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On introduction of two diffeomorphism invariant Colombeau algebras

JIŘÍ JELÍNEK

Abstract. Equivalent definitions of two diffeomorphism invariant Colombeau algebras introduced in [7] and [5] (Grosser et al.) are listed and some new equivalent definitions are presented. The paper can be treated as tools for proving in [8] the equality of both algebras.

Keywords: Colombeau algebra of generalized functions, representative, diffeomorphism invariance

Classification: 46F, 46F05

In [4] a diffeomorphism invariant Colombeau-type algebra was proposed. Such an algebra was consistently introduced in [7], then the authors of [5] have very carefully examined it and, in addition to this algebra denoted by \mathcal{G}^d , they have introduced another diffeomorphism invariant Colombeau algebra \mathcal{G}^2 , apparently larger than \mathcal{G}^d and more close to the algebra that Colombeau and Meril intended in [4]. However, it was not discovered that these two algebras are identical. Thanks to this equality, we can use the simpler definition of \mathcal{G}^d knowing that we do not lose generality. As the proof of equality of both algebras is rather complicated, we postpone it in a separate paper [8]. In this paper, we recapitulate basic definitions and notations and give new equivalent definitions of these algebras. Although the aim of this paper is to give tools for proving the identity $\mathcal{G}^2 = \mathcal{G}^d$, the transparent list of equivalent definitions can be useful also for readers that do not take interest in this identity. E.g. the condition (0°) in §8 discovered by the authors of [5] is a surprisingly simple tool for verifying that a representative is negligible: in [5] the equivalence is proved for \mathcal{E}_M^d , here for \mathcal{E}_M^2 , too.

Basic definitions and notations

We will use mostly the same notations as in [7], [5]. In [5, p.14], operators T_x , S_ε on \mathcal{D} and T on $\mathcal{D} \times \mathbb{R}^d$ are introduced: If φ is a test function on an

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Euclidean space \mathbb{R}^d , $x \in \mathbb{R}^d$, $\varepsilon > 0$, then the functions $T_x\varphi$ and $S_\varepsilon\varphi$ on \mathbb{R}^d and $T(\varphi, x) \in \mathcal{D} \times \mathbb{R}^d$ are defined as follows:

$$T_x\varphi(y) := \varphi(y - x), \quad S_\varepsilon\varphi(y) := \varepsilon^{-d}\varphi\left(\frac{y}{\varepsilon}\right), \quad T(\varphi, x) := (T_x\varphi, x).$$

Thanks to this notation we do not need to use Colombeau’s notation φ_ε meaning $S_\varepsilon\varphi$.

We deal with test functions $\varphi \in \mathcal{D}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set. The notation $\mathcal{A}_q(\Omega)$ has its usual sense by Colombeau and we write \mathcal{A}_q instead if Ω is clear from the context or not important. We denote $\mathcal{A} := \mathcal{A}_0 - \mathcal{A}_0 = \{\varphi \in \mathcal{D}; \int \varphi = 0\}$. The topologies on \mathcal{A}_q and \mathcal{A} are induced by \mathcal{D} .

Note that in [7] a different formalism is used assigning representatives to a generalized function. In [5] this is called J-formalism unlike Colombeau’s C-formalism: A function $(\varphi, x) \mapsto R(\varphi, x)$ is considered in [7] to be a representative of a generalized function in the case when $R \circ T : \{(\varphi, x) \mapsto R(T_x\varphi, x)\}$ is a representative of this generalized function in Colombeau’s sense. The new formalism is convenient when dealing with generalized functions on a \mathcal{C}^∞ manifold different from \mathbb{R}^d and is used e.g. in [6]. In this paper we will use the classical Colombeau’s formalism, because it is sufficient for our aim and the calculations will be simpler. However, while referring to [7], a change of formalism is needed.

§1. Definition. If R is a representative, we denote by $(R)_\varepsilon$ or simply by R_ε the function $(R)_\varepsilon(\varphi, x) = R(S_\varepsilon\varphi, x)$ while in [7] $(R)_\varepsilon(\varphi, x) = R(T_x \circ S_\varepsilon\varphi, x)$ as a consequence of another formalism and thus, for a given generalized function, the notation $(R)_\varepsilon(\varphi, x)$ has the same meaning in both formalisms.

In this paper a representative R of a generalized function is a function of specific properties (see below) on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$, while in [5] (similarly in [7] with another formalism) a representative is defined only on $U(\Omega) := \{(\varphi, x); \varphi \in \mathcal{A}_0(\Omega - x), x \in \Omega\}$. This is legitimized by the following

Proposition. *Every generalized function in $\mathcal{G}^d(\Omega)$ resp. $\mathcal{G}^2(\Omega)$ with a representative $R_0 \in \mathcal{E}_M^d(\Omega)$ resp. $\in \mathcal{E}_M^2(\Omega)$ defined on $U(\Omega)$ has another representative $R \in \mathcal{E}_M^d(\Omega)$ resp. $\in \mathcal{E}_M^2(\Omega)$ that is defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$. The equivalence means that after restriction on $U(\Omega)$ it is $R - R_0 \in \mathcal{N}$.*

The proof is below.

Remarks. For representatives defined on $U(\Omega)$ moderateness is defined in [5, 7.2 resp. 17.1] while for representatives defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ the definitions are below §4, (1°) resp. §7 (1°). However these definitions are the same or equivalent. The only difference is that in the former case on a given bounded set resp. path in $\mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}_0(\mathbb{R}^d))$ and a given $K \Subset \Omega$ (means compact subset), $(R_0)_\varepsilon(\varphi, x)$ is only defined for sufficiently small ε , while in the latter case this is defined always.

So for moderateness of a representative defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$, only its values on $U(\Omega)$ matter.

Proposition says that we obtain the same algebra if we admit only representatives defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$. For \mathcal{G}^d this follows directly from [7, Theorem 21]. In our formalism this theorem can be formulated as follows. For a family of numbers $\{q_i \in \mathbb{N}_0\}_{i \in I}$ and an open covering $\{V_i\}_{i \in I}$ of Ω with $V_i \subseteq \Omega$ denote

$$\begin{aligned} \mathfrak{V}((V_i, q_i)_{i \in I}) &:= \{(\varphi, x); \exists i \in I \text{ such that } x \in V_i, \varphi \in \mathcal{A}_{q_i}(V_i - x)\} \\ &= \bigcup_i U(V_i) \cap \mathcal{A}_{q_i}. \end{aligned}$$

If R_0 is a \mathcal{C}^∞ function on $\mathfrak{V}((V_i, q_i)_{i \in I})$, moderate in a certain way defined in that theorem, then there is a moderate smooth function R on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ coinciding with R_0 on some set $\mathfrak{V}((V'_i, q'_i)_{i \in I'})$ of the above type.

It follows from this assertion that R and R_0 define the same generalized function. There is a lack in [7] that the notion of smoothness on $\mathfrak{V}((V_i, q_i)_{i \in I})$ is not explained and with the formalism used in [7] we cannot apply the differentiation theory used there. Here we can follow the method of [5, Chapter 5] for defining differentials of R_0 on $U(V_i) \cap \mathcal{A}_{q_i}$ ($\forall i$). The appropriate topology on $U(V_i)$ is τ_2 but we can simply choose the topology τ_1 induced by $\mathcal{D}(\mathbb{R}^d) \times \Omega$. This follows from the fact that we can choose a finer covering $\{V'_{i'}\}_{i' \in I'}$ such that every $\overline{V'_{i'}}$ is compact in some V_i . On the other hand, in [7] with the formalism used there we use no tools to define differentials on \mathfrak{V} , but fortunately it is not needed to do so. It suffices to suppose (approach of [9]) that R_0 is smooth on smooth curves in \mathfrak{V} (see Remark 3 below) because the only property concerning smoothness we need is: the composition of smooth mappings on smooth curves is smooth on smooth curves.

Theorem 21 in [7] is stronger than we need. $q_i = 0$ would satisfy our task and the reasoning would be much simpler. The authors of [5] used this method in Chapter 8 for verifying chief properties of \mathcal{G}^d and by way they proved our assertion, too. More precisely: The representative R obtained on $U(\Omega)$ while proving S2 is in fact defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$. R is even continuously infinitely differentiable, but we will not use this result; we only note that the same algebras can be constructed with continuously infinitely differentiable representatives.

In [5] this method is not applied to \mathcal{G}^2 . So we are going to give in brief a proof that is a copy of the proof in [5, Chapter 8]. The details are left to the reader.

PROOF of the proposition for \mathcal{G}^2 : Choose a locally finite covering $(W_j)_{j \in \mathbb{N}}$ of Ω with $\overline{W_j} \Subset \Omega$ and a partition of unity $(\chi_j)_{j \in \mathbb{N}}$ subordinate to $(W_j)_{j \in \mathbb{N}}$. Moreover, for each $j \in \mathbb{N}$ choose functions $\vartheta_j \in \mathcal{D}$, $\vartheta_j = 1$ on a neighbourhood of $\overline{W_j}$, and

$\psi_j \in \mathcal{A}_0(W_j)$. The map $\pi_j : \mathcal{A}_0(\mathbb{R}^d) \rightarrow \mathcal{A}_0(\Omega)$ defined by

$$\pi_j(\varphi) := \vartheta_j \varphi + (1 - \int \vartheta_j \varphi) \psi_j$$

is smooth on $\mathcal{A}_0(\mathbb{R}^d)$ and identical on $\mathcal{A}_0(W_j)$. Then for each j the function R_j on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ defined by

$$R_j(\varphi, x) := \begin{cases} \chi_j(x) R_0(T_{-x} \circ \pi_j \circ T_x(\varphi), x) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega \end{cases}$$

is smooth. To show that $R := \sum R_j$ is moderate we first note that in a neighbourhood of any $K \Subset \Omega$ only finitely many R_j do not vanish identically, so it is enough to show that one single R_j is moderate. For this, it is enough to show that the function (element of $\mathcal{E}(W_j)$ by the following definition)

$$\mathcal{A}_0(\mathbb{R}^d) \times W_j \ni (\varphi, x) \mapsto R_0(T_{-x} \circ \pi_j \circ T_x(\varphi), x)$$

is moderate. If $W \subset \Omega$ is open and R_0 is defined on $U(\Omega)$, following Grosser et al. [5] we denote by $R_0|_W$ the restriction of R_0 to $U(W)$. We left to the reader to prove that $R_0|_W$ is moderate provided R_0 is moderate. To see that $R_0(T_{-x} \circ \pi_j \circ T_x(\varphi), x)$ is moderate, it is enough to realize that for a given compact $K \Subset W_j$ and a given bounded path

$$\{(\varphi_x^\varepsilon)_{x \in \Omega}; \varepsilon \in]0, 1]\} \subset \mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}_0(\mathbb{R}^d)),$$

$\forall x \in K$ and ε small enough, we have $S_\varepsilon \varphi_x^\varepsilon \in \mathcal{A}_0(W_j - x)$, so $T_x S_\varepsilon \varphi_x^\varepsilon \in \mathcal{A}_0(W_j)$, where π_j is identical. Thus $R_0(T_{-x} \circ \pi_j \circ T_x(\varphi), x) = R_0(\varphi, x)$ for $\varphi = \varphi_x^\varepsilon$, $R(\varphi, x) = R_0(\varphi, x)$ is moderate and $R - R_0$ is negligible. \square

§2. Definition. We denote by $\mathcal{E}[\Omega]$ or $\mathcal{E}(\Omega)$ the space of functions

$$\begin{aligned} \mathcal{A}_0(\mathbb{R}^d) \times \Omega &\rightarrow \mathbb{C} \\ (\varphi, x) &\mapsto R(\varphi, x) \end{aligned}$$

that are \mathcal{C}^∞ simultaneously in both variables. As we do not use Schwartz's notation $\mathcal{E}(\Omega)$ for $\mathcal{C}^\infty(\Omega)$, we can use the notation $\mathcal{E}(\Omega)$ (unlike Colombeau) with this meaning. Like in [7], we denote by $\mathbf{d}R$ the total differential of the function R of two variables and by $\mathbf{d}R$ the partial differential with respect to the first variable running mostly over a part of \mathcal{A}_0 . The derivatives with respect to the second variable are denoted ∂^α and we distinguish them from $(\frac{\partial}{\partial x})^\alpha$ e.g. if the first variable depends on x , too. So we do not use indices for distinguishing partial differentials and we can use them to indicate the direction of the derivative; e.g. $\mathbf{d}_{\psi_1, \psi_2}^2 R(\varphi, x)$ is the same as $\mathbf{d}^2 R(\varphi, x)[\psi_1, \psi_2]$. Moreover, if we denote $\psi = (\psi_1, \psi_2)$, then $\mathbf{d}_\psi^2 R(\varphi, x)$ denotes the same, as well. If the function is given as a composition, e.g. $R(S(\varphi), x)$, then $\mathbf{d}R(S(\varphi), x)$ signifies the differential of this composition and is thus distinguished from $(\mathbf{d}R)(S(\varphi), x)$.

Remarks. There are divers notions of differentiability of mappings of locally convex spaces; some of them are equivalent in many cases investigated in this paper: we mostly deal with \mathcal{C}^∞ functions defined on an open part of a subspace of \mathcal{D} or $\mathcal{D} \times \mathbb{R}^d$. Without explicitly mentioned, “differential” means the Fréchet differential: If F is a vector-valued function defined on an open part of a locally convex space \mathcal{F} , the Fréchet differentiability of F at $\varphi \in \mathcal{F}$ means that $dF(\varphi)$ is a continuous linear mapping and

$$(1) \quad \lim_{t \searrow 0} \frac{F(\varphi + t\psi) - F(\varphi)}{t} = dF(\varphi)[\psi]$$

uniformly if ψ runs over any bounded subset \mathcal{B} of \mathcal{F} .

Note that a differentiable mapping (at every point of its domain) need not be continuous, but it is continuous (see Yamamuro [13, §1.7]) in the case \mathcal{F} is metrizable. Following [1] we denote by \mathcal{C}^n the class of differentiable mappings up to order n , unlike [13] where in addition the continuity of the differentials is required. For a \mathcal{C}^∞ mapping on a metrizable space both notions coincide.

The differential of a higher order at a fixed point is a hypo-continuous multi-linear mapping. If \mathcal{F} is a Fréchet space, such a mapping is (jointly) continuous (Robertson A.P.-Robertson W.J. [11, VII, Proposition 11]) and evidently this holds for (LF)-spaces, too.

Some authors prefer other notions of differentiability. In Colombeau [1] Silva differential and Silva differential in enlarged sense are introduced and is proved (1.4.7, 1.4.8) that for \mathcal{C}^∞ both notions coincide if \mathcal{F} is a co-Schwartz locally convex space. \mathcal{D} is even co-nuclear, see Pietsch [10, 6.2.6, 4.1.6]. Silva differential in enlarged sense is by definition the Fréchet one with the only exception that dF is only bounded on bounded sets (not necessarily continuous). However on a bornological space \mathcal{F} (our case) such a mapping is separately continuous; in our case continuous. The authors of [5] choose a direct definition of \mathcal{C}^∞ by Kriegl-Michor [9]: F is by definition \mathcal{C}^∞ iff for every \mathcal{C}^∞ curve C in the domain of F , the curve $F \circ C$ is \mathcal{C}^∞ . It is said in Chapter 4 that this notion of smoothness is weaker than Silva-smoothness but is equivalent if \mathcal{F} is a complete Montel space. Hence in our case all the above mentioned notions of \mathcal{C}^∞ smoothness coincide.

The last definition of smoothness has the advantage that it can also be applied when the domain of F is a part of a linear space with a non-induced topology. The domain even need not be open. We distinguish this case saying that F is smooth on smooth curves, regardless if there is any non-trivial curve in its domain. However only in the case the domain is an open subset of \mathcal{F} with the induced topology, it is proved in Kriegl-Michor [9] that F has smooth differentials; only in that case we have the above mentioned equivalence of smoothness.

The following proposition says in brief that continuous differentials on a Fréchet space are locally equi-continuous; this can be easily generalized for mappings into a locally convex space, but we do not need such a generalization. The formulation is a bit complicated in order to correspond to our purposes.

Proposition. *Let \mathcal{F} be a Fréchet space, $\omega \in \mathcal{F}$, $\mathcal{A} \subset \mathcal{F}$ a closed vector subspace (with the induced topology), F a complex function on an open neighbourhood of ω in the affine space $\omega + \mathcal{A}$, continuously differentiable up to order L ($L \in \mathbb{N}$). Then there is a neighbourhood \mathcal{U} of zero in \mathcal{A} such that for all $\varphi \in \omega + \mathcal{U}$ and $\psi_\ell \in \mathcal{U}$, ($\ell = 1, \dots, L$) it is $|d_{\psi_1, \dots, \psi_L}^L F(\varphi)| \leq 1$.*

More generally, if $\mathcal{K} \in \omega + \mathcal{A}$ is a compact contained in the domain of F , $L \in \mathbb{N}$, under the same hypotheses there is a neighbourhood \mathcal{U} of zero in \mathcal{A} such that for all $\varphi \in \mathcal{K} + \mathcal{U}$ and $\psi_\ell \in \mathcal{U}$, ($\ell = 1, \dots, L$) it is $|d_{\psi_1, \dots, \psi_L}^L F(\varphi)| \leq 1$.

PROOF BY INDUCTION: We change the last inequality with $|d_{\psi_1, \dots, \psi_L}^L F(\varphi)| \leq 1 + |F(\omega)|$. This is equivalent and holds evidently for $L = 0$, too. Let $L \in \mathbb{N}$ be given, and let (induction assumption) for any \mathcal{C}^{L-1} function F it is $|d_{\psi_1, \dots, \psi_{L-1}}^{L-1} F(\varphi)| \leq 1 + |F(\omega)|$ under the hypotheses of the proposition. Now, let F be a \mathcal{C}^L function, $\omega \in \mathcal{F}$. Choose a basis of absolutely convex neighbourhoods of zero $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots$ in \mathcal{A} and denote (for $n \in \mathbb{N}$)

$$\mathcal{B}_n := \left\{ \psi \in \mathcal{A}; \forall \varphi \in \omega + \mathcal{U}_n, \psi_1, \dots, \psi_{L-1} \in \mathcal{U}_n : |d_{\psi_1, \dots, \psi_{L-1}, \psi}^L F(\varphi)| \leq 1 + |F(\omega)| \right\}.$$

\mathcal{B}_n are absolutely convex and closed. $d_\psi F$ is a \mathcal{C}^{L-1} function, hence by the induction assumption

$$\forall \psi \in \mathcal{A} \exists \mathcal{U}_n \forall \varphi \in \omega + \mathcal{U}_n, \psi_1, \dots, \psi_{L-1} \in \mathcal{U}_n : |d_{\psi_1, \dots, \psi_{L-1}, \psi}^L F(\varphi)| \leq 1.$$

This means $\bigcup \mathcal{B}_n = \mathcal{A}$. It is known for Fréchet spaces that in that case some \mathcal{B}_n is a neighbourhood of zero in \mathcal{A} , what we wanted to prove. (Proof: Some \mathcal{B}_n is not nowhere-dense because a Fréchet space is not of the first category. As \mathcal{B}_n is close, it is a neighbourhood of some point. Being absolutely convex, it is a neighbourhood of zero.)

Now we are going to prove the second part. As \mathcal{K} is compact, it can be covered with a finite number of sets $\omega_m + \frac{1}{2}\mathcal{U}_m$ where \mathcal{U}_m is an absolutely convex open neighbourhood of zero in \mathcal{A} assigned to ω_m by the first part of Proposition. Then $\mathcal{U} := \bigcap \mathcal{U}_m$ is the desired neighbourhood. □

Corollary. *Under the same hypotheses, if $\lim_{n \rightarrow \infty} \varphi_n = \varphi$ in $\omega + \mathcal{A}$ and $\lim_{n \rightarrow \infty} \psi_{\ell n} = \psi_\ell$ in \mathcal{A} ($\ell = 1, \dots, L$), then $\lim_{n \rightarrow \infty} d_{\psi_{1n}, \dots, \psi_{Ln}}^L F(\varphi_n) = d_{\psi_1, \dots, \psi_L}^L F(\varphi)$.*

This holds more generally if \mathcal{F} is an (LF)-space, because then the convergent sequences are contained in a Fréchet subspace of \mathcal{F} .

§3. Definition. For a locally convex space \mathcal{F} , we denote by $\mathcal{C}^\infty(\Omega \rightarrow \mathcal{F})$ the locally convex space of all \mathcal{C}^∞ maps

$$\begin{aligned} \Phi &= (\varphi_x)_{x \in \Omega} : \Omega \rightarrow \mathcal{F} \\ &x \mapsto \varphi_x \end{aligned}$$

with the usual topology of uniform convergence of every derivative with respect to x on every compact $K \Subset \Omega$.

Notation. The diffeomorphism invariant algebra \mathcal{G} that I have defined in [7] will be denoted here following Grosser et al. [5] by \mathcal{G}^d . In this paper we investigate the other algebra \mathcal{G}^2 as well and denote the algebra of representatives of \mathcal{G}^d resp. \mathcal{G}^2 by \mathcal{E}_M^d resp. \mathcal{E}_M^2 . On the other hand, the ideal of negligible representatives for \mathcal{G}^2 will be denoted simply by \mathcal{N} because $\mathcal{N} \cap \mathcal{E}_M^d$ is then the ideal of negligible representatives for \mathcal{G}^d .

§4. Equivalent definitions of $\mathcal{E}_M^d(\Omega)$. $\mathcal{E}_M^d(\Omega)$ is the set of all $R \in \mathcal{E}[\Omega]$ with moderate growth, which means that one of the following equivalent conditions is satisfied.

(1°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d \exists N \in \mathbb{N}$:

$$\left(\frac{\partial}{\partial x}\right)^\alpha R_\varepsilon(\varphi_x, x) = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly if $x \in K$ and $(\varphi_x)_{x \in \Omega}$ runs over any bounded subset of $\mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}_0(\mathbb{R}^d))$ (this space is the topological subspace of $\mathcal{C}^\infty(\Omega \rightarrow \mathcal{D}(\mathbb{R}^d))$).

(2°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \exists N \in \mathbb{N}$:

$$\partial^\alpha d^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly if $x \in K, \varphi$ runs over any bounded subset of $\mathcal{A}_0(\mathbb{R}^d)$ and ψ_1, \dots, ψ_k are in a bounded subset of $\mathcal{A}(\mathbb{R}^d)$.

(3°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \exists N \in \mathbb{N} \forall B \Subset \mathbb{R}^d, \mathcal{B}$ (bounded) $\subset \mathcal{A}_0(B)$
 $\exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B), C > 0, C = 1$
 if $k \geq 1, \forall x \in K, \varepsilon \in]0, 1], \varphi \in \mathcal{B} + \mathcal{U}, \psi_1, \dots, \psi_k \in \mathcal{U}$:

$$\partial^\alpha d^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] \leq C \varepsilon^{-N}.$$

PROOF OF EQUIVALENCES: The equivalence (1°) \Leftrightarrow (2°) is proved in [7, Theorem 17] (with another formalism) or in [5, Theorem 7.12]. (3°) \Rightarrow (2°) being evident, we only have to prove (3°) \Leftarrow (2°), first for the case \mathcal{B} is a singleton, $\mathcal{B} = \{\omega\}, \omega \in \mathcal{A}_0(B)$. This proof is left to the reader. It could be the same or simpler than the similar proofs in §7 below for the algebra \mathcal{E}_M^2 . □

§5. For the following definition of the null ideal in \mathcal{G}^2 , we use the notion of bounded path introduced in Colombeau-Meril [4] in order to define the moderate growth and the negligibility of representatives. It is explained in [7] that a bounded path should depend on $x \in \Omega$, so sometimes its values should belong to $\mathcal{C}^\infty(\Omega \rightarrow \mathcal{D})$ rather than to \mathcal{D} .

Definition. A path in this paper is a mapping of the interval $]0, 1]$ into a topological linear space (or its part), mostly

$$\begin{aligned} &]0, 1] \rightarrow \mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}_0) \\ \text{or } &]0, 1] \rightarrow \mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}) \\ &\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega}, \end{aligned}$$

however paths with values in \mathcal{A}_0 or in \mathcal{A} (independent of $x \in \Omega$) will be used, too. Adjectives like \mathcal{C}^q , \mathcal{C}^∞ refer to this mapping of the variable ε . Like in [4], we use upper indices, however this will be the only case of using an upper index for a variable.

Remark. Evidently, for a locally convex space \mathcal{F} , a path $\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{F})$ is \mathcal{C}^∞ iff the mapping $\varepsilon, x \mapsto \varphi_x^\varepsilon \in \mathcal{F}$ is \mathcal{C}^∞ .

Also it is useful to consider paths without any smoothness requirement. In that case a path even need not be continuous. A path is said to be *bounded* if its range is bounded; a path $\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{F})$ is bounded iff for every $K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$ the set $\left\{ \left(\frac{\partial}{\partial x} \right)^\alpha \varphi_x^\varepsilon; x \in K, \varepsilon \in]0, 1] \right\}$ is bounded in \mathcal{F} .

§6. Definition. We say (by [5], introduced in [4]) that a path

$$\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{D})$$

has *asymptotically vanishing moments of order $N \in \mathbb{N}$* iff for every $K \Subset \Omega$ and $\beta \in \mathbb{N}_0^d$ with $1 \leq |\beta| \leq N$ it is

$$\sup_{x \in K} \left| \int_{\mathbb{R}^d} \varphi_x^\varepsilon(\xi) x^{i\beta} d\xi \right| = O(\varepsilon^N) \quad (\varepsilon \searrow 0).$$

For a path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{D}$ the same means that for all $\beta \in \mathbb{N}_0^d$ with $1 \leq |\beta| \leq N$ it is

$$\int \varphi^\varepsilon(\xi) \xi^\beta d\xi = O(\varepsilon^N) \quad (\varepsilon \searrow 0).$$

In [5, Theorem 16.5] is proved (formulated only for \mathcal{A}_0 instead of \mathcal{D}): If $\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{D})$ is a bounded \mathcal{C}^∞ path with asymptotically vanishing moments of order $q \geq 2$, then $\forall \alpha$ the path

$$\varepsilon \mapsto \left(\left(\frac{\partial}{\partial x} \right)^\alpha \varphi_x^\varepsilon \right)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{D})$$

has asymptotically vanishing moments of order $q - 1$.

§7. Now we could define the negligible ideal and then the algebra \mathcal{G}^d as the quotient algebra. However, the definition of the negligible ideal for both algebras \mathcal{G}^d and \mathcal{G}^2 is the same, so we defer it and define first the algebra of representatives for \mathcal{G}^2 . This one is introduced in [5], is larger than \mathcal{E}_M^d and more closed to the algebra that Colombeau and Meril intended to introduce in [4].

Equivalent definitions of \mathcal{E}_M^2 . If $\Omega \subset \mathbb{R}^d$ is an open set, $\mathcal{E}_M^2(\Omega)$ is defined to be the set of all elements $R \in \mathcal{E}(\Omega)$ fulfilling one of the following equivalent conditions (\mathcal{A}_q means $\mathcal{A}_q(\mathbb{R}^d)$).

(1°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d \exists N \in \mathbb{N}$: for every bounded \mathcal{C}^∞ path

$$(2) \quad \varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}_0)$$

that has asymptotically vanishing moments of order N , we have

$$(3) \quad \left(\frac{\partial}{\partial x}\right)^\alpha R_\varepsilon(\varphi_x^\varepsilon, x) = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

(1'°) = condition (1°) without \mathcal{C}^∞ requirement for the path $\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega}$. In that case the bounded path even need not be continuous with respect to ε .

(1''°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d \exists N \in \mathbb{N}$: (3) holds uniformly if $x \in K$ and (2) runs over a set of paths that are uniformly bounded and have uniformly vanishing moments.

For the following equivalent conditions (2'°) and (3'°) similar equivalent conditions like (1°)–(1''°) can be easily formulated and proved; we will not do it for the sake of brevity.

(2'°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \exists N \in \mathbb{N}$: for every bounded paths

$$(4) \quad \varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0, \quad \varepsilon \mapsto \psi_i^\varepsilon \in \mathcal{A} \quad (i = 1, 2, \dots, k)$$

that all have asymptotically vanishing moments of order N , we have

$$(5) \quad \partial^\alpha d^k R_\varepsilon(\varphi^\varepsilon, x)[\psi_1^\varepsilon, \dots, \psi_k^\varepsilon] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

(3'°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \exists N \in \mathbb{N}$: (5) holds whenever the first of bounded paths (4) has asymptotically vanishing moments of order N .

For the following equivalent definitions, we use a function V_N on \mathcal{A}_0 ($\forall N \in \mathbb{N}$) estimating moments up to order N . This function should satisfy:

$\forall \mathcal{B}$ (bounded) $\subset \mathcal{A}_0 \exists C_1, C_2 > 0 \forall \varphi \in \mathcal{B}$ we have

$$(6) \quad C_2 \sum_{\substack{\beta \in \mathbb{N}_0^d \\ 1 \leq |\beta| \leq N}} \left| \int \xi^\beta \varphi(\xi) d\xi \right| \leq V_N(\varphi) \leq C_1 \sum_{\substack{\beta \in \mathbb{N}_0^d \\ 1 \leq |\beta| \leq N}} \left| \int \xi^\beta \varphi(\xi) d\xi \right|.$$

(4°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \exists N \in \mathbb{N} \forall B \Subset \mathbb{R}^d, \omega \in \mathcal{A}_0(\mathbb{R}^d), V_N$ (fulfilling (6)) $\exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B), C > 0, C = 1$ if $k \geq 1$:

$$(7) \quad \left| \partial^\alpha d^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] \right| \leq C\varepsilon^{-N}$$

whenever

$$(8) \quad x \in K, 0 < \varepsilon \leq 1, \varphi \in \omega + \mathcal{U}, V_N(\varphi) \leq \varepsilon^N \text{ and } \psi_1, \dots, \psi_k \in \mathcal{U}.$$

(5°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \exists N \in \mathbb{N} \forall B \Subset \mathbb{R}^d, \mathcal{B}$ (bounded) $\subset \mathcal{A}_0(\mathbb{R}^d), V_N$ (fulfilling (6)) $\exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B), C > 0, C = 1$ if $k \geq 1$: (7) holds whenever

$$x \in K, 0 < \varepsilon \leq 1, \varphi \in \mathcal{B} + \mathcal{U}, V_N(\varphi) \leq \varepsilon^N \text{ and } \psi_1, \dots, \psi_k \in \mathcal{U}.$$

Remark. By §1, Definition of R_ε , we can replace the expression $d^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k]$ with

$$d^k R(S_\varepsilon \varphi, x)[\psi_1, \dots, \psi_k] = (d^k R)(S_\varepsilon \varphi, x)[S_\varepsilon \psi_1, \dots, S_\varepsilon \psi_k].$$

This equality is a special case of the chain rule (formula for the derivation of a composition, e.g. [7, §12] or Yamamuro [13, (1.8.3)]) where the inner function S_ε is linear. In that case the sum in the chain rule has one term only containing the first differentials of the inner function $d_\psi S_\varepsilon(\varphi) = S_\varepsilon(\psi)$.

PROOF OF EQUIVALENCES: The equivalence of (1°), (1'°) and (1''°) can be easily seen (for (1°) \Rightarrow (1'°) see the proof of Theorem 3 in [7] or [5, 10.5] the proof of (C) \Rightarrow (A)).

(1°) \Leftrightarrow (2°) is said in in Grosser et al. [5, Theorem 17.4] and proved at the end of Chapter 17. The proof is based on the same proof for \mathcal{G}^d in [7].

(3'°) \Rightarrow (2'°) is evident. □

PROOF OF (2'°) \Rightarrow (4°): by contradiction. If (4°) does not hold for some K, α, k , take N for these K, α, k by (2'°). In non(4°) put $(k + 1)N + 1$ instead of N and so get $B \Subset \mathbb{R}^d, \omega \in \mathcal{A}_0(\mathbb{R}^d)$ and a function V_N fulfilling (6). Choose a basis $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots$ of absolutely convex open neighbourhoods of zero in $\mathcal{A}(B)$. By non(4°), for every $j = 1, 2, \dots$ there are

$$(9) \quad \varepsilon_j \in]0, 1], x_j \in K, \varphi_j \in \omega + \mathcal{U}_j \quad \text{with} \quad V_{(k+1)N+1}(\varphi_j) \leq \varepsilon_j^{(k+1)N+1}$$

$$\text{and} \quad \psi_{ij} \in \mathcal{U}_j \quad (i = 1, 2, \dots, k)$$

such that

$$(10) \quad \left| \partial^\alpha d^k R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1j}, \dots, \psi_{kj}] \right| > C \varepsilon_j^{-(k+1)N-1}$$

where $C = j$ for $k = 0$, $C = 1$ for $k \geq 1$.

As $\{\mathcal{U}_j\}$ is an increasing basis, we have by (9)

$$(11) \quad \lim_{j \rightarrow \infty} \varphi_j = \omega, \quad \lim_{j \rightarrow \infty} \psi_{ij} = 0 \quad (i = 1, 2, \dots, k).$$

Consequently, the sets $\{\varphi_j; j = 1, 2, \dots\}$, $\{\psi_{ij}; j = 1, 2, \dots\}$ ($i = 1, 2, \dots, k$) are bounded in $\mathcal{A}(B)$. As we can take subsequences instead, we can suppose without loss of generality that either $\{\varepsilon_j\}$ has a limit $\varepsilon_0 \in]0, 1]$, or

$$(12) \quad \varepsilon_1 > \varepsilon_2 > \dots \searrow 0$$

and (in both cases) $\lim x_j = x_0 \in K$. In the former case, we have by Corollary 2, due to (11),

$$(13) \quad \lim_{j \rightarrow \infty} \left| \partial^\alpha d^k R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1j}, \dots, \psi_{kj}] \right| = \left| \partial^\alpha d^k R_{\varepsilon_0}(\omega, x_0)[0, \dots, 0] \right|$$

$= 0$ if $k \geq 1$ resp. $= |\partial^\alpha R_{\varepsilon_0}(\omega, x_0)|$ if $k = 0$.

This contradicts (10).

Now only the case (12) remains and we can suppose without loss of generality that $\varepsilon_1 = 1$ in (12). In this case we define paths $\varepsilon \mapsto \varphi^\varepsilon$, $\varepsilon \mapsto \psi_i^\varepsilon$, ($i = 1, \dots, k$) as follows:

$$(14) \quad \varphi^\varepsilon = \varphi_j, \quad \psi_i^\varepsilon = \varepsilon^N \cdot \psi_{ij} \quad \text{for } \varepsilon \in [\varepsilon_j, \varepsilon_{j-1}[$$

$(j = 2, 3, \dots \text{ resp. } j = 1 \text{ and } \varepsilon = 1).$

By (9), for $\varepsilon \in [\varepsilon_j, \varepsilon_{j-1}[$ we have

$$V_{(k+1)N+1}(\varphi^\varepsilon) = V_{(k+1)N+1}(\varphi_j) \leq \varepsilon_j^{(k+1)N+1} \leq \varepsilon^{(k+1)N+1}$$

and so due to (6) the path $\varepsilon \mapsto \varphi^\varepsilon$ has asymptotically vanishing moments of order $(k + 1)N + 1$; the more of order N . The paths are bounded. On bounded sets $\{\psi_{ij}; j = 1, 2, \dots\}$, moments are bounded, so the paths $\varepsilon \mapsto \psi_i^\varepsilon$, ($i = 1, \dots, k$) have asymptotically vanishing moments of order N , too. On the other hand, if $\varepsilon = \varepsilon_j$, we estimate due to (10):

$$\left| \partial^\alpha d^k R_\varepsilon(\varphi^\varepsilon, x_j)[\psi_1^\varepsilon, \dots, \psi_k^\varepsilon] \right| = \left| \partial^\alpha d^k R_{\varepsilon_j}(\varphi_j, x_j)[\varepsilon_j^N \psi_{1j}, \dots, \varepsilon_j^N \psi_{kj}] \right|$$

$> \varepsilon_j^{kN} \cdot C \varepsilon_j^{-(k+1)N-1} = C \varepsilon_j^{-N-1}.$

This contradicts (2'°). □

PROOF OF (4°) ⇒ (5°): Bounded sets in the space \mathcal{D} are relatively compact (see [12, III.2.2., Theorem 7]). Hence (5°) follows easily from the fact that the set \mathcal{B} can be covered with a finite number of sets $\omega_1 + \frac{1}{2}\mathcal{U}_1, \dots, \omega_m + \frac{1}{2}\mathcal{U}_m$, where the neighbourhoods $\mathcal{U}_1, \dots, \mathcal{U}_m$ and the points $\omega_1, \dots, \omega_m$ have the properties described in (4°). Put $\mathcal{U} = \frac{1}{2} \bigcap_{j=1}^m \mathcal{U}_j$. Then the sets $\omega_1 + \mathcal{U}_1, \dots, \omega_m + \mathcal{U}_m$ cover $\mathcal{B} + \mathcal{U}$ and the proof is evident. □

PROOF OF (5°) ⇒ (3'°): Getting N from (5°), we are proving (3'°) for $N + 1$ instead of N . Let the first of bounded paths (4) has asymptotically vanishing moments of order $N + 1$, let the compact $B \Subset \mathbb{R}^d$ contain all supports of the values of the bounded paths (4) and denote $\mathcal{B} = \{\varphi^\varepsilon; \varepsilon \in]0, 1]\}$. Choose e.g., by (6),

$$V_N = \sum_{1 \leq |\beta| \leq N} \left| \int \xi^\beta \varphi(\xi) d\xi \right|$$

and so we get \mathcal{U} by (5°). As the sets

$$\{\psi_i^\varepsilon; \varepsilon \in]0, 1]\} \quad (i = 1, 2, \dots, k)$$

are bounded, there is a $c > 0$ such that $c\psi_i^\varepsilon \in \mathcal{U} \quad (\forall i, \varepsilon)$. Then the condition (5°) gives

$$\left| \partial^\alpha d^k R_\varepsilon(\varphi^\varepsilon, x)[c\psi_1^\varepsilon, \dots, c\psi_k^\varepsilon] \right| \leq C\varepsilon^{-N}$$

whenever

$$x \in K \quad \text{and} \quad V_N(\varphi^\varepsilon) \leq \varepsilon^N.$$

Thanks to (6), this condition is fulfilled for ε small enough, as the path $\varepsilon \mapsto \varphi^\varepsilon$ has asymptotically vanishing moments of order $N + 1$. Hence

$$\begin{aligned} \left| \partial^\alpha d^k R_\varepsilon(\varphi^\varepsilon, x)[\psi_1^\varepsilon, \dots, \psi_k^\varepsilon] \right| &= c^{-k} \left| \partial^\alpha d^k R_\varepsilon(\varphi^\varepsilon, x)[c\psi_1^\varepsilon, \dots, c\psi_k^\varepsilon] \right| \\ &\leq c^{-k} \cdot C\varepsilon^{-N} = O(\varepsilon^{-N-1}) \end{aligned}$$

what we had to prove. Thus the equivalence of all equivalent definitions is proved. □

§8. Equivalent definitions of the null ideal \mathcal{N} , i.e. the ideal of the negligible representatives for algebra \mathcal{G}^2 , is the set of all $R \in \mathcal{E}_M^2(\Omega)$ fulfilling one of the following equivalent conditions (\mathcal{A}_q means $\mathcal{A}_q(\mathbb{R}^d)$, \mathcal{D} means $\mathcal{D}(\mathbb{R}^d)$, ...). As $\mathcal{E}_M^d \subset \mathcal{E}_M^2$, the more this equivalences hold for $R \in \mathcal{E}_M^d$ and we can use any of the

following conditions to define the ideal $\mathcal{N} \cap \mathcal{E}_M^d$ of negligible representatives for the algebra \mathcal{G}^d .

(0°) $\forall K \Subset \Omega, n \in \mathbb{N} \exists q \in \mathbb{N} \forall \mathcal{B}$ (bounded) $\subset \mathcal{D}$:

$$R_\varepsilon(\varphi, x) = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K, \varphi \in \mathcal{B} \cap \mathcal{A}_q$.

(1°) (classical Colombeau's definition, only the uniformity with respect to φ is added here) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, n \in \mathbb{N} \exists q \in \mathbb{N} \forall \mathcal{B}$ (bounded) $\subset \mathcal{D}$:

$$\partial^\alpha R_\varepsilon(\varphi, x) = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K, \varphi \in \mathcal{B} \cap \mathcal{A}_q$.

(2°) (the same for the differentials with respect to φ) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0, n \in \mathbb{N} \exists q \in \mathbb{N} \forall \mathcal{B}$ (bounded) $\subset \mathcal{D}$:

$$\partial^\alpha d^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K, \varphi \in \mathcal{B} \cap \mathcal{A}_q, \psi_1, \dots, \psi_k \in \mathcal{B} \cap (\mathcal{A}_q - \mathcal{A}_q)$.

(3°) $\forall K \Subset \Omega, n \in \mathbb{N} \exists q \in \mathbb{N}$: for every bounded \mathcal{C}^∞ path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$ that has asymptotically vanishing moments of order q , we have

$$R_\varepsilon(\varphi^\varepsilon, x) = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

(4°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, n \in \mathbb{N} \exists q \in \mathbb{N}$: for every bounded \mathcal{C}^∞ path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$ that has asymptotically vanishing moments of order q , we have

$$\partial^\alpha R_\varepsilon(\varphi^\varepsilon, x) = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

(5°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0, n \in \mathbb{N} \exists q \in \mathbb{N}$: for every bounded \mathcal{C}^∞ paths $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0, \varepsilon \mapsto \psi_i^\varepsilon \in \mathcal{A} \ (i = 1, \dots, k)$ that all have asymptotically vanishing moments of order q , we have

$$\partial^\alpha d^k R_\varepsilon(\varphi^\varepsilon, x)[\psi_1^\varepsilon, \dots, \psi_k^\varepsilon] = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

Evidently, equivalent conditions (3'°), (4'°), (5'°) resp. (3''°), (4''°), (5''°) can be added where the \mathcal{C}^∞ requirement for paths is omitted resp. in addition the uniformity condition is supplied like in §7, Equivalent definitions.

(6°) $\forall K \Subset \Omega, \alpha \in \mathbb{N}_0^d, n \in \mathbb{N} \exists q \in \mathbb{N}$: for every bounded \mathcal{C}^∞ path

$$\varepsilon \mapsto (\varphi_x^\varepsilon)_{x \in \Omega} \in \mathcal{C}^\infty(\Omega \rightarrow \mathcal{A}_0)$$

that has asymptotically vanishing moments of order q , we have

$$\left(\frac{\partial}{\partial x}\right)^\alpha R_\varepsilon(\varphi_x^\varepsilon, x) = O(\varepsilon^n) \quad (\varepsilon \searrow 0).$$

uniformly for $x \in K$.

Remarks. 8.1. The equivalence $(1^\circ) \Leftrightarrow (2^\circ)$ is proved in [7, Theorem 18], while the condition (0°) is added only in [5, Theorem 13.1] (both equivalences are proved in [5] and [7] only in \mathcal{E}_M^d , here we have to prove them). It is surprising that there is such a simple tool for proving the negligibility that can be applied to the original Colombeau algebra as well (see [5, Chapter 12, 13]).

8.2. Although we have to consider paths depending on $x \in \Omega$ to define the moderateness, we see that paths not depending on x are sufficient for defining the negligibility. There is an error in [7, Theorem 18.4 $^\circ$] discovered and corrected in [5]: first the formulation does not correspond to the definition of negligible representatives in [4], where the paths do not depend on x , second the equivalence does not hold. Now we see that the condition 18.4 $^\circ$ in [7], dealing with paths depending on x , need not be corrected, it can be omitted.

PROOF OF EQUIVALENCES: The ideas of the proofs are the same that were used already in [7]. \mathcal{D} in these proofs means $\mathcal{D}(\mathbb{R}^d)$, \mathcal{A}_q means $\mathcal{A}_q(\mathbb{R}^d)$, $(3^\circ) \Leftrightarrow (4^\circ) \Leftrightarrow (5^\circ)$ follow from [5, Theorem 17.9]. □

PROOF OF $(0^\circ) \Leftrightarrow (3^\circ)$: We know that (3°) is equivalent to the similar condition $(3'^\circ)$ without the \mathcal{C}^∞ requirement for the path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$. $\text{non}(0^\circ) \Rightarrow \text{non}(3'^\circ)$ being evident, we are going to prove $(0^\circ) \Rightarrow (3'^\circ)$. For a given K take first a number N by 7(2'' $^\circ$) for $\alpha = 0, k = 1$ such that for every bounded path $\varepsilon \mapsto \psi^\varepsilon \in \mathcal{A}$ that has asymptotically vanishing moments of order N we have

$$(15) \quad d_{\psi^\varepsilon} R_\varepsilon(\tilde{\varphi}^\varepsilon, x) = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly if $x \in K$ and $\varepsilon \mapsto \tilde{\varphi}^\varepsilon$ runs over a set of equi-bounded paths having uniformly asymptotically vanishing moments of order N . Then, having chosen n , let q satisfies (0°) and at the same time

$$(16) \quad q \geq n + 2N.$$

Let a path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$ satisfy the hypotheses of (3°) and let $B \Subset \mathbb{R}^d$ be a bounded set containing the supports of all φ^ε . Recall a known lemma of functional analysis (Robertson A.P.-Robertson W.J. [11, II.3, Lemma 5]). If linear forms f_0, f_1, \dots, f_k on a linear space E are linearly independent then there is a point $x \in E$ such that $f_0(x) = 1, f_1(x) = \dots = f_k(x) = 0$. Since the functions $x \mapsto x^\beta$ ($\beta \in \mathbb{N}_0^d, 0 \leq |\beta| \leq q$) considered as distributions $\in \mathcal{D}'(B)$ are linearly independent, there are test functions $\psi_\alpha \in \mathcal{D}(B)$ ($\alpha \in \mathbb{N}_0^d, 1 \leq |\alpha| \leq q$) fulfilling

$$(17) \quad \int \psi_\alpha(\xi) \xi^\alpha \, d\xi = 1,$$

$$(18) \quad \int \psi_\alpha(\xi) \xi^\beta \, d\xi = 0 \quad \text{for } \beta \neq \alpha, 0 \leq |\beta| \leq q.$$

By (18), $\psi_\alpha \in \mathcal{A}(B)$ (note that $\alpha \neq 0$). If we denote

$$(19) \quad c_{\alpha\varepsilon} := \int \varphi^\varepsilon(\xi) \xi^\alpha \, d\xi,$$

we obtain that

$$(20) \quad \kappa^\varepsilon := \varphi^\varepsilon - \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ 1 \leq |\alpha| \leq q}} c_{\alpha\varepsilon} \psi_\alpha \in \mathcal{A}_q(B).$$

As $\varepsilon \mapsto \varphi^\varepsilon$ has asymptotically vanishing moments of order q ,

$$(21) \quad c_{\alpha\varepsilon} = O(\varepsilon^q) \quad (\varepsilon \searrow 0).$$

Let us order the summation indices α in (20) into a sequence $\alpha_1, \dots, \alpha_m$. Then

$$\begin{aligned} R_\varepsilon(\varphi^\varepsilon, x) - R_\varepsilon(\kappa^\varepsilon, x) &= \sum_{j=1}^m \left(R_\varepsilon(\kappa^\varepsilon + \sum_{i=1}^j c_{\alpha_i\varepsilon} \psi_{\alpha_i}, x) - R_\varepsilon(\kappa^\varepsilon + \sum_{i=1}^{j-1} c_{\alpha_i\varepsilon} \psi_{\alpha_i}, x) \right) \end{aligned}$$

and by the mean value theorem (e.g. [7, Theorem 11]) the term of this sum belongs to the closed convex hull of the set

$$\begin{aligned} &\left\{ dR_\varepsilon(\kappa^\varepsilon + \sum_{i=1}^{j-1} c_{\alpha_i\varepsilon} \psi_{\alpha_i} + t \cdot c_{\alpha_j\varepsilon} \psi_{\alpha_j}, x)[c_{\alpha_j\varepsilon} \psi_{\alpha_j}]; t \in]0, 1[\right\} \\ &= \left\{ \varepsilon^{q-N} dR_\varepsilon(\kappa^\varepsilon + \sum_{i=1}^{j-1} c_{\alpha_i\varepsilon} \psi_{\alpha_i} + t \cdot c_{\alpha_j\varepsilon} \psi_{\alpha_j}, x)[\varepsilon^{N-q} c_{\alpha_j\varepsilon} \psi_{\alpha_j}]; t \in]0, 1[\right\}. \end{aligned}$$

By (21) ($N \leq q$ due to (16)) the path $\varepsilon \mapsto \varepsilon^{N-q} c_{\alpha_j\varepsilon} \psi_{\alpha_j}$ has asymptotically vanishing moments of order N , so it follows from (15) that

$$R_\varepsilon(\varphi^\varepsilon, x) - R_\varepsilon(\kappa^\varepsilon, x) = \varepsilon^{q-N} \cdot O(\varepsilon^{-N}) = O(\varepsilon^{q-2N}) = O(\varepsilon^n)$$

(the last equality follows from (16)) uniformly if $x \in K$. By (20) and (0°), we have $R_\varepsilon(\kappa^\varepsilon, x) = O(\varepsilon^n)$ uniformly for $x \in K$, hence so is $R_\varepsilon(\varphi^\varepsilon, x)$. Thus the equivalence (0°) \Leftrightarrow (3°) is proved. \square

PROOF OF (2°) \Leftrightarrow (1°) \Leftrightarrow (0°): (2°) \Rightarrow (1°) \Rightarrow (0°) being obvious, we are going to prove (0°) \Rightarrow (2°). For this purpose, we write (2°) in the following equivalent form using the total differential \mathbf{d} of R :

(2'°) $\forall K \Subset \Omega, k \in \mathbb{N}_0, n \in \mathbb{N} \exists q \in \mathbb{N}$ such that $\forall \mathcal{B}$ (bounded) $\subset \mathcal{D}$ we have

$$(22) \quad \mathbf{d}^k R_\varepsilon(\varphi, x)[(\psi_1, h_1), \dots, (\psi_k, h_k)] = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

uniformly for

$$(23) \quad \begin{aligned} x \in K, \varphi \in \mathcal{B} \cap \mathcal{A}_q, \psi_i \in \mathcal{B} \cap (\mathcal{A}_q - \mathcal{A}_q), \\ h_i \in \mathbb{R}^d, |h_i| \leq 1 \quad (\text{Euclidean norm, } i = 1, \dots, k). \end{aligned}$$

Similarly, we will write §7, the definition (2^o) in the form using the total differential:

$\forall K^* \Subset \Omega, k \in \mathbb{N}_0 \exists N \in \mathbb{N}$ such that for every bounded paths

$$\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0, \quad \varepsilon \mapsto \psi_i^\varepsilon \in \mathcal{A} \quad (i = 1, 2, \dots, k)$$

that all have asymptotically vanishing moments of order N , we have

$$\mathbf{d}^k R_\varepsilon(\varphi^\varepsilon, x)[(\psi_1^\varepsilon, h_1), \dots, (\psi_k^\varepsilon, h_k)] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K^*, h_i \in \mathbb{R}^d, |h_i| \leq 1 \ (i = 1, \dots, k)$. Let us write $k + 1$ instead of k and apply this definition to test functions belonging to \mathcal{A}_N resp. $\mathcal{A}_N - \mathcal{A}_N$ only. We easily obtain the following consequence:

$\forall K^* \Subset \Omega, k \in \mathbb{N}_0 \exists N \in \mathbb{N}$ such that for every bounded $\mathcal{B} \subset \mathcal{D}$, we have

$$(24) \quad \mathbf{d}^{k+1} R_\varepsilon(\varphi, x)[(\psi_1, h_1), \dots, (\psi_{k-1}, h_{k-1}), (\psi_k, h_k), (\psi_k, h_k)] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K^*, \varphi \in \mathcal{B} \cap \mathcal{A}_N, \psi_i \in \mathcal{B} \cap (\mathcal{A}_N - \mathcal{A}_N), h_i \in \mathbb{R}^d, |h_i| \leq 1 \ (i = 1, \dots, k)$.

In the following, we will write Φ for (φ, x) and Ψ_i for (ψ_i, h_i) . The proof will be done by induction. Denote by $S(k) \ (k \in \mathbb{N}_0)$ the statement

$S(k) : \forall K \Subset \Omega, n \in \mathbb{N} \exists q \in \mathbb{N}$ such that $\forall \mathcal{B}$ (bounded) $\subset \mathcal{D}$, (22) holds uniformly under conditions (23).

$S(0)$ is (0°) . Choosing $K \Subset \Omega, k \in \mathbb{N}, n \in \mathbb{N}$, we have to deduce $S(k)$ from $S(k-1)$. First, for the chosen K and k , we get N from the consequence containing (24), where we substitute a larger compact

$$K^* := \left\{ x \in \mathbb{R}^d; \text{dist}(x, K) \leq \Delta \right\} \subset \Omega$$

with an appropriate $\Delta > 0$. Then, for this K^* by the statement $S(k-1)$, we get an integer $q \geq N$ such that

$$(25) \quad \mathbf{d}^{k-1} R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}] = O(\varepsilon^{2n+N}) \quad (\varepsilon \searrow 0)$$

uniformly under conditions: $x \in K^*, \varphi, \psi_i, h_i$ by (23) for any bounded $\mathcal{B} \subset \mathcal{D}$. Under these conditions and for $t \in [0, \Delta]$, we have by (24)

$$\mathbf{d}^{k+1} R_\varepsilon(\Phi + t\Psi_k)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k, \Psi_k] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly. From the mean value theorem it follows

$$\begin{aligned} & \left| \mathbf{d}^k R_\varepsilon(\Phi + t\Psi_k)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k] - \mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k] \right| \\ & \leq \sup_{t' \in [0, t]} \left| \mathbf{d}^{k+1} R_\varepsilon(\Phi + t'\Psi_k)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k, t'\Psi_k] \right| = tO(\varepsilon^{-N}) \quad (\varepsilon \searrow 0) \end{aligned}$$

uniformly under the above conditions. Denoting by $\overline{B}(a, r) \subset \mathbb{C}$ the closed ball of center a and radius r , we can write this

$$\mathbf{d}^k R_\varepsilon(\Phi + t\Psi_k)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k] \in \overline{B}(\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k], t\varepsilon^{-N} \cdot c)$$

with a constant c depending on \mathcal{B} but neither on $t \in [0, \Delta]$ nor on $\varphi, \psi_i \in \mathcal{B}$. It follows from the mean value theorem again:

$$\begin{aligned} & \mathbf{d}^{k-1} R_\varepsilon(\Phi + \varepsilon^{n+N} \Psi_k)[\Psi_1, \dots, \Psi_{k-1}] - \mathbf{d}^{k-1} R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}] \\ & \in \overline{\text{conv}} \left\{ \mathbf{d}^k R_\varepsilon(\Phi + t\Psi_k)[\Psi_1, \dots, \Psi_{k-1}, \varepsilon^{n+N} \Psi_k]; t \in [0, \varepsilon^{n+N}] \right\} \\ & \subset \bigcup_{t \in [0, \varepsilon^{n+N}]} \overline{B}(\varepsilon^{n+N} \mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k], \varepsilon^{n+N} \cdot t\varepsilon^{-N} c) \\ & = \overline{B}(\varepsilon^{n+N} \mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k], \varepsilon^{2n+N} \cdot c). \end{aligned}$$

The radius is $O(\varepsilon^{2n+N})$ uniformly under (23); the left-hand side is $O(\varepsilon^{2n+N})$ as well, thanks to (25). Hence the center $\varepsilon^{n+N} \mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k]$ must be $O(\varepsilon^{2n+N})$, too. Thus

$$\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k] = O(\varepsilon^n) \quad (\varepsilon \searrow 0)$$

what we had to prove.

It remains to prove the equivalence with (6°). (5°) \Rightarrow (6°) follows from the chain rule (differentiation of the composition, e.g. [7, Theorem 12] or [13, (1.8.3)]). (6°) \Rightarrow (4°) is obvious. □

9. Now, we can define the quotient algebras $\mathcal{G}^2 := \mathcal{E}_M^2 / \mathcal{N}$ and $\mathcal{G}^d := \mathcal{E}_M^d / \mathcal{N} \cap \mathcal{E}_M^d$. The equality of both algebras is proved in [8]. The set of representatives \mathcal{E}_M^2 is strictly larger than \mathcal{E}_M^d , as is shown in [5, 17.11].

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