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Rings of continuous functions vanishing at infinity

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Abstract. We prove that a Hausdorff space X is locally compact if and only if its topology coincides with the weak topology induced by $C_\infty(X)$. It is shown that for a Hausdorff space X, there exists a locally compact Hausdorff space Y such that $C_\infty(X) \cong C_\infty(Y)$. It is also shown that for locally compact spaces X and Y, $C_\infty(X) \cong C_\infty(Y)$ if and only if $X \cong Y$. Prime ideals in $C_\infty(X)$ are uniquely represented by a class of prime ideals in $C^*(X)$. ∞-compact spaces are introduced and it turns out that a locally compact space X is ∞-compact if and only if every prime ideal in $C_\infty(X)$ is fixed. The existence of the smallest ∞-compact space in βX containing a given space X is proved. Finally some relations between topological properties of the space X and algebraic properties of the ring $C_\infty(X)$ are investigated. For example we have shown that $C_\infty(X)$ is a regular ring if and only if X is an ∞-compact P_∞-space.

Keywords: σ-compact, pseudocompact, ∞-compact, ∞-compactification, P_∞-space, P-point, regular ring, fixed and free ideals

Classification: 54C40

1. Introduction

Throughout this article, the space X stands for a nonempty completely regular Hausdorff space. We denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real valued continuous functions on the space X, ideals are assumed to be proper ideals and the reader is referred to [7] for undefined terms and notations. Kohls in [9] has proved that the intersection of all free maximal ideals in $C^*(X)$ is precisely the set $C_\infty(X)$ consisting of all continuous functions f in $C(X)$ which vanish at infinity, in the sense that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for each $n \in \mathbb{N}$. Kohls has also shown that the set $C_K(X)$ of all functions in $C(X)$ with compact support is the intersection of all the free ideals in $C(X)$ and of all the free ideals in $C^*(X)$. $C_K(X)$ is an ideal of $C(X)$ and it is easy to see that $C_\infty(X)$ is an ideal in $C^*(X)$ but not in $C(X)$, see also [4], [9] and 7D in [7]. In fact $C_\infty(X)$ is a subring of $C(X)$ and topological spaces X for which $C_\infty(X)$ is an ideal of $C(X)$ are characterized in [4]. Our main purpose in this article is the study of the ring structure of $C_\infty(X)$ and of the relations between topological properties of the space X and algebraic properties of the ring $C_\infty(X)$.

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This article consists of four sections. In Section 2, we will characterize locally compact spaces X by the structure of the ring $C_\infty(X)$. We will see that for studying the ring $C_\infty(X)$, it suffices to consider the topological space X to be a locally compact space. It is shown that whenever X and Y are locally compact, then $C_\infty(X) \cong C_\infty(Y)$ if and only if $X \cong Y$. This part of article is also presented in ICM 2002, see [11]. Section 3 is devoted to the ideal structure of the ring $C_\infty(X)$ and to a new compactness concept, namely the ∞-compactness. In this section prime ideals of $C_\infty(X)$ are investigated and using a special class of prime ideals in $C^*(X)$, a unique representation for prime ideals of $C_\infty(X)$ is given. ∞-*compact* spaces are those spaces X for which $C_K(X) = C_\infty(X)$. We show that for a locally compact space X, every prime ideal in $C_\infty(X)$ is fixed if and only if X is an ∞-compact space. The existence of the smallest ∞-compact space in βX containing X is also proved in this section. We denote this smallest ∞-compact space by ∞X and we call it the ∞-*compactification* of the space X. In the last results of the Section 3, we have characterized the type of points in $\infty X \setminus X$. We have shown that every point in $\infty X \setminus X$ is a non-P-point in βX. In Section 4, the relations between algebraic properties of $C_\infty(X)$ and topological properties of the space X are studied. We have shown that the ring $C_\infty(X)$ is regular if and only if X is an ∞-compact P_∞-*space* (a space X for which $Z(f)$ is open for every $f \in C_\infty(X)$). We will also observe that the ring $C_\infty(X)$ has a finite Goldie dimension if an only if the only open locally compact subsets of X are finite sets. Finally, locally compact spaces X are characterized for which the ring $C_\infty(X)$ is a Baer ring or a p.p. ring.

The following proposition and its corollary are proved in [4]. They will be used in the next sections.

Proposition 1.1. *$C_\infty(X)$ is an ideal in $C(X)$ if and only if every open locally compact subset of X is relatively pseudocompact. (A subset U of X is called relatively pseudocompact if $f(U)$ is bounded for all $f \in C(X)$.)*

Corollary 1.2. *Let X be a locally compact Hausdorff space. Then $C_\infty(X)$ is an ideal in $C(X)$ if and only if X is a pseudocompact space.*

We also need the following lemma.

Lemma 1.3. *No point of $A \subseteq X$ has a compact neighborhood in X if and only if $f(A) = \{0\}$ for all $f \in C_\infty(X)$.*

Proof: If $a \in A$ and $f(a) \neq 0$ for some $f \in C_\infty(X)$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < |f(a)|$ and hence $H = \{x \in X : |f(x)| \geq \frac{1}{n+1}\}$ is a compact neighborhood of a, a contradiction. Now suppose that the point a has a compact neighborhood H. Then there exists $f \in C(X)$ such that $f(a) = 1$ and $f(X \setminus \operatorname{int} H) = \{0\}$. Since for every $n \in \mathbb{N}$ we have $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq H$, the closed set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact and hence $f \in C_\infty(X)$. This proves the converse. $\square$

For proof of the following proposition, see Corollary 3.6 in [12].

Proposition 1.4. *Let $\mathcal{A}$ be a commutative algebra over the rationals with unity. Let I be an ideal of $\mathcal{A}$. Then an ideal D of I is a maximal ideal of I if and only if $D = M \cap I$ for some maximal ideal M in $\mathcal{A}$.*

2. Characterization of locally compact spaces X by the ring $C_\infty(X)$

We recall that for any topological space X, the set of all continuous real valued functions which vanish at infinity is a ring, which is denoted by $C_\infty(X)$. In fact for every $f, g \in C_\infty(X)$, we have $\{x \in X : |f(x)+g(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{2n}\} \cup \{x \in X : |g(x)| \geq \frac{1}{2n}\}$ and $\{x \in X : |f(x)g(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{\sqrt{n}}\} \cup \{x \in X : |g(x)| \geq \frac{1}{\sqrt{n}}\}$. By the following propositions and corollaries, for studying the ring $C_\infty(X)$, we may consider the space X to be a locally compact space.

Proposition 2.1. *For a Hausdorff space X, the following statements are equivalent:*

1. *X is locally compact;*
2. *$\mathfrak{B} = \{X \setminus Z(f) : f \in C_\infty(X)\}$ is a base for open sets in X;*
3. *the collection $C_\infty(X)$ separates points from closed sets (i.e., whenever F is a closed set in X and $x_0 \notin F$, then there exists $f \in C_\infty(X)$ such that $f(x_0) = 1$ and $f(F) = \{0\}$).*

Proof: (1)→(2). Let G be an open set in X and $x_0 \in G$. Then there exists a compact set H such that $x_0 \in \operatorname{int} H \subseteq H \subseteq G$. Now define $f \in C(X)$ with $f(x_0) = 1$ and $f(X \setminus \operatorname{int} H) = \{0\}$. Since $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq X \setminus Z(f) \subseteq H$, $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact, $\forall n \in \mathbb{N}$, i.e., $f \in C_\infty(X)$ and clearly $x_0 \in X \setminus Z(f) \subseteq G$, i.e., $\mathfrak{B}$ is a base for open sets in X.

(2)→(3). Is clear.

(3)→(1). For every open set G and $x_0 \in G$, there exists $f \in C_\infty(X)$ such that $f(X \setminus G) = \{0\}$ and $f(x_0) = 1$. Therefore $x_0 \in \{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq G$ and by letting $H = \{x \in X : |f(x)| \geq \frac{1}{2}\}$, H is compact and $x_0 \in \operatorname{int} H \subseteq H \subseteq G$ which means that X is locally compact. □

Corollary 2.2. *If X is a Hausdorff space, then X is locally compact if and only if its topology coincides with the weak topology induced by $C_\infty(X)$.*

Proposition 2.3. *For every Hausdorff space X, whenever $C_\infty(X) \neq (0)$, then there exists a locally compact space Y such that $C_\infty(X) \cong C_\infty(Y)$. In fact the space Y may be considered as a nonempty open locally compact subspace of X.*

Proof: Let Y be the set of all points in X which have a compact neighborhood. Clearly Y is a locally compact open subspace of X and since $C_\infty(X) \neq (0)$,

$Y \neq \emptyset$. We may also assume that $Y \neq X$, for otherwise X itself would be a locally compact space. Define $\sigma : C_\infty(X) \to C_\infty(Y)$ by $\sigma(f) = f|_Y$, $\forall f \in C_\infty(X)$. Since by Lemma 1.3, $f(X \setminus Y) = 0$, evidently σ is a one to one function. σ is also onto, for if $g \in C_\infty(Y)$, then we define $g^* : X \to \mathbb{R}$ such that $g^*(x) = g(x)$, $\forall x \in Y$ and $g^*(x) = 0$, $\forall x \in X \setminus Y$. To see the continuity of g^*, it is enough to show that g^* is continuous on the nonempty set $X \setminus Y$. Given $x \in X \setminus Y$ and $\epsilon > 0$, the set $\{x \in Y : |g(x)| \geq \epsilon\}$ is compact in Y and hence in X. Therefore $G = X \setminus \{x \in Y : |g(x)| \geq \epsilon\} = \{x \in X : |g^*(x)| < \epsilon\}$ is an open set in X and $g^*(G) \subseteq (-\epsilon, \epsilon)$, i.e., g^* is continuous at $x \in X \setminus Y$. On the other hand, $\{x \in X : |g^*(x)| \geq \frac{1}{n}\} = \{x \in Y : |g(x)| \geq \frac{1}{n}\}$ implies that $g^* \in C_\infty(X)$. Now $\sigma(g^*) = g$, i.e., σ is onto. Finally, for every $f, g \in C_\infty(X)$ it is easy to see that $\sigma(f + g) = \sigma(f) + \sigma(g)$ and $\sigma(fg) = \sigma(f)\sigma(g)$, i.e., $C_\infty(X) \cong C_\infty(Y)$. □

Proposition 2.4. *If X is a completely regular Hausdorff space, then every maximal ideal of $C_\infty(X)$ is fixed. In fact every maximal ideal of $C_\infty(X)$ is of the form $M_x \cap C_\infty(X)$, where M_x is a fixed maximal ideal in $C(X)$ and the point x has a compact neighborhood.*

Proof: Since $C_\infty(X)$ is the intersection of all free maximal ideals in $C^*(X)$, by Proposition 1.4, every maximal ideal in $C_\infty(X)$ is of the form $M_p^* \cap C_\infty(X)$, where $p \in X$ and $C_\infty(X) \nsubseteq M_p^*$. But if $C_\infty(X) \subseteq M_p^*$ for some $p \in X$, then $f(p) = 0$ for all $f \in C_\infty(X)$ and by Lemma 1.3, the point p has no compact neighborhood. Hence if we consider A to be the set of all points of X which have no any compact neighborhood, then the collection of all maximal ideals of $C_\infty(X)$ is $\{M_x^* \cap C_\infty(X) : x \in X \setminus A\}$. On the other hand, $M_x^* = C^*(X) \cap M_x$, for all $x \in X$, see 4.7 in [7]. This implies that every maximal ideal of $C_\infty(X)$ is of the form $M_x \cap C_\infty(X)$, where $x \in X \setminus A$. □

By the above proposition, whenever X is locally compact, the only maximal ideals of $C_\infty(X)$ are of the form $M_p \cap C_\infty(X)$, where $p \in X$, i.e., we have a one-to-one correspondence between X and the set $\mathfrak{M}$ of all maximal ideals of $C_\infty(X)$. If $\mathfrak{M}$ is equipped with the hull-kernel topology, then using this topological space, as in [7, Theorem 4.9], we have the following theorem.

Theorem 2.5. *Two locally compact spaces X and Y are homeomorphic if and only if $C_\infty(X)$ and $C_\infty(Y)$ are isomorphic.*

We conclude this section by the following proposition which is evident by Corollary 2.2 and the fact that every idempotent of $C_\infty(X)$ is in $C_K(X)$. We recall that a topological space X is said to be zero-dimensional if it has a base consisting of open-closed sets. We refer the reader to [6] for more facts about the zero-dimensional spaces.

Proposition 2.6. *A Hausdorff space X is a locally compact zero-dimensional space if and only if its topology coincides with the weak topology induced by the set of idempotents of $C_\infty(X)$.*

3. Prime ideals of $C_\infty(X)$ and ∞-compact spaces

We devote this section to some important ideals related to $C_\infty(X)$. Prime ideals in $C_\infty(X)$, the z-ideal $C_{l\sigma}(X)$, the ideal $C_K(X)$ and the ideal $C_R(X) = \bigcap_{p\in \upsilon X\setminus X} M^p$ are important ideals related to $C_\infty(X)$. First of all we show that $C_{l\sigma}(X)$ is the smallest z-ideal in $C(X)$ containing $C_\infty(X)$. Next we will characterize topological spaces X for which $C_\infty(X) = C_K(X)$ or $C_\infty(X) = C_R(X)$. Studying the prime ideals of $C_\infty(X)$ and characterization of the type of points in the remainder $\infty X \setminus X$ are the final parts of this section.

We need the following useful lemma which is also proved in [4].

Lemma 3.1. *Let A be an open subset of X. Then $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$ if and only if A is a σ-compact locally compact subset of X.*

Proof: Let $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$. Then $A = \bigcup_{n=1}^\infty A_n$, where $A_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}$. Since each A_n is compact, A is σ-compact. If $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x \in \{y \in X : |f(y)| > \frac{1}{n_0}\} \subseteq A_{n_0}$. Thus we get A is a locally compact subset of X and this proves the necessity. For sufficiency, let A be a σ-compact locally compact subset of X. Then $A = \bigcup_{n=1}^\infty A_n$, where A_n is compact and $A_n \subseteq \operatorname{int} A_{n+1}$ for all $n \in \mathbb{N}$, see [6, p. 250]. Now for each $n \in \mathbb{N}$, there exists $f_n \in C(X)$ such that $f(X) \subseteq [0,1]$, $f_n(A_n) = \{1\}$ and $f_n(X \setminus \operatorname{int} A_{n+1}) = \{0\}$. Then $f = \sum_{n=1}^\infty f_n/2^n$ is an element of $C(X)$ by the Weierstrass M-test. Clearly $A = X \setminus Z(f)$. We claim that $f \in C_\infty(X)$. Let $x_0 \notin A_{n+1}$. Then $f_1(x_0) = \cdots = f_n(x_0) = 0$ and so $f(x_0) \leq \frac{1}{2^{n+1}} + \cdots \leq \frac{1}{2^n} < \frac{1}{n}$. Therefore $x_0 \notin \{x \in X : |f(x)| \geq \frac{1}{n}\}$, and hence $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq A_{n+1}$ and so we get $f \in C_\infty(X)$. ☐

In fact the collection of all the complement of σ-compact locally compact subsets of X is a z-filter $\mathcal{F}$ in X containing $Z[C_\infty(X)]$. By the next proposition, $Z^{-1}[\mathcal{F}]$ is the smallest z-ideal in $C(X)$ containing $C_\infty(X)$.

Proposition 3.2. *Let*

$$C_{l\sigma}(X) = \{f \in C(X) : X \setminus Z(f) \text{ is locally compact } \sigma\text{-compact}\}.$$

Then $C_{l\sigma}(X)$ is the smallest z-ideal in $C(X)$ containing $C_\infty(X)$ or $C_{l\sigma}(X)$ is all of $C(X)$.

Proof: If $g \in C(X)$ and $f \in C_{l\sigma}(X)$, then $X \setminus Z(fg) \subseteq X \setminus Z(f)$ and clearly $X \setminus Z(fg)$ is also locally compact σ-compact, i.e., $fg \in C_{l\sigma}(X)$. Since $X \setminus Z(f+g) \subseteq (X \setminus Z(f)) \cup (X \setminus Z(g))$, we have $f + g \in C_{l\sigma}(X)$ for every $f, g \in C_{l\sigma}(X)$. Hence $C_{l\sigma}(X)$ is an ideal in $C(X)$ and it is evident that $C_{l\sigma}(X)$ is a z-ideal containing $C_\infty(X)$. Now suppose that I is a z-ideal in $C(X)$ such that $C_\infty(X) \subseteq I$. If $f \in C_{l\sigma}(X)$, then $X \setminus Z(f)$ is locally compact σ-compact and hence by Lemma 3.1, there exists $g \in C_\infty(X)$ such that $Z(f) = Z(g)$. But

$g \in C_\infty(X) \subseteq I$ and I is a z-ideal, hence $f \in I$, i.e., $C_{l\sigma}(X) \subseteq I$. We note that $C_{l\sigma}(X) = C(X)$ if and only if X is a locally compact σ-compact space. $\square$

We recall that $C_K(X) = \bigcap_{p\in\beta X\setminus X} O^{*^p} = \bigcap_{p\in\beta X\setminus X} O^p$ and $C_\infty(X) = \bigcap_{p\in\beta X\setminus X} M^{*^p}$, see 7E and 7F in [7]. Obviously $C_K(X) \subseteq C_\infty(X)$ and $C_K(X) = C_\infty(X)$ if and only if every open locally compact σ-compact subset of X is contained in a compact set in X, see [4, Proposition 2.1]. For convenience, whenever $C_K(X) = C_\infty(X)$ we call X an *∞-compact* space. For example, $\mathbb{N}$ and $\mathbb{Q}$ are ∞-compact spaces. Moreover, if we denote $C_R(X) = \bigcap_{p\in\upsilon X\setminus X} M^p$, where υX is the realcompactification of X, then $C_\infty(X) \subseteq C_{l\sigma}(X) \subseteq C_R(X)$. To show the second inclusion, $C_\infty(X) = \bigcap_{p\in\beta X\setminus X} M^{*^p}$ implies that

$$C_\infty(X)C(X) = (\bigcap_{p\in\beta X\setminus X} M^{*^p})C(X) \subseteq \bigcap_{p\in\beta X\setminus X} M^{*^p}C(X).$$

Now by parts b and c of 7.9 in [7], $M^{*^p}C(X) = C(X)$, $\forall p \in \beta X \setminus \upsilon X$ and $M^{*^p}C(X) = M^p$, $\forall p \in \upsilon X$; hence $C_\infty(X)C(X) \subseteq \bigcap_{p\in\upsilon X\setminus X} M^p = C_R(X)$. Since $C_{l\sigma}(X)$ is the smallest z-ideal containing $C_\infty(X)$ and $C_R(X)$ is also a z-ideal containing $C_\infty(X)$, we have $C_{l\sigma}(X) \subseteq C_R(X)$.

The following proposition shows that for a locally compact space X, the equality $C_\infty(X) = C_R(X)$ is equivalent to pseudocompactness of the space X.

Proposition 3.3. *For a locally compact space X, $C_\infty(X) = C_R(X)$ if and only if X is a pseudocompact space.*

Proof: If X is pseudocompact, then $\upsilon X = \beta X$, see 8A in [7]. Hence

$$C_\infty(X) = \bigcap_{p\in\beta X\setminus X} M^{*^p} = \bigcap_{p\in\upsilon X\setminus X} M^{*^p} = \bigcap_{p\in\upsilon X\setminus X} M^p = C_K(X).$$

Conversely, suppose that $C_\infty(X) = \bigcap_{p\in\upsilon X\setminus X} M^p$; then $C_\infty(X)$ is an ideal in $C(X)$ and hence X should be a pseudocompact space by Corollary 1.2. $\square$

Proposition 3.4. *Every locally compact ∞-compact space is a pseudocompact space.*

Proof: Let X be a locally compact ∞-compact space. Then $C_\infty(X) = C_K(X)$, i.e., $C_\infty(X)$ is an ideal in $C(X)$. Now by Corollary 1.2, X is a pseudocompact space. $\square$

Corollary 3.5. *Every locally compact ∞-compact and realcompact space is compact.*

The converse of the Proposition 3.4 is not true, i.e., not every locally compact pseudocompact space has to be an ∞-compact space.

Example 3.6. Consider the Tychonoff plank space T. T is a locally compact pseudocompact space and the ring $C(T)$ has only one free maximal ideal M^t, where $t = (\omega_1, \omega)$ and $M^t \neq O^t$, see 8.20 in [7]. Now since T is pseudocompact, ${M^*}^t = M^t$ and $C_\infty(X) = {M^*}^t \neq O^t = C_K(X)$, i.e., T is not ∞-compact.

Next we are going to characterize prime ideals of the subring $C_\infty(X)$ via prime ideals of $C^*(X)$. By $\mathrm{Spec}(C_\infty(X))$, we mean the set of all prime ideals of the ring $C_\infty(X)$. For details of spectrum for general rings, see [8]. The spectrum of $C_\infty(X)$ might be empty only whenever $C_\infty(X) = (0)$.

Proposition 3.7. *For every completely regular Hausdorff space X, we have*

$$\mathrm{Spec}(C_\infty(X)) = \{P^* \cap C_\infty(X) : P^* \text{ is a prime ideal in } C^*(X) \text{ and } C_\infty(X) \nsubseteq P^*\}.$$

We have $C_\infty(X) \neq (0)$ if and only if $\mathrm{Spec}(C_\infty(X)) \neq \emptyset$.

Proof: For every prime ideal P^* in $C^*(X)$ with $C_\infty(X) \nsubseteq P^*$, clearly $P^* \cap C_\infty(X)$ is a prime ideal in $C_\infty(X)$. Conversely, let P_∞ be a prime ideal in $C_\infty(X)$. Then P_∞ is an ideal in $C^*(X)$, for if $f \in P_\infty$ and $g \in C^*(X)$, then $fg = f^{1/3} f^{2/3} g$ and $f^{2/3} g \in C_\infty(X), f^{1/3} \in P_\infty$ imply that $fg \in P_\infty$. Now suppose that P^* is a prime ideal in $C^*(X)$ minimal over P_∞ and disjoint from the multiplicatively closed set $C_\infty(X) - P_\infty$. It goes without saying that $P_\infty = P^* \cap C_\infty(X)$. To prove the second part of the proposition, suppose that $C_\infty(X) \neq (0)$. Then by Proposition 2.3, there exists a nonempty locally compact space Y such that $C_\infty(X) \cong C_\infty(Y)$. Hence it is enough to show that $\mathrm{Spec}(C_\infty(Y)) \neq \emptyset$. If Y is compact, then $C_\infty(X) = C^*(X)$ and clearly $\mathrm{Spec}(C_\infty(X)) \neq \emptyset$. Thus suppose that Y is not compact. Since Y is locally compact and noncompact, then by 4D in [7], $C_K(Y)$ is free and hence no fixed prime ideal of $C^*(Y)$ contains $C_\infty(Y)$. On the other hand, since $C_\infty(Y)$ is a free ideal of $C^*(X)$, by Theorem 3.1 in [2], $C_\infty(Y)$ intersects every nonzero ideal in $C^*(X)$ nontrivially. Therefore if P^* is a fixed prime ideal in $C^*(Y)$, we have $C_\infty(Y) \nsubseteq P^*$ and $P^* \cap C_\infty(Y) \neq (0)$ which means that $\mathrm{Spec}(C_\infty(Y))$ contains at least a nonzero prime ideal. The converse is evident, for $C_\infty(X) = (0)$ implies that $\mathrm{Spec}(C_\infty(X)) = \emptyset$. □

To establish a one-to-one correspondence between prime ideals of $C_\infty(X)$ and a subclass of prime ideals of $C^*(X)$, we need the following lemma which will also be used in Section 4.

Lemma 3.8. *Let I be an ideal in a commutative ring R. Suppose that Q and P are ideals in R and P is prime. If P does not contain I and $Q \cap I \subseteq P \cap I$, then $Q \subseteq P$. In particular, if Q is also a prime ideal and $Q \cap I = P \cap I$, then $P = Q$.*

Proof: $Q \cap I \subseteq P \cap I$ implies that $Q \cap I \subseteq P$. Since P is prime and $I \nsubseteq P$, we have $Q \subseteq P$. □

The following proposition shows that every prime ideal P_∞ of $C_\infty(X)$ has a unique representation of the form $P_\infty = P^* \cap C_\infty(X)$, where P^* is a prime ideal in $C^*(X)$.

Proposition 3.9. *Let $\mathcal{D}$ be the collection of all prime ideals of $C^*(X)$ which do not contain $C_\infty(X)$. Then $\Phi : \mathcal{D} \to \mathrm{Spec}(C_\infty(X))$ defined by $\Phi(P^*) = P^* \cap C_\infty(X)$ is a one-to-one correspondence.*

Proof: Using Proposition 3.7 and Lemma 3.8 the proof is evident. □

If X has no point with compact neighborhood, then $C_\infty(X) = (0)$ is contained in every ideal of $C^*(X)$. Even if the space X is locally compact, many prime ideals of $C^*(X)$ may contain $C_\infty(X)$. In the following proposition, we show that whenever X is a locally compact ∞-compact space, then all free prime ideals of $C^*(X)$ contain $C_\infty(X)$.

Proposition 3.10. *A locally compact Hausdorff space X is ∞-compact if and only if every prime ideal in $C_\infty(X)$ is fixed.*

Proof: Let X be an ∞-compact space and P_∞ be a prime ideal in $C_\infty(X)$. By Proposition 3.7, there exists a prime ideal P^* in $C^*(X)$ such that $P_\infty = P^* \cap C_\infty(X)$, where $C_\infty(X) \nsubseteq P^*$. P^* is not free, for otherwise $C_\infty(X) = C_K(X) \subseteq P^*$, by ∞-compactness of X and 4D in [7], a contradiction. Hence P^* is fixed and therefore P_∞ is fixed too. Conversely suppose that every prime ideal in $C_\infty(X)$ is fixed but X is not ∞-compact, i.e., $C_\infty(X) \neq C_K(X)$. Hence there exists $f \in C_\infty(X)$ such that $f \notin C_K(X)$. Now consider the prime ideal P^* in $C^*(X)$ containing $C_K(X)$ but not f. Since X is locally compact, then by 4D in [7], $C_K(X)$ is free, so P^* is free. Since $C_\infty(X) \nsubseteq P^*$, $P_\infty = P^* \cap C_\infty(X)$ is a prime ideal in $C_\infty(X)$ by Proposition 3.7. Now $C_K(X) \subseteq P^* \cap C_\infty(X) = P_\infty$ implies that P_∞ is also free which contradicts our hypothesis. □

Remark 3.11. $C_\infty(X)$ may be contained in no prime ideal of $C(X)$. In fact this happens if and only if X is a locally compact σ-compact space. To see this, let P be a prime ideal in $C(X)$ such that $C_\infty(X) \subseteq P$. Thus there exists a maximal ideal M in $C(X)$ such that $C_\infty(X) \subseteq M$. Since $C_{l\sigma}(X)$ is the smallest z-ideal containing $C_\infty(X)$, $C_{l\sigma}(X) \subseteq M$ by Proposition 3.2, which implies that $C_{l\sigma}(X)$ is an ideal in $C(X)$. By definition of the ideal $C_{l\sigma}(X)$, this shows that X is not locally compact or X is not σ-compact. Conversely, suppose that X is either not locally compact or not σ-compact. Then $C_{l\sigma}(X)$ is an ideal of $C(X)$. Now $C_{l\sigma}(X)$ is contained in a maximal ideal of $C(X)$. Clearly, that maximal ideal which is also a prime ideal in $C(X)$ contains $C_\infty(X)$.

$C_\infty(X)$ may contain a prime ideal of $C^*(X)$. If P^* is a prime ideal in $C^*(X)$ and $P^* \subseteq C_\infty(X)$, then $P^* \subseteq \bigcap_{x \in \beta X \setminus X} M^{*x}$ and since every prime ideal in $C^*(X)$ is contained in a unique maximal ideal in $C^*(X)$, $C_\infty(X) = M^{*x}$, where $\beta X \setminus X = \{x\}$. This shows that $C_\infty(X)$ contains a prime ideal of $C^*(X)$ if and only if the cardinal number of the remainder $\beta X \setminus X$ is 1. In this case $C_\infty(X)$ itself is a maximal ideal in $C^*(X)$.

It is time to show the existence of the smallest ∞-compact space in βX containing the space X. To avoid the confusion, we denote the ideals M^p and O^p in $C(X)$ by $M^p(X)$ and $O^p(X)$, respectively. The corresponding ideals in $C^*(X)$ are also denoted by $M^{*^p}(X)$ and $O^{*^p}(X)$.

Theorem 3.12. *Let $\{Y_\alpha\}_{\alpha\in S}$ be a collection of ∞-compact spaces such that $X \subseteq Y_\alpha \subseteq \beta X$, $\forall \alpha \in S$. Then $Y = \bigcap_{\alpha\in S} Y_\alpha$ is also an ∞-compact space.*

Proof: First suppose that $X \subseteq T \subseteq \beta X$ and define the map $\varphi : C^*(X) \to C^*(T)$ by $\varphi(f) = f^\beta|_T$ (denote $f^\beta|_T$ by f^T). It is clear that φ is an isomorphism. Moreover, for every $p \in \beta X$, we have $\varphi(O^{*^p}(X)) = O^{*^p}(T)$ and $\varphi(M^{*^p}(X)) = M^{*^p}(T)$. To see this let $\varphi(f) \in \varphi(O^{*^p}(X))$, where $f \in O^{*^p}(X)$. Then $p \in \operatorname{int}_{\beta X} Z(f^\beta) = \operatorname{int}_{\beta X} Z(f^T)^\beta$ and hence $f^T \in O^{*^p}(T)$ implies that $\varphi(O^{*^p}(X)) \subseteq O^{*^p}(T)$. Since φ is an isomorphism, similarly $\varphi^{-1}(O^{*^p}(T)) \subseteq O^{*^p}(X)$ and hence $\varphi(O^{*^p}(X)) = O^{*^p}(T)$. The proof of $\varphi(M^{*^p}(X)) = M^{*^p}(T)$ is similar. More generally, whenever $A \subseteq \beta X$ we have also $\varphi(O^{*^A}(X)) = O^{*^A}(T)$ and $\varphi(M^{*^A}(X)) = M^{*^A}(T)$. Now for every $\alpha \in S$, let $\varphi_\alpha : C^*(Y) \to C^*(Y_\alpha)$ be an isomorphism defined by $\varphi_\alpha(f) = f^{Y_\alpha}$, $\forall f \in C^*(Y)$. By the above argument we have

$$\begin{aligned}
C_K(Y) &= O^{*^{\beta Y \setminus Y}}(Y) = O^{*^{\beta Y \setminus \cap Y_\alpha}}(Y) = O^{*^{\bigcup(\beta Y_\alpha \setminus Y_\alpha)}}(Y) = \bigcap_{\alpha\in S} O^{*^{\beta Y_\alpha \setminus Y_\alpha}}(Y)\\
&= \bigcap_{\alpha\in S} \varphi_\alpha^{-1}(O^{*^{\beta Y_\alpha \setminus Y_\alpha}}(Y_\alpha)) = \bigcap_{\alpha\in S} \varphi_\alpha^{-1}(C_K(Y_\alpha)) = \bigcap_{\alpha\in S} \varphi_\alpha^{-1}(C_\infty(Y_\alpha))\\
&= \bigcap_{\alpha\in S} \varphi_\alpha^{-1}(M^{*^{\beta Y_\alpha \setminus Y_\alpha}}(Y_\alpha)) = \bigcap_{\alpha\in S} M^{*^{\beta Y_\alpha \setminus Y_\alpha}}(Y) = M^{*^{\bigcup(\beta Y_\alpha \setminus Y_\alpha)}}(Y)\\
&= M^{*^{\beta Y \setminus \cap Y_\alpha}}(Y) = C_\infty(Y).
\end{aligned}$$

□

Corollary 3.13. *For every completely regular Hausdorff space X, there is an smallest ∞-compact space in βX containing X.*

Proof: By Theorem 3.12, this smallest ∞-compact space is the intersection of all ∞-compact spaces in βX containing X. □

We conclude this section by the following lemmas and proposition which characterize the type of points in $\infty X \setminus X$. First we note that, if $X \subseteq Y \subseteq \beta X$, then a point $p \in \beta X$ is said to be a *P-point with respect to* Y if $O^p(Y) = M^p(Y)$. In case $Y = X$, we apply $O^p = M^p$ instead of $O^p(X) = M^p(X)$ and briefly we say that p is a P-point.

Lemma 3.14. *Suppose that $p \in \beta X$ and $X \subseteq Y \subseteq \beta X$. Then for every $f \in C^*(X)$, $f \in O^p(X)$ if and only if $f^Y \in O^p(Y)$.*

Proof: We consider $\varphi_Y : C^*(X) \to C^*(Y)$ defined by $\varphi_Y(f) = f^Y$, $\forall f \in C^*(X)$. As was pointed out in the proof of Theorem 3.12, $\varphi_Y(M^{*^p}(X)) = M^{*^p}(Y)$ and $\varphi_Y(O^{*^p}(X)) = O^{*^p}(Y)$. Hence for every $f \in C^*(X)$, $\varphi_Y(f) = f^Y \in O^p(Y) \cap C^*(Y) = O^{*^p}(Y)$ if and only if $f \in \varphi_Y^{-1}(O^{*^p}(Y)) = O^{*^p}(X)$ which is equivalent to $f \in O^p(X)$. □

Lemma 3.15. *Suppose that $p \in \beta X$ and $X \subseteq Y \subseteq \beta X$. If p is a P-point with respect to Y, then it is also a P-point with respect to X.*

Proof: We suppose that $f \in M^p(X)$ and consider $g = \frac{f^2}{1+f^2}$. Hence $Z(f) = Z(g)$ and therefore $g \in M^p(X) \cap C^*(X)$. Thus $p \in \mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X} Z(g) \subseteq \mathrm{cl}_{\beta X}(Z(g^Y))$ implies that $g^Y \in M^p(Y) = O^p(Y)$ and by Lemma 3.14, $g \in O^p(X)$. Hence $f \in O^p(X)$, i.e., p is a P-point with respect to X. □

Proposition 3.16. *If $p_\circ \in \infty X \setminus X$, then $p_\circ$ is a non-P-point with respect to ∞X and hence it is a non-P-point with respect to βX.*

Proof: We put $Y = \infty X$ and $T = Y \setminus \{p_\circ\}$. Thus T is not ∞-compact and therefore there exists $f \in C_\infty(T) - C_K(T)$. For every $p \in \beta Y \setminus Y = \beta X \setminus \infty X \subseteq \beta X \setminus T = \beta T \setminus T$ we have $f^\beta(p) = 0$. However, if we let $g = f^Y$, then $g^\beta(p) = f^\beta(p) = 0$, $\forall p \in \beta Y \setminus Y$ and hence $g \in C_\infty(Y)$ implies that $g \in C_K(Y)$. Therefore $p \in \mathrm{int}_{\beta X} Z(g^\beta) = \mathrm{int}_{\beta X} Z(f^\beta)$, $\forall p \in \beta Y \setminus Y$ and hence $f \in O^{*^p}(T)$, $\forall p \in (\beta T \setminus T) \setminus \{p_\circ\}$. Now $f \notin O^{*^{p_\circ}}(T)$ since $f \notin C_K(T)$, and by Lemma 3.14, $g = f^Y \notin O^{p_\circ}(Y)$. But $g(p_\circ) = f^\beta(p_\circ) = 0$ and hence $g \in M^{p_\circ}(Y)$, i.e., $p_\circ$ is not a P-point with respect to Y. Finally, by Lemma 3.15, $p_\circ$ is not also a P-point with respect to βX. □

Corollary 3.17. *If for a topological space X, we put*

$$\Pi = \{p \in \beta X \setminus X : p \text{ is a P-point in } \beta X\}$$

then $\infty X \subseteq \beta X \setminus \Pi$. Moreover if $\beta X \setminus \Pi \subseteq Y \subseteq \beta X$, then Y is an ∞-compact space containing ∞X.

4. Relations between algebraic properties of $C_\infty(X)$ and topological properties of X

In this section we present topological characterizations of some algebraic properties of the ring $C_\infty(X)$. We will characterize topological spaces X for which the ring $C_\infty(X)$ is a regular ring, has a finite Goldie dimension, a p.p. ring and a Baer ring. First of all we consider $C_\infty(X)$ to be a regular ring. A ring R is called regular if for every $a \in R$, there exists $b \in R$ with $a = a^2 b$. A completely regular Hausdorff space X is said to be a P-space if every G_δ-set (zero-set) in X is an open set. It is well-known that $C(X)$ is a regular ring if and only if X is a P-space, see Theorem 14.29 and 4J in [7]. Whenever $Z(f)$ is open for every $f \in C_\infty(X)$, we call X a P_∞*-space*. The following theorem shows that $C_\infty(X)$ is a regular ring if and only if X is an ∞-compact P_∞-space.

Theorem 4.1. *The following statements are equivalent:*

(1) $C_\infty(X)$ *is a regular ring;*
(2) *every open locally compact* σ*-compact set in* X *is compact;*
(3) $\forall f \in C_\infty(X)$, $X \setminus Z(f)$ *is compact;*
(4) X *is an* ∞*-compact* P_∞*-space;*
(5) $\forall p \in X$, $M_p \cap C_\infty(X) = O_p \cap C_K(X)$.

Proof: (1)→(2). By Lemma 3.1, every open locally compact σ-compact set is of the form $X \setminus Z(f)$ for some $f \in C_\infty(X)$. Since $C_\infty(X)$ is regular, there exists $g \in C_\infty(X)$ such that $f^2g = f$. Now $f(fg-1) = 0$ implies that $\{x : (fg)(x) \neq 1\} = Z(f)$, i.e., $Z(f)$ is open. On the other hand, $g(x) = \frac{1}{f(x)}$ for every $x \in X \setminus Z(f)$ and hence $g(x) \geq \frac{1}{N}$, where N is an upper bound for $|f|$ (note that every member of $C_\infty(X)$ is bounded). Therefore

$$X \setminus Z(f) \subseteq \{x \in X : |g(x)| \geq \frac{1}{N}\} = A_N.$$

Since $X \setminus Z(f)$ is closed and A_N is compact, $X \setminus Z(f)$ is also compact.

(2)→ (3) → (4) → (5). Evident.

(5)→ (1). (5) implies that for every $f \in C_\infty(X)$, $Z(f)$ is open and $X \setminus Z(f)$ is compact. Now for every $f \in C_\infty(X)$, we define $g(x) = 0$ for $x \in Z(f)$ and $g(x) = \frac{1}{f(x)}$ for $x \in X \setminus Z(f)$. By pasting lemma, $g \in C(X)$ and $\{x \in X : |g(x)| \geq \frac{1}{n}\} \subseteq X \setminus Z(f)$ implies that $\{x \in X : |g(x)| \geq \frac{1}{n}\}$ is compact, i.e., $g \in C_\infty(X)$ and $f^2g = f$ means that $C_\infty(X)$ is regular. □

Remark 4.2. Clearly every P-space is a P_∞-space but every P_∞-space is not necessarily a P-space. For example let S be a P-space and consider the space X, the free union of spaces S and $\mathbb{Q}$ ($\mathbb{Q}$ with usual topology). By Lemma 1.3, for every $f \in C_\infty(X)$, we have $f(\mathbb{Q}) = 0$ and since S is a P-space, $Z(f)$ is open $\forall f \in C_\infty(X)$, i.e., X is a P_∞-space. But $\mathbb{Q}$ is not a P-space and hence X is not a P-space either.

Proposition 4.3. *Let* X *be a locally compact Hausdorff space. If* X *is a* P_∞*-space, then it is also a P-space.*

Proof: If X is a P_∞-space, then $M_x^* \cap C_\infty(X) = O_x^* \cap C_\infty(X)$, $\forall x \in X$. Since M_x^* is prime in $C^*(X)$, then by Lemma 3.8, either $M_x^* = O_x^*$ or $C_\infty(X) \subseteq O_x^*$. But $C_\infty(X) \subseteq O_x^*$ does not happen, for if K and H are compact neighborhoods of

x such that $K \subseteq \operatorname{int} H$, then define $g \in C(X)$ with $g(K) = \{1\}$ and $g(X \setminus \operatorname{int} H) = \{0\}$. Since $X \setminus Z(g) \subseteq H$, we have $g \in C_K(X) \subseteq C_\infty(X)$ but $g \notin O_x^*$. Hence $M_x^* = O_x^*$, $\forall x \in X$ and therefore X is a P-space. □

Corollary 4.4. *Let X be a locally compact Hausdorff space. Then $C_\infty(X)$ is a regular ring if and only if X is finite.*

Proof: If X is finite, then clearly $C_\infty(X)$ is a regular ring. Conversely, if $C_\infty(X)$ is a regular ring, then by Theorem 4.1, X is an ∞-compact P_∞-space and hence it is a P-space by Proposition 4.3. Now according to Proposition 3.4, X is a pseudocompact P-space which should be finite by 4K in [7]. □

Next we characterize spaces X for which the ring $C_\infty(X)$ has a finite Goldie dimension. Before doing this, we need to characterize uniform ideals and essential ideals in $C_\infty(X)$. A nonzero ideal I in a commutative ring R is called *essential* if it intersects every nonzero ideal nontrivially, and it is called *uniform* if any two nonzero ideals contained in I intersect nontrivially. In [2, Proposition 1.1], it is shown that the ideal I in $C(X)$ is uniform if and only if it is minimal, i.e., I is generated by an idempotent $e \in C(X)$ such that $X \setminus Z(e)$ is singleton. In [2, Proposition 3.1], it is also shown that an ideal E in $C(X)$ is essential if and only if $\operatorname{int}_X \cap Z[E] = \emptyset$, i.e., $\bigcap Z[E]$ is nowhere dense. By the following proposition, analogous criteria hold for essential ideals and uniform ideals in $C_\infty(X)$. First we need the following lemma.

Lemma 4.5. *Let $f, g \in C_\infty(X)$.*

(a) *If there exists $n_0 \in \mathbb{N}$ such that $\{x \in X : |g(x)| < \frac{1}{n_0}\} \subseteq Z(f)$, then f is a multiple of g in $C_\infty(X)$.*
(b) *If $|f| \leq |g|^r$ for some $r > 1$, then f is a multiple of g in $C_\infty(X)$.*

Proof: (a) We define $h(x) = f(x)/g(x)$ for $|g(x)| \geq \frac{1}{2n_0}$ and $h(x) = 0$ for $|g(x)| \leq \frac{1}{2n_0}$. Clearly $h \in C(X)$ and $f = gh$. But for every $n \in \mathbb{N}$, we have

$$\{x \in X : |h(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{2n_0 n}\}$$

which implies that $\{x \in X : |h(x)| \geq \frac{1}{n}\}$ is compact for any $n \in \mathbb{N}$, i.e., $h \in C_\infty(X)$.

(b) By problem 1D in [7], there exists $h \in C(X)$ such that $f = gh$. Now $|gh| \leq |g|^r$ implies that $\{x \in X : |h(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |g(x)|^{r-1} \geq \frac{1}{n}\}$ and hence $h \in C_\infty(X)$. □

Proposition 4.6. (a) *An ideal E in $C_\infty(X)$ is essential if and only if $\bigcap Z[E]$ is nowhere compact (i.e., $\bigcap Z[E]$ does not contain any nonempty compact neighborhood).*

(b) *An ideal I in $C_\infty(X)$ is uniform if and only if $I = (f)$ for some $f \in C_\infty(X)$, where $X \setminus Z(f)$ is a singleton.*

Proof: (a) Suppose E is an essential ideal in $C_\infty(X)$ and $B = \bigcap Z[E]$ is not nowhere compact. Then there exists a compact set A with $A \subseteq B$ and $\operatorname{int} A \neq \emptyset$. Let $a \in \operatorname{int} A$ and define $f \in C(X)$ such that $f(X \setminus \operatorname{int} A) = \{0\}$ and $f(a) = 1$. Hence $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq A$ implies that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact, i.e., $f \in C_\infty(X)$. Now if there exists $g \in C_\infty(X)$ such that $g \in (f) \cap E$, then $Z(f) \subseteq Z(g)$ implies that $X \setminus Z(g) \subseteq X \setminus Z(f) \subseteq A \subseteq B \subseteq Z(g)$ and hence $g = 0$ which contradicts the essentiality of E in $C_\infty(X)$. Conversely, let $\bigcap Z[E]$ be nowhere compact, $0 \neq f \in C_\infty(X)$ and $a \in X \setminus Z(f)$. Then there exists $n \in \mathbb{N}$ such that $|f(a)| \geq \frac{1}{n}$ and hence a is in the compact set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$. Since $\bigcap Z[E]$ is nowhere compact, there exists $b \in \{x \in X : |f(x)| \geq \frac{1}{n}\} \setminus \bigcap Z[E]$ which implies that there is $g \in E$, such that $g(b) \neq 0$ and hence $0 \neq fg \in (f) \cap E$, i.e., E is essential in $C_\infty(X)$.

(b) Let I be a uniform ideal in $C_\infty(X)$ and $f \in I$. First we show that $X \setminus Z(f)$ is a singleton. Suppose that $x_0, y_0 \in X \setminus Z(f)$ and $x_0 \neq y_0$. By Lemma 3.1, $X \setminus Z(f)$ is a locally compact subspace of X and hence there exist two disjoint compact neighborhoods G and H in $X \setminus Z(f)$ of points x_0 and y_0 respectively. Since $X \setminus Z(f)$ is open in X, G and H are also compact neighborhoods in X. Now we define two functions $g, h \in C(X)$ such that $g(x_0) = 1 = h(y_0)$ and $g(X \setminus \operatorname{int} G) = \{0\} = h(X \setminus \operatorname{int} H)$. Since $\{x \in X : |g(x)| \geq \frac{1}{n}\} \subseteq G$ and G is compact, $\{x \in X : |g(x)| \geq \frac{1}{n}\}$ is also compact, i.e., $g \in C_\infty(X)$. Similarly, $h \in C_\infty(X)$. Now consider the principal subideals (fg) and (fh) of I. Since I is a uniform ideal, there exists $0 \neq k \in (fg) \cap (fh)$ and hence there exists $z \in X \setminus Z(g)$ with $k(z) \neq 0$. Now $kg = 0$ contradicts $k(z)g(z) \neq 0$ and therefore $X \setminus Z(f)$ is a singleton, say $X \setminus Z(f) = \{x_0\}$. Next we show that for every $g \in I$, we have also $X \setminus Z(g) = \{x_0\}$. Let $X \setminus Z(g) = \{y_0\}$ and $y_0 \neq x_0$. For the principal subideals (f) and (g) of I, we have $(f) \cap (g) = (0)$, for if $h \in (f) \cap (g)$, then $Z(f) \cup Z(g) = X \subseteq Z(h)$ implies that $h = 0$. This contradicts the uniformity of I and hence $X \setminus Z(g) = \{x_0\}$. Therefore we have shown that there exists an isolated point $x_0 \in X$ such that $X \setminus Z(f) = \{x_0\}$, $\forall f \in I$. Finally, suppose that $f, g \in I$ and $f(x_0) = \alpha$. Then there exists $n \in \mathbb{N}$ such that $|\alpha| \geq \frac{1}{n}$ and hence $\{x \in X : |f(x)| < \frac{1}{n}\} \subseteq Z(g)$ which implies that g is a multiple of f by Lemma 4.5. This shows that $I = (f)$. The converse is evident. □

It is well-known that if a ring R has a finite Goldie dimension, then there exists an integer $n > 0$ such that any direct sum of nonzero ideals in R has always m terms, where $m \leq n$ and there is a direct sum of uniform ideals with n terms which is essential in R, see [8] and [10].

Proposition 4.7. *$C_\infty(X)$ has a finite Goldie dimension if and only if every open locally compact set in X is finite.*

Proof: If $C_\infty(X) = (0)$, then every locally compact set in X is empty. Now suppose that $C_\infty(X) \neq (0)$ has a finite Goldie dimension and let G be a locally compact open set in X. Hence there exists $n > 0$ such that the direct sum of n uniform ideals $I_1, I_2, \dots, I_n$ in $C_\infty(X)$ is an essential ideal E in $C_\infty(X)$. By Proposition 4.6, there is an isolated point $x_i \in X$ and $f_i \in I_i$ such that $I_i = (f_i)$, where $X \setminus Z(f_i) = \{x_i\}$, for $i = 1, 2, \dots, n$. This implies that $\bigcap Z[I] = X \setminus \{x_1, x_2, \dots, x_n\}$ and again by Proposition 4.6, $X \setminus \{x_1, x_2, \dots, x_n\}$ does not contain any nonempty compact neighborhood. Thus $G \cap (X \setminus \{x_1, x_2, \dots, x_n\}) = \emptyset$ and hence $G \subseteq \{x_1, x_2, \dots, x_n\}$, i.e., G is finite. The converse is obvious. □

Corollary 4.8. *If X is a locally compact Hausdorff space, then $C_\infty(X)$ has a finite Goldie dimension if and only if X is finite.*

Finally we characterize the locally compact spaces X for which $C_\infty(X)$ is a p.p. ring or a Baer ring. A topological space X is called *extremally* (*basically*) *disconnected* if each open (cozero) set in X has an open closure. A commutative ring R is a *p.p.* (*Baer*) *ring* if for any $a \in R$ ($S \subseteq R$), $\mathrm{Ann}(a)$ ($\mathrm{Ann}(S)$) is the principal ideal generated by an idempotent. In [1] and [3], it is shown that X is basically (extremally) disconnected if and only if $C(X)$ is a p.p. (Baer) ring.

Theorem 4.9. *Let X be a locally compact space.*

(a) *$C_\infty(X)$ is a p.p. ring if and only if X is a basically disconnected compact space.*
(b) *$C_\infty(X)$ is a Baer ring if and only if X is an extremally disconnected compact space.*

Proof: (a) Let $C_\infty(X)$ be a p.p. ring. Then for every $0 \neq f \in C_\infty(X)$, there exists an idempotent $e \in C_\infty(X)$ such that $\mathrm{Ann}(f) = (e)$. Therefore $X \setminus Z(e) \subseteq \mathrm{int}\, Z(f)$. We show that $X \setminus Z(e) = \mathrm{int}\, Z(f)$. Let $x \in \mathrm{int}\, Z(f)$ but $x \notin X \setminus Z(e)$ and define $g \in C(X)$ such that $g(X \setminus \mathrm{int}\, K) = \{0\}$ and $g(x) = 1$, where K is a compact neighborhood of x contained in $\mathrm{int}\, Z(f) \cap Z(e)$. Hence $g \in C_\infty(X)$ and $gf = 0$ but $g \notin (e)$, for $Z(e) \nsubseteq Z(g)$ ($g(x) = 1$, $e(x) = 0$), a contradiction. This implies that $X \setminus Z(e) = \mathrm{int}\, Z(f)$ and hence $Z(e) = \mathrm{cl}_X(X \setminus Z(f))$. Now if we take $f \in C_K(X)$, then $Z(e)$ and $X \setminus Z(e)$ are compact, i.e., X is compact. We have also shown that for every $f \in C_\infty(X)$, $\mathrm{int}\, Z(f)$ is closed. Since X is compact, $C_\infty(X) = C(X)$ and hence for every $f \in C(X)$, $\mathrm{int}\, Z(f)$ is closed, i.e., X is basically disconnected. Conversely, if X is a compact space, then $C_\infty(X) = C(X)$ and since X is basically disconnected, $C_\infty(X)$ is a p.p. ring by [1, Lemma 3].

(b) If $C_\infty(X)$ is a Baer ring, then it is p.p. ring and hence by part (a), X is compact, i.e., $C_\infty(X) = C(X)$. Now part (b) is well-known for compact spaces, see [5]. □

Corollary 4.10. *Let X be a locally compact non-compact space. Then $C_\infty(X)$ is never a p.p. (Baer) ring.*

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