\]

(iii) \( \inf_{T \times J} f > -\infty \).

Then, for each \( \varepsilon > 0 \) there exists \( K \in \mathcal{L}(T) \) such that \( m(T \setminus K) \leq \varepsilon \) and the function \( f|_K \times J \) is lower semicontinuous at each point \((t, x) \in K \times (J \setminus E)\).

**Proof:** Without loss of generality we can assume that \( f(t, x) \geq 0 \) for all \((t, x) \in T \times J\). For each \( n \in \mathbb{N} \), let \( f_n : T \times J \to [0, +\infty[ \) be the function defined by putting, for each \((t, x) \in T \times J\),

\[
f_n(t, x) := \inf_{y \in J} \left[ n|x - y| + f(t, y) \right].
\]
Of course, for each $n \in \mathbb{N}$ and each $(t, x) \in T \times J$ one has $f_n(t, x) \leq f(t, x)$. Consequently, the function $f^* : T \times J \to [0, +\infty[$ defined by

$$f^*(t, x) := \sup_{n \in \mathbb{N}} f_n(t, x)$$

satisfies the inequality

$$f^*(t, x) \leq f(t, x) \quad \text{for all} \quad (t, x) \in T \times J.$$ 

Now, let us observe the following facts.

(a) For each $n \in \mathbb{N}$ and each $x \in J$, the function $f_n(\cdot, x)$ is measurable. This follows from Lemma III.39 of [5], since the function

$$(t, y) \to n |x - y| + f(t, y)$$

is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable.

(b) For each $n \in \mathbb{N}$ and each $t \in T$, the function $f_n(t, \cdot)$ is $n$-Lipschitzian over $J$. Indeed, for each $x, z \in J$ one has

$$f_n(t, x) \leq \inf_{y \in J} \left[ n |x - z| + n |z - y| + f(t, y) \right]$$

$$= n |x - z| + f_n(t, z),$$

hence the claim follows easily.

(c) One has

$$f^*(t, x) = f(t, x) \quad \text{for all} \quad (t, x) \in T \times (J \setminus E).$$

To see this, choose any $(t, x) \in T \times (J \setminus E)$ and $\eta > 0$. Since the function $f(t, \cdot)$ is lower semicontinuous at $x$, there exists $\delta > 0$ such that for each $y \in J$ with $|x - y| < \delta$ one has

$$f(t, y) > \beta := f(t, x) - \eta.$$

Fix $n^* > \beta/\delta$. Then, for each $y \in J$ one has

$$\begin{cases} 
    n^* |x - y| + f(t, y) \geq f(t, y) > \beta & \text{if } |x - y| < \delta \\
    n^* |x - y| + f(t, y) \geq n^* \delta + f(t, y) > \beta + f(t, y) \geq \beta & \text{if } |x - y| \geq \delta.
\end{cases}$$

It follows that $f_{n^*}(t, x) \geq \beta$, hence the claim follows.

Now, choose any $\varepsilon > 0$. By Theorem 2 of [15], for each $n \in \mathbb{N}$ there exists a set $K_n \in \mathcal{L}(T)$ such that

$$m(T \setminus K_n) \leq \frac{\varepsilon}{2^n}.$$
and the function $f_n|_{K_n \times J}$ is continuous. If we put $K := \bigcap_{n \in \mathbb{N}} K_n$, then $K \in \mathcal{L}(T)$, $m(T \setminus K) \leq \varepsilon$ and the function $f^*|_{K \times J}$ is lower semicontinuous. Fix any point $(t^*, x^*) \in K \times (J \setminus E)$, and let us show that the function $f|_{K \times J}$ is lower semicontinuous at $(t^*, x^*)$. To this aim, let $\gamma > 0$. By the lower semicontinuity of $f^*|_{K \times J}$, there exists a neighborhood $U$ of $(t^*, x^*)$ in $K \times J$ such that

$$f^*(t^*, x^*) - \gamma < f^*(t, x) \quad \text{for all} \quad (t, x) \in U.$$ 

By (8) and (9), it follows that

$$f(t, x) \geq f^*(t, x) > f^*(t^*, x^*) - \gamma = f(t^*, x^*) - \gamma \quad \text{for all} \quad (t, x) \in U,$$

as desired. $\Box$

**Lemma 7.** Let $T \in \mathcal{L}(I)$, let $S$ be a separable metric space, $F : T \times J \to 2^S$ a multifunction with nonempty values and $E \subseteq J$ a given set. Assume that:

(i) $F$ is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable;

(ii) for each $t \in T$ one has

$$\{ x \in J : F(t, \cdot) \text{ is not lower semicontinuous at } x \} \subseteq E.$$

Then, for each $\varepsilon > 0$ there exists a set $K \in \mathcal{L}(T)$ such that $m(T \setminus K) \leq \varepsilon$ and the multifunction $F|_{K \times J}$ is lower semicontinuous at each point $(t, x) \in K \times (J \setminus E)$.

**Proof:** Let $\rho$ be an equivalent distance over $S$ such that $\rho \leq 1$, and let $\{y_n\}$ be a dense sequence in $S$. By Proposition 13.2.2 of [14], for each $y \in S$ the function $\rho(y, F(\cdot , \cdot))$ is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable. Moreover, by Lemma 5, for each $t \in T$ and each $y \in S$ one has that

$$\{ x \in J : \rho(y, F(t, \cdot)) \text{ is not upper semicontinuous at } x \} \subseteq E.$$

Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, applying Lemma 6 to the function $-\rho(y_n, F(\cdot , \cdot))$, we have that there exists $K_n \in \mathcal{L}(T)$ such that

$$m(T \setminus K_n) \leq \frac{\varepsilon}{2^n}$$

and the function

$$\rho(y_n, F(\cdot , \cdot))|_{K_n \times J}$$

is upper semicontinuous at each point $(t, x) \in K_n \times (J \setminus E)$. Putting $K := \bigcap_{n \in \mathbb{N}} K_n$, we have that $m(T \setminus K) \leq \varepsilon$ and for each $n \in \mathbb{N}$ the function

$$\rho(y_n, F(\cdot , \cdot))|_{K \times J}$$

is upper semicontinuous at each point $(t, x) \in K \times (J \setminus E)$. By Lemma 5 our claim follows. $\Box$
Lemma 8. Let $S$ be a separable metric space, $F : I \times J \to 2^S$ a multifunction with nonempty complete values, $E \subseteq J$ a given set. Assume that:

(i) $F$ is $\mathcal{L}(I) \otimes \mathcal{B}(J)$-measurable;
(ii) for each $t \in I$ one has

$$\{ x \in J : F(t, \cdot) \text{ is not lower semicontinuous at } x \} \subseteq E.$$

Then, there exists a selection $\phi : I \times J \to S$ of $F$ such that:

(a) for a.a. $t \in I$, one has

$$\{ x \in J : \phi(t, \cdot) \text{ is discontinuous at } x \} \subseteq E \cup \lambda H;$$

(b) for each $x \in J \setminus (E \cup \lambda H)$, the function $\phi(\cdot, x)$ is measurable.

Proof: By Lemma 7, the interval $I$ can be partitioned into a sequence of measurable sets $\{K_n\}$ and in one negligible set $Y$ such that for each $n \in \mathbb{N}$ the multifunction $F|_{K_n \times J}$ is lower semicontinuous at each point $(t, x) \in K_n \times (J \setminus E)$. By Lemma 4, for each $n \in \mathbb{N}$ there exists a function $g_n : K_n \times J \to S$ such that

$$g_n(t, x) \in F(t, x) \quad \text{for all } (t, x) \in K_n \times J$$

and $g_n$ is continuous at each point $(t, x) \in [K_n \times (J \setminus E)] \cap \Omega$. For each $t \in Y$, let $h_t : J \to S$ be any selection of the multifunction $F(t, \cdot)$. Now, let the function $\phi : I \times J \to S$ be defined by putting, for each $(t, x) \in I \times J$,

$$\phi(t, x) = \begin{cases} g_n(t, x) & \text{if } t \in K_n \\ h_t(x) & \text{if } t \in Y. \end{cases}$$

Of course, $\phi$ is a selection of $F$. To show conclusion (a), choose $t^* \in I \setminus (Y \cup H)$, and let $n \in \mathbb{N}$ be such that $t^* \in K_n$. Since $t^* \notin H$, we have that $g_n : K_n \times J \to S$ is continuous at each point $(t^*, x)$ with $x \in J \setminus (E \cup \lambda H)$. Hence, we have that

$$\{ x \in J : g_n(t^*, \cdot) \text{ is discontinuous at } x \} \subseteq E \cup \lambda H.$$

Since one has $\phi(t^*, \cdot) = g_n(t^*, \cdot)$, (a) follows. To show (b), fix $\hat{x} \in J \setminus (E \cup \lambda H)$. Observe that for each $n \in \mathbb{N}$ the function $g_n : K_n \times J \to S$ is continuous at each point $(t, \hat{x})$ such that $t \in K_n \setminus H$. It follows that $g_n(\cdot, \hat{x}) : K_n \to S$ is continuous at each point $t \in K_n \setminus H$, hence the function $g_n(\cdot, \hat{x})|_{K_n \setminus H}$, being continuous, is measurable. Since $H$ and $Y$ are negligible, the conclusion follows. \qed
3. Proof of Theorem 1

Without loss of generality we can assume that (5), (6) and (7) hold for all \( t \in I \). Moreover, we can assume \( j < +\infty \).

Firstly, let us show that \( v(t) \) and \( z(t) \) are measurable in \( I \). Indeed, by assumption (i) it is not difficult to check that for each \( t \in I \) one has

\[
\begin{align*}
v(t) &= \inf_{x \in J \setminus E_2} f^*(t, x), \\
z(t) &= \sup_{x \in J \setminus E_2} f^*(t, x).
\end{align*}
\]

Again by (i), the set \( P \cap (J \setminus E_2) \) is dense in \( J \setminus E_2 \) and countable. Hence, the function \( f^*|_{I \times (J \setminus E_2)} \) is \( \mathcal{L}(I) \otimes \mathcal{B}(J \setminus E_2) \)-measurable by the Lemma at p. 198 of [15]. By Lemma III.39 of [5] our claim follows.

Let \( l : I \to \mathbb{R} \) be any measurable function such that

\[
(11) \quad v(t) \leq l(t) \leq z(t) \quad \text{for all} \quad t \in I,
\]

and let \( \hat{f} : I \times J \to \mathbb{R} \) be defined by

\[
\hat{f}(t, x) = \begin{cases} f^*(t, x) & \text{if } x \notin E_2 \\ l(t) & \text{if } x \in E_2. \end{cases}
\]

Since \( E_2 \) is closed, (6) implies that for each \( t \in I \) one has

\[
(12) \quad \{ x \in J : \hat{f}(t, \cdot) \text{ is discontinuous at } x \} \subseteq E_2.
\]

Moreover, the function \( \hat{f} \) is \( \mathcal{L}(I) \otimes \mathcal{B}(J) \)-measurable and by (10) and (11), one has

\[
(13) \quad v(t) \leq \hat{f}(t, x) \leq z(t) \quad \text{for all} \quad (t, x) \in I \times J.
\]

Now, observe that by (ii) and by Theorem 2.4 of [18] the function \( h \) is inductively open. That is, there exists a set \( Y \in \mathcal{B}(A) \) such that \( h|_Y \) is open and \( h(Y) = h(A) \). It follows that the multifunction \( T : h(A) \to 2^Y \) defined by

\[
T(s) = h^{-1}(s) \cap Y
\]

is lower semicontinuous in \( h(A) \) with nonempty values. Let \( G : I \times J \to 2^Y \) be defined by

\[
G(t, x) = T(\hat{f}(t, x)) = h^{-1}(\hat{f}(t, x)) \cap Y
\]

(\( G \) is well defined by (7) and (13)). Then \( G \) is \( \mathcal{L}(I) \otimes \mathcal{B}(J) \)-measurable and, by (12), for all \( t \in I \) one has

\[
\{ x \in J : G(t, \cdot) \text{ is not lower semicontinuous at } x \} \subseteq E_2.
\]
Consequently, the multifunction

\[(t, x) \in I \times J \rightarrow \overline{G(t, x)}\]

is \(\mathcal{L}(I) \otimes \mathcal{B}(J)\)-measurable and for each \(t \in I\) one has

\[\{x \in J : \overline{G(t, \cdot)}\text{ is not lower semicontinuous at } x\} \subseteq E_2.\]

By Lemma 8, there exists a selection \(k : I \times J \rightarrow \mathbb{R}\) of the multifunction (14) such that for a.a. \(t \in I\) one has

\[\{x \in J : k(t, \cdot) \text{ is discontinuous at } x\} \subseteq E_2 \cup \lambda H,\]

and for each \(x \in J \setminus (E_2 \cup \lambda H)\) the function \(k(\cdot, x)\) is measurable. For each \(t \in I\), let us put

\[\alpha(t) := \inf h^{-1}([v(t), z(t)]).\]

By the continuity of \(h\) and by (7) and (13) we get

\[k(t, x) \in h^{-1}(\hat{f}(t, x)) \text{ for all } (t, x) \in I \times J\]

and

\[0 < \alpha(t) \leq k(t, x) \leq \beta(t) \text{ for all } (t, x) \in I \times J.\]

Let \(T_1 \subseteq I\) be such that \(m(T_1) = 0\) and (15) holds for all \(t \in I \setminus T_1\). Let \(\psi : I \times \mathbb{R} \rightarrow \mathbb{R}\) be defined by

\[\psi(t, x) = \begin{cases} k(t, x) & \text{if } (t, x) \in (I \setminus T_1) \times (J \setminus E_2) \\ \beta(t) & \text{otherwise.} \end{cases}\]

Then, for each \(t \in I \setminus T_1\) one has

\[\{x \in \mathbb{R} : \psi(t, \cdot) \text{ is discontinuous at } x\} \subseteq E_2 \cup \lambda H.\]

Let \(P' := \lambda ((\mathbb{Q} \cap I) \setminus H)\) (where \(\mathbb{Q}\) denotes the set of rational real numbers). Then \(P'\) is countable and dense in \(J\). If \(P''\) is any countable dense subset of \(\mathbb{R} \setminus J\), then the set \(P^* := P' \cup P''\) is countable and dense in \(\mathbb{R}\), and by the above construction the function \(\psi(\cdot, x)\) is measurable for all \(x \in P^*\).

Thus, all the assumptions of Proposition 2 of [6] are satisfied. Consequently, the multifunction \(F : I \times \mathbb{R} \rightarrow \mathbb{R}\) defined by

\[F(t, x) := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{y \in P''} \{\psi(t, y)\}}_{|y-x| \leq \frac{1}{m}}\]
satisfies the conclusion of the same proposition. Moreover, by the above construction it follows that

\[(18) \quad F(t, x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all} \quad (t, x) \in I \times \mathbb{R}.\]

Now we want to apply Theorem 1 of [17], with \(T = I\), \(X = Y = \mathbb{R}\), \(p = s\), \(q = j'\), \(V = L^s(I)\), \(\Psi(u) = u\), \(r = \|\beta\|_{L^s(I)}\), \(\varphi \equiv +\infty\),

\[\Phi(u)(t) = \int_I g(t, z) u(z) \, dz,\]

and \(F : I \times \mathbb{R} \to 2^\mathbb{R}\) as defined above. To this aim, we argue as in [6] and observe the following facts.

(a) \(\Phi(L^s(I)) \subseteq C^0(I)\). This follows from our assumptions (v) and (vi) and the Lebesgue’s dominated convergence theorem.

(b) If \(v \in L^s(I)\) and \(\{v^k\}\) is a sequence in \(L^s(I)\), weakly convergent to \(v\) in \(L^{j'}(I)\), then the sequence \(\{\Phi(v^k)\}\) converges to \(\Phi(v)\) strongly in \(L^1(I)\). This follows by Theorem 2 at p.359 of [13], since \(g\) is \(j\)-th power summable in \(I \times I\) (note that \(g\) is measurable on \(I \times I\) by the classical Scorza-Dragoni’s theorem; see [20] or also [12]).

(c) By (18), the function

\[\omega : t \in I \to \sup_{x \in \mathbb{R}} d(0, F(t, x))\]

belongs to \(L^s(I)\) and \(\|\omega\|_{L^s(I)} \leq \|\beta\|_{L^s(I)}\) (for what concerns the measurability of \(\omega\), we refer to [17]).

Thus, all the assumptions of Theorem 1 of [17] are satisfied. Consequently there exist \(\hat{u} \in L^s(I)\) and a set \(T_2 \subseteq I\), with \(m(T_2) = 0\), such that

\[(19) \quad \hat{u}(t) \in F(t, \Phi(\hat{u})(t)) \quad \text{for all} \quad t \in I \setminus T_2.\]

We now want to prove that \(\hat{u}(t)\) is a solution of equation (4). To this aim, we argue as in [6]. Firstly, let us observe that by (18) we have

\[(20) \quad \hat{u}(t) \in [\alpha(t), \beta(t)] \quad \text{for all} \quad t \in I \setminus (T_1 \cup T_2).\]

For each \(t \in I\), put

\[\gamma(t) := \Phi(\hat{u})(t) = \int_I g(t, z) \, \hat{u}(z) \, dz.\]
By assumptions (iv) and (v), taking into account (20), for each $t \in I$ we get

$$0 \leq \gamma(t) \leq \|\phi_0\|_{L^s(I)} \cdot \|\hat{u}\|_{L^s(I)} \leq \frac{\lambda}{\|\beta\|_{L^s(I)}} \cdot \|\beta\|_{L^s(I)} = \lambda,$$

hence $\gamma(I) \subseteq J$. By assumptions (v) and (vi), by (20) and by Lemma 2.2 at p.226 of [16], we get

$$\gamma'(t) = \int_I \frac{\partial g}{\partial t}(t, z) \hat{u}(z) \, dz > 0 \text{ for all } t \in ]0, 1[.$$

In particular, the continuous function $\gamma$ is strictly increasing in $I$. Hence, by Theorem 2 of [21] the function $\gamma^{-1}$ is absolutely continuous. Let us put

$$S := \gamma^{-1}[(E_1 \cup E_2 \cup \lambda H) \cap \gamma(I)].$$

By assumption (i) and by Theorem 18.25 of [11] we have that $m(S) = 0$. Let

$$S^* := S \cup T_1 \cup T_2.$$

For each $t \in I \setminus S^*$, since $\gamma(t) \in J \setminus (E_1 \cup E_2 \cup \lambda H)$ and taking into account (17), (19) and Proposition 2 of [6], we get

$$\hat{u}(t) \in F(t, \gamma(t)) = \{\psi(t, \gamma(t))\} = \{k(t, \gamma(t))\}.$$

Consequently, taking into account (5) and (16), for each $t \in I \setminus S^*$ we get

$$h(\hat{u}(t)) = \hat{f}(t, \gamma(t)) = \hat{f}(t, \gamma(t)) = f(t, \gamma(t)) = f(t, \int_I g(t, z) \hat{u}(z) \, dz).$$

This ends our proof. \qed

**Remark.** The example at p.245 of [4] shows that in the assumption (vi) of Theorem 1 one cannot assume that

$$0 \leq \frac{\partial g}{\partial t}(t, z) \leq \phi_1(z).$$

Moreover, the Example at the end of [6] shows that none of the sets $E_1$, $E_2$ in the statement of Theorem 1 can depend on $t$. 
References

[17] Naselli Ricceri O., Ricceri B., An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t, \Phi(u)(t))$ and application to a multivalued boundary value problem, Appl. Anal. 38 (1990), 259–270.

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