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Equality of two diffeomorphism invariant Colombeau algebras

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The paper is a continuation of [9] and its only aim is to prove that both diffeomorphism invariant Colombeau-type algebras $G^d$ and $G^2$ introduced in [8] and [6] (Grosser et al.) coincide. In [6] diverse possibilities to define Colombeau-type algebras are researched; our result shows that there is only one diffeomorphism invariant Colombeau-type algebra among them for a given domain $\Omega \subset \mathbb{R}^d$.

§1. In this paper, we use notations introduced in [9] and mostly we refer to [9]. This paper is devoted to prove the following

Theorem. For every open $\Omega \subset \mathbb{R}^d$, the algebras $G^2(\Omega)$ and $G^d(\Omega)$ coincide.

As the algebras are quotient algebras $G^2 := \mathcal{E}_M^2/\mathcal{N}$ and $G^d := \mathcal{E}_M^d/\mathcal{N} \cap \mathcal{E}_M^d$, the theorem says that for any representative $R \in \mathcal{E}_M^2$, another representative $\tilde{R} \in \mathcal{E}_M^d$ can be found with $R - \tilde{R} \in \mathcal{N}$. To prove it, in all what follows, we assume that $R \in \mathcal{E}_M^2$ is given and we are going to construct $\tilde{R}$. This will be done in several steps. In every step functions of variables $\varphi, x$ are constructed and their properties are presented with the aim to construct at last the required representative $\tilde{R}$. First we show that it is sufficient to do it for representatives with compact support.

Proposition. Suppose that for any representative $R \in \mathcal{E}_M^2(\mathbb{R}^d)$ such that there is a compact $K \subset \mathbb{R}^d$ fulfilling $R(\varphi, x) = 0$ whenever $x \in \mathbb{R}^d \setminus K$, a representative

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\( \tilde{R} \in \mathcal{E}^d_M(\mathbb{R}^d) \) can be found with \( R - \tilde{R} \in \mathcal{N} \). Then for any open \( \Omega \subset \mathbb{R}^d \) the algebras \( \mathcal{G}^2(\Omega) \) and \( \mathcal{G}^d(\Omega) \) coincide.

**Proof:** Let \( R \in \mathcal{E}^2_M(\Omega) \). Recall that unlike in [8] and [6] here a representative is defined on \( \mathcal{E}(\Omega) = \mathcal{A}_0(\mathbb{R}^d) \times \Omega \) and we do not lose generality with this assumption. Choose a locally finite covering
\[
\Omega = \bigcup \Omega_m \quad \text{with} \quad \overline{\Omega}_m \subset \Omega
\]
and a partition of unity \( 1 = \sum \chi_m \) on \( \Omega \) subordinated to this covering, \( \chi_m \in \mathcal{D}(\Omega_m) \), \( K_m := \text{supp} \chi_m \). Then \( R = \sum R_m \) if we denote \( R_m(\varphi, x) := R(\varphi, x) \cdot \chi_m(x) \) and we have \( R_m(\varphi, x) = 0 \) whenever \( x \in \Omega \setminus K_m \). \( R_m \) can be easily extended to belong to \( \mathcal{E}^d_M(\mathbb{R}^d) \) putting \( R_m(\varphi, x) = 0 \) whenever \( x \in \mathbb{R}^d \setminus K_m \).

By hypothesis, we can find \( \tilde{R}_m \in \mathcal{E}^d_M(\mathbb{R}^d) \) with \( \tilde{R}_m - R_m \in \mathcal{N} \). Then, for every \( m \), we choose a test function \( \sigma_m \in \mathcal{D}(\Omega_m) \) that is \( = 1 \) on a neighbourhood of \( K_m \). The functions \( \chi_m \) and \( \sigma_m \) are considered to be elements of \( \mathcal{E}(\mathbb{R}^d) \) as functions independent of the first variable. Consequently, \( \sigma_m \in \mathcal{E}^d_M \). Considering all representatives to be elements of \( \mathcal{E}^2_M(\Omega) \) (i.e. restricted to \( \mathcal{A}_0(\mathbb{R}^d) \times \Omega \)), we have (note that \( \mathcal{N} \) is an ideal) \( R - \sum \tilde{R}_m \sigma_m = \sum (R_m - \tilde{R}_m \sigma_m) = \sum (R_m - \tilde{R}_m) \sigma_m \in \mathcal{N} \), the sum being locally finite. From the same reason, \( \tilde{R} := \sum \tilde{R}_m \sigma_m \in \mathcal{E}^d_M \). \( \tilde{R} \) is thus a required representative. \( \Box \)

§2. **Remark.** For \( B \subset \mathbb{R}^d \), it is known that \( \mathcal{D}(B) \) is a Fréchet space. Its topology can be generated by a countable system of norms defined by continuous scalar products, e.g.

\[
\varphi, \psi \mapsto \int \frac{\partial^{dn}}{\partial \xi_1^m \ldots \partial \xi_d^m} \varphi(\xi) \cdot \frac{\partial^{dn}}{\partial \xi_1^m \ldots \partial \xi_d^m} \psi(\xi) \, d\xi.
\]

A continuous scalar product \( \varphi, \psi \mapsto (\varphi, \psi) \) is \( \mathcal{C}^\infty \), being sesqui-linear, and we have

\[
d_\psi (\varphi, \varphi) = 2 \mathcal{R}(\varphi, \psi),
\]
\[
d_{\psi_1, \psi_2}^2 (\varphi, \varphi) = 2 \mathcal{R}(\psi_1, \psi_2);
\]

the derivatives of higher orders are zero. Hence the norm generated by a continuous scalar product is \( \mathcal{C}^\infty \) in all points except of origin. The function \( \psi \mapsto (\psi, \psi) = ||\psi||^2 \) is \( \mathcal{C}^\infty \) always.

**Notation.** The function \( V_N \), used in Equivalent Definition [9, §7, (4°) and (5°)] of \( \mathcal{E}^2_M \), can be

\[
V_N (\varphi) = \left( \sum_{\beta \in \mathbb{N}_0^d \atop 1 \leq |\beta| \leq N} \left| \int \xi^\beta \varphi(\xi) \, d\xi \right|^2 \right)^{1/2} \quad (\varphi \in \mathcal{A}_0).
\]
Evidently, this function fulfills [9, (6)]:

\[ \forall N \in \mathbb{N}, \mathcal{B} \text{ (bounded)} \subset A_0 \quad \exists C_1, C_2 > 0 \quad \forall \varphi \in \mathcal{B} : \]

\[ C_2 \sum_{\beta \in \mathbb{N}_0^d, 1 \leq |\beta| \leq N} \left| \int \xi^\beta \varphi(\xi) \, d\xi \right| \leq V_N(\varphi) \leq C_1 \sum_{\beta \in \mathbb{N}_0^d, 1 \leq |\beta| \leq N} \left| \int \xi^\beta \varphi(\xi) \, d\xi \right|. \]

Evidently every multiple \( c \cdot V_N(\varphi) \) satisfies these inequalities, so it can be used in Equivalent Definition [9, §7, (4°) and (5°)]. Also the function

\[ V'_N(\varphi) = \left( \sum_{\beta \in \mathbb{N}_0^d, 1 \leq |\beta| \leq N} \| \varphi \|^{4|\beta|/d} \left| \int \xi^\beta \varphi(\xi) \, d\xi \right|^2 \right)^{1/2} \quad (\varphi \in A_0) \]

fulfills the above inequalities [9, (6)] that allows us to use it in Equivalent Definitions [9, §7, (4°), (5°)]. Indeed, a bounded set is relatively compact, so \( \exists c_1, c_2 > 0 \) (depending on \( \mathcal{B} \)) \( \forall \varphi \in \mathcal{B} \) we have \( c_2 \leq \| \varphi \|_{L^2} \leq c_1 \); [9, (6)] follows easily.

It can be checked that

\[ (2) \quad \| S_\varepsilon \varphi \|^{-2/d} = \varepsilon \cdot \| \varphi \|^{-2/d}, \quad \int S_\varepsilon \varphi(\xi) \xi^\beta \, d\xi = \varepsilon |\beta| \int \varphi(\xi) \xi^\beta \, d\xi \quad (\beta \in \mathbb{N}_0^d), \]

so \( V'_N(S_\varepsilon \varphi) = V'_N(\varphi) \).

For all what follows, a function \( \rho \in A_0([-1,1]) \) is fixed such that

\[ \rho(\xi) > 0 \quad \text{iff} \quad \xi \in ]-1,1[, \]

\[ \rho_\varepsilon := S_\varepsilon \rho, \quad \vartheta := \rho_{1/2} \ast \chi_{[-3/2,3/2]} \]

(convolution with the characteristic function),

\[ \vartheta_m(\xi) := \vartheta(2^{-m} \xi) \quad (m \text{ integer}). \]

So \( \vartheta_m(\xi) = 1 \) iff \( \xi \in [-2^m,2^m], \) \( 0 < \vartheta_m(\xi) \leq 1 \) iff \( \xi \in ]-2^{m+1},2^{m+1}[ \) and \( \vartheta_m \) is decreasing on \( [2^m,2^{m+1}] \). Denote furthermore

\[ K_m := [-2^m,2^m]^d \]

and for \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \) denote \( \vartheta_m^\otimes d(\xi) := \vartheta_m(\xi_1) \ldots \vartheta_m(\xi_d). \) If there is no danger of confusion, we will write simply \( \vartheta_m(\xi) \) instead of \( \vartheta_m^\otimes d(\xi). \)
Thanks to §1, Proposition we can assume that the given representative \( R \) belongs to \( E^2_M(\mathbb{R}^d) \) and that there is a compact \( K \subseteq \Omega \) fulfilling \( R(\varphi, x) = 0 \) for \( x \in \mathbb{R}^d \setminus K \). In this case, in the equivalent definitions of \( E^2_M \) and \( N \) ([9, §7, 8]) we can omit \( \forall K \subseteq \Omega \) and replace the uniformity on \( K \) with the uniformity on the whole of \( \mathbb{R}^d \). Denote by \( N_L \) the number \( N \) from Equivalent Definition [9, §7, (5)] holding at the same time for all \( |\alpha| \leq L \) and for all differentials of order \( k \leq L \). Certainly, this equivalent definition remains valid if we take any greater number for \( N_L \). We replace our representative with another one determining the same generalized function, if needed, to obtain the following

**Properties of \( R \)**

(1°) There is an increasing sequence \( \{N_L\}_{L \in \mathbb{N}} \subseteq \mathbb{N}, N_L \geq L \), fulfilling:

\[
\forall B \in \mathbb{R}, \mathcal{B} (\text{bounded}) \subseteq \mathcal{A}_0(B), L \in \mathbb{N} \quad \exists U (\text{absolutely convex open neighbourhood of zero}) \subseteq \mathcal{A}(B), C > 0 \quad \forall \ell = 1, 2, \ldots, L, \psi_1, \ldots, \psi_\ell \in U, \varphi \in \mathcal{B} + 2U, \varepsilon \in [0, 1], \varepsilon^{N_L} \geq V_{N_L}(\varphi), \alpha \in \mathbb{N}^d_0, |\alpha| \leq L, x \in \mathbb{R}^d:
\]

\[
\left| \partial^\alpha (d_{S_\varepsilon \psi_1, \ldots, S_\varepsilon \psi_\ell} R(S_\varepsilon \varphi, x)) \right| = \left| \partial^\alpha d_{\psi_1, \ldots, \psi_\ell} R(S_\varepsilon \varphi, x) \right| \leq \varepsilon^{-N_L}, \quad \left| \partial^\alpha R(S_\varepsilon \varphi, x) \right| \leq C \varepsilon^{-N_L}.
\]

(2°) The first inequality in (3) can be written in the form

\[
\left| \partial^\alpha d_{\psi_1, \ldots, \psi_\ell} R(S_\varepsilon \varphi, x) \right| \leq \varepsilon^{-N_L} \|\psi_1\|_U \cdots \|\psi_\ell\|_U
\]

if we omit the hypothesis \( \psi_1, \ldots, \psi_\ell \in U \), only supposing \( \psi_1, \ldots, \psi_\ell \in \mathcal{A}(B) \) (\( \|\cdot\|_U \) denotes the Minkowski functional assigned to \( U \)).

(3°) If \( L = 1 \), the hypothesis \( \varepsilon^{N_L} \geq V_{N_L}(\varphi) \) can be omitted, so that (3) and (4) hold for every \( \varepsilon \in [0, 1] \).

(4°) Consequently, if \( \mathcal{B} \) is convex, \( \varphi_1, \varphi_2 \in \mathcal{B} + 2U \), we have to consider two cases.

If \( L = 1 \), \( |\alpha| \leq 1 \) then

\[
\left| \partial^\alpha (R(S_\varepsilon \varphi_2, x) - R(S_\varepsilon \varphi_1, x)) \right| \leq \varepsilon^{-N_1} \|\varphi_2 - \varphi_1\|_U.
\]

Otherwise if \( |\alpha| \leq L \) and \( \varepsilon^{N_L} \geq V_{N_L}(\varphi_1), \varepsilon^{N_L} \geq V_{N_L}(\varphi_2), \ell = 1, \ldots, L, \psi_1, \ldots, \psi_{\ell - 1} \in \mathcal{A}(B) \), then

\[
\left| \partial^\alpha d_{\psi_1, \ldots, \psi_{\ell - 1}} (R(S_\varepsilon \varphi_2, x) - R(S_\varepsilon \varphi_1, x)) \right| \leq \varepsilon^{-N_L} \|\psi_1\|_U \cdots \|\psi_{\ell - 1}\|_U \|\varphi_2 - \varphi_1\|_U.
\]

**Proof of (3°)**: The items (1°) and (2°) are consequences of [9, §7, (5°)]. The equality in (3) follows from the chain rule where the inner function is linear, so
Applying with \( \psi \) \( R \) possibly we have to choose the number to choose a suitable representative determining a given generalized function and possibly we have to choose the number \( N_1 \), too. Let \( R \) be a given representative. Applying \( \frac{1}{2} \| x_B \|^{-2N_1/d} V'_{N_1} \) instead of \( V_{N_1} \) in Equivalent Definition [9, §7, (5°)] (or in Properties, item (1°)) for \( L = 1 \), we get \( U \) and \( C > 0 \) such that (3) holds if \( \ell = 1 \), \( |\alpha| \leq 1 \), \( \psi_1 \in U \), \( \varphi \in \mathcal{B} + 2U \) and

\[
\varepsilon N_1 \geq \frac{1}{2} \| x_B \|^{-2N_1/d} V'_{N_1}(\varphi).
\]

Let us define \( R'(\varphi, x) := \vartheta(\|\varphi\|^{2N_1/d} V'_{N_1}(\varphi)) \cdot R(\varphi, x) \) \( (\mathcal{L}^2 \text{ norms}) \).

First we prove that \( R' \in \mathcal{E}_M^2 \) and \( R - R' \in \mathcal{N} \). Let \( \varepsilon \mapsto \varphi \varepsilon \) (see [9, §7, (2°), §8, (4°)]) be a bounded path with asymptotically vanishing moments of order \( N_1 + 1 \).

This means that the set \( \{ \varphi \varepsilon; \ v \in [0, 1]\} \) is bounded and

\[
V'_{N_1+1}(\varphi \varepsilon) = O(\varepsilon^{N_1+1}) \quad (\varepsilon \searrow 0).
\]

Consequently the set \( \{ \|\varphi \varepsilon\|; \ v \in [0, 1]\} \) is bounded and by (2), as \( V'_{N_1} \leq V'_{N_1+1} \), we have

\[
\| S \varepsilon \varphi \varepsilon \|^{2N_1/d} V'_{N_1}(S \varepsilon \varphi) \leq \|\varphi \varepsilon\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1+1}(\varphi \varepsilon) = O(\varepsilon).
\]

It follows that for a sufficiently small \( \varepsilon \), \( S \varepsilon \varphi \varepsilon \) belongs to the open set (independent of \( x \)) \( \{ \varphi; \ \|\varphi\|^{2N_1/d} V'_{N_1}(\varphi) < 1 \} \) where \( R'(\bullet, x) = R(\bullet, x) \). Hence the assertions \( R' \in \mathcal{E}_M^2 \) and \( R - R' \in \mathcal{N} \) are proved.

Now we want to prove that \( R' \) fulfills (3°). This means that the relations (3) with \( R' \) hold for all \( \varepsilon \in [0, 1] \), provided \( L = \ell = 1 \) \( (|\alpha| \leq 1) \). To this aim, for \( \psi \in \mathcal{A}(B) \) we first estimate

\[
d_\psi \partial^\alpha R'(S \varepsilon \varphi, x) = d_\psi \partial^\alpha \left( \vartheta(\|S \varepsilon \varphi\|^{2N_1/d} V'_{N_1}(S \varepsilon \varphi)) \cdot R(S \varepsilon \varphi, x) \right)
\]

\[
= d_\psi \vartheta \left( \|S \varepsilon \varphi\|^{2N_1/d} V'_{N_1}(S \varepsilon \varphi) \right) \cdot \partial^\alpha R(S \varepsilon \varphi, x)
\]

\[
+ \vartheta \left( \|S \varepsilon \varphi\|^{2N_1/d} V'_{N_1}(S \varepsilon \varphi) \right) \cdot d_\psi \partial^\alpha R(S \varepsilon \varphi, x)
\]

\[
= d_\psi \vartheta \left( \|\varphi\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1}(\varphi) \right) \cdot \partial^\alpha R(S \varepsilon \varphi, x)
\]

\[
+ \vartheta \left( \|\varphi\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1}(\varphi) \right) \cdot d_\psi \partial^\alpha R(S \varepsilon \varphi, x).
\]

If \( \|\varphi\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1}(\varphi) > 2 \) then \( R'(S \varepsilon \varphi, x) = 0 \), hence we have to estimate (8) only if

\[
\frac{1}{2} \|\varphi\|^{2N_1/d} V'_{N_1}(\varphi) \leq \varepsilon N_1.
\]
By the Hölder inequality we have (χ denotes the characteristic function):

\[ 1 = \int \varphi \chi_B \leq \|\varphi\| \|\chi_B\|, \text{ i.e. } \|\varphi\| \geq \|\chi_B\|^{-1}. \tag{9} \]

Consequently

\[ \frac{1}{2} \|\chi_B\|^{-2N_1/d} V'_{N_1}(\varphi) \leq \varepsilon^{N_1} \]

and this is exactly our hypothesis (7) assuring that (3) holds. Hence two terms of (8) are estimated:

\[ \left| \partial^\alpha R(S_\varepsilon \varphi, x) \right| \leq C\varepsilon^{-N_1}, \quad \left| \partial^\alpha d_\psi R(S_\varepsilon \varphi, x) \right| \leq C\varepsilon^{-N_1}. \]

It remains to estimate

\[ d_\psi \vartheta \left( \|\varphi\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1}(\varphi) \right) \leq \max_t \left| \frac{d}{dt} \vartheta(t) \right| \varepsilon^{-N_1} \cdot d_\psi (\|\varphi\|^{2N_1/d} V'_{N_1}(\varphi)). \tag{10} \]

By [9, §2, Proposition] about local equicontinuity of differentials there is an absolutely convex open neighbourhood of zero \( U \subset A(B) \) such that

\[ d_\psi (\|\varphi\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1}(\varphi)) \leq 1 \]

whenever \( \varphi \in \mathcal{B} + 2U, \psi \in U \). Under these hypotheses, we have got

\[ d_\psi \vartheta \left( \|\varphi\|^{2N_1/d} \varepsilon^{-N_1} V'_{N_1}(\varphi) \right) \leq C_1 \varepsilon^{-N_1} \]

with a constant \( C_1 \) depending only on \( \vartheta \). Due to (8) and (3), it follows

\[ d_\psi \partial^\alpha R'(S_\varepsilon \varphi, x) \leq C_1 C \varepsilon^{-2N_1} + C\varepsilon^{-N_1} \leq (C_1 + 1)C \varepsilon^{-2N_1}. \]

Replacing \( U \) with a smaller one, we get \( \leq \varepsilon^{-2N_1} \). It remains to estimate \( \partial^\alpha R'(S_\varepsilon \varphi, x) \). This is similar or simpler, so we let it to the reader. \( \square \)

**Proof of (4°):** Using the mean value theorem, we have for some \( \tau \in ]0, 1[ \)

\[ \left| \partial^\alpha d_\psi^{t_1-1, \ldots, t_{\ell-1}} R(S_{2-n} \varphi, x) - \partial^\alpha d_\psi^{t_1-1, \ldots, t_{\ell-1}} R(S_{2-n} \varphi, x) \right| \]

\[ \leq \left| \partial^\alpha d_\psi^{t_1} R(S_{2-n} (\tau \varphi + (1 - \tau) \varphi), x) \right| \]

The function \( \tau \mapsto V_N(\tau \varphi + (1 - \tau) \varphi) \) is convex because \( V_N(\varphi) \) is the Euclidean norm of the point with coordinates \( \int \xi^2 \varphi(\xi) \, d\xi \). Thus in the second case of (4°), \( V_{N_L}(\tau \varphi + (1 - \tau) \varphi) \leq \varepsilon^{N_L} \) holds for all \( \tau \in [0, 1] \) and we can apply (4). \( \square \)
§4. Notation. Let a number \( r \in \mathbb{N} \) be given, let us consider the Fourier series of a function \( \psi \in \mathcal{A}(K_r) \) on the cube \( K_{r+1} \subset \mathbb{R}^d \) (§2, Notation, the function \( \vartheta_{r}^{\otimes d} \) will be denoted simply by \( \vartheta_r \))

\[
\psi(\xi) = \sum_{\beta \in \mathbb{Z}^d} c'_\beta e^{2^{-r-1} \pi i \beta \cdot \xi} \quad \text{where} \quad \beta \cdot \xi := \beta_1 \xi_1 + \cdots + \beta_d \xi_d,
\]

(12)

\[
c'_\beta = 2^{-d(r+2)} \int \psi(\xi) e^{-2^{-r-1} \pi i \beta \cdot \xi} d\xi.
\]

As \( \int \psi = 0 \), we have \( c'_0 = 0 \). As \( \vartheta_r = 1 \) on \( K_r \), we have as well

\[
\psi(\xi) = \sum_{\beta \neq 0} c'_\beta e^{2^{-r-1} \pi i \beta \cdot \xi} \vartheta_r(\xi).
\]

We will use another expansion \( \psi = \sum_{\beta \neq 0} c'_\beta \gamma'_\beta \) where the functions \( \gamma'_\beta \) are defined:

\[
\gamma'_\beta(\xi) = e^{2^{-r-1} \pi i \beta \cdot \xi} \vartheta_r(\xi) - c''_\beta \vartheta_r(\xi)
\]

with constants \( c''_\beta \) such that \( \gamma'_\beta \in \mathcal{A} \). This means that

(13)

\[
c''_\beta \int \vartheta_r(\xi) d\xi = \int e^{2^{-r-1} \pi i \beta \cdot \xi} \vartheta_r(\xi) d\xi.
\]

It is known that the Fourier coefficients (12) of a test function tend rapidly to zero if \( |\beta| \to \infty \). By (13), \( c''_\beta \) tend rapidly to zero as well.

We arrange the multi-indices \( \beta \neq 0 \) into a sequence \( \{\beta_j\}_{j=1}^\infty \) in such a way that the sequence \( \{|\beta_j|\} \) is non-decreasing; then we change the notation writing \( \gamma'_j, c'_j, \ldots \) rather than \( \gamma'_\beta, c'_\beta, \ldots \). Then the above expansion takes the form

\[
\psi = \sum_{j=1}^\infty c'_j \gamma'_j.
\]

Evidently \( |\beta_j| \leq j \leq (2|\beta_j|+1)^d \) (\( \Leftrightarrow j \) does not exceed the number of the indices \( \beta \) with \( |\beta| \leq |\beta_j| \)). Hence any multi-sequence \( \{a_\beta\}_{\beta} \) is moderated (i.e. \( |a_\beta| \leq c|\beta|^m \) for some \( c \) and \( m \)) iff the sequence \( \{a_{\beta_j}\}_{j} \) is moderated. \( \{a_\beta\}_{\beta} \) tends rapidly to zero iff \( \{a_{\beta_j}\}_{j} \) tends rapidly to zero.

If \( \mathcal{U} \subset \mathcal{A}(K_{r+1}) \) is an absolutely convex open neighbourhood of zero, then \( \|\gamma'_j\|_{\mathcal{U}} \) is a moderate sequence (this can be calculated e.g. if \( \|\gamma'_\beta\|_{\mathcal{U}} \) is the norm \( \|\gamma'_\beta\| \) from §2, Remark). So we get the following
Result. If $U \subset A(K_{r+1})$ is an absolutely convex open neighbourhood of zero, then there are $\gamma_j \in A(K_{r+1})$ ($j \in \mathbb{N}$) such that

$$
\sum_{j=1}^{\infty} \|\gamma_j\|_U \leq 1
$$

and any function $\psi \in A(K_r)$ has an expansion

$$
\psi = \sum c_j \gamma_j
$$

with coefficients $c_j$ tending rapidly to zero.

Indeed, choose a moderate sequence $\lambda_j \rightarrow \infty$ such that the functions $\gamma_j := \gamma_j' \lambda_j \in A(K_{r+1})$ fulfill (14) and then put $c_j = c'_j \lambda_j$.

Definition of $R_{kr\omega}$. Let to any $k, r, n \in \mathbb{N}$ and $\omega \in A_0(K_r)$, a neighbourhood of zero $U$ in the space $A(K_{r+1})$ be assigned which is the unit ball for a smooth norm (see §2, Remark), independent of $n$, following Properties of $R$ (§3) with $B = \{\omega\} \subset A(K_{r+1})$ and all $L \leq k$. Assume furthermore that $U$ is as small as $|d_\psi V_{N_L}(\varphi)| \leq 1$ whenever $L = 1, \ldots, k$, $\varphi \in \omega + 2U$, $\psi \in U$, due to the local equicontinuity of the differentials of $C^\infty$ functions, [9, §2, Proposition].

Then the function $\varphi, x \mapsto R_{kr\omega}(\varphi, x)$ is defined on the domain

$$
S_{2-n}(\omega + (U \cap A(K_r))) \times \mathbb{R}^d = S_{2-n}(\omega + U) \cap A_0(K_{r-n}) \times \mathbb{R}^d
$$

as follows.

$$
R_{kr\omega} := \lim_{J \in \mathbb{N}} R_J,
$$

$$
R_J(S_{2-n} \varphi, x) := \int \cdots \int R(S_{2-n}(\varphi + \sum_{j=1}^{J} t_j \gamma_j), x) \prod_{j=1}^{J} \rho_\delta(t_j) \, dt_j,
$$

$\varphi \in (\omega + U) \cap A_0(K_r)$, $\rho_\delta := S_\delta \rho$ (§2, following (2)), $\delta = \delta_{kn} := 2^{-n(k+1)N_k}$.

For the sake of simplicity of the notation, we do not indicate the dependence of $R_J$ on $k, r, n, \omega$.

Properties of $R_{kr\omega}$.

(1°) If $k, r, n \in \mathbb{N}$, $\omega \in A_0(K_r)$, then $R_{kr\omega}$ is well defined on its domain (15). If $x \in \mathbb{R}^d$, $\varphi \in \omega + (U \cap A(K_r))$, then

$$
R_{kr\omega}(S_{2-n} \varphi, x) = \lim_{J \rightarrow \infty} R_J(S_{2-n} \varphi, x)
$$
uniformly with respect to \( \varphi, x \) and

\[
|R_{krn\omega}(S_{2-n} \varphi, x)| \leq C \cdot 2^{nN_1},
\]

with a constant \( C \) not depending on \( n \).

(2°) If \( n \in \mathbb{N}, x \in \mathbb{R}^d, \ell = 0, 1, \ldots, L - 1, L \leq k, \alpha \in \mathbb{N}_0^d, |\alpha| \leq L, \varphi \in \omega + (U \cap A(K_r)), 2^{-nN_1 - 1} > V_{N_1} (\varphi), \psi_1, \ldots, \psi_\ell \in U \cap A(K_r), \) then

\[
(20) \quad \left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R_{krn\omega}(S_{2-n} \varphi, x) \right| \leq C \cdot 2^{nN_1L}
\]

uniformly with respect to \( x, \varphi, \psi_1, \ldots, \psi_\ell \) under the above conditions \((k, r, n \) fixed). Consequently \( \varphi, x \mapsto \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R_{krn\omega}(S_{2-n} \varphi, x) \) is continuous, hence the order of derivatives (under the above conditions) does not matter. Furthermore we have

\[
(21) \quad \left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} (R_{krn\omega}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x)) \right| \leq 2^{nN_1L} \cdot \delta_{kn}.
\]

(3°) \( R_{krn\omega} \) is \( C^\infty \) with respect to the first variable on its domain (15) and there is an absolutely convex open neighbourhood of zero \( \mathcal{V} = V_{krn\omega} \subset A(K_r) \) not depending on \( n \) such that if \( x \in \mathbb{R}^d, n \in \mathbb{N}, L \in \mathbb{N}, \psi_\ell \in \mathcal{V} \ (\ell = 1, \ldots, L) \), \( \varphi \in \omega + U \), then

\[
(22) \quad \left| d^L_{\psi_1, \ldots, \psi_L} R_{krn\omega}(S_{2-n} \varphi, x) \right| \leq b_L \cdot 2^{nN_1} \cdot \delta_{kn}^{-L}
\]

with a constant \( b_L \) depending only on \( L \) and \( \rho \).

**Proof of (17) and (19):** By the definition (16) we have

\[
\partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R_J(S_{2-n} \varphi, x)
= \int \ldots \int \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R\left(S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_j \gamma_j \right), x \right) \prod_{j=1}^{J} \rho_\delta(t_j) \, dt_j
\]

and we have as well

\[
\partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R_J(S_{2-n} \varphi, x)
= \int \ldots \int \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R\left(S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_j \gamma_j \right), x \right) \prod_{j=1}^{J+1} \rho_\delta(t_j) \, dt_j,
\]
because \( \int \rho_\delta(t_{J+1}) \, dt_{J+1} = 1 \). It follows
\[
\left| \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_{J+1}(S_{2-n} \varphi, x) - \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_J(S_{2-n} \varphi, x) \right|
\]
\begin{equation}
= \left| \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \left[ \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J+1} t_j \gamma_j \right) , x \right)
- \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_j \gamma_j \right) , x \right) \right] \cdot \prod_{j=1}^{J+1} \rho_\delta(t_j) \, dt_j \right| .
\end{equation}

Now we want to apply §3, Property of \( R \) (4°) for \( \varepsilon = 2^{-n} \). By hypotheses of §3, (4°), there are two cases. For \( L = 1 \) this gives estimation
\[
\left| R_{J+1}(S_{2-n} \varphi, x) - R_J(S_{2-n} \varphi, x) \right|
\leq \left| \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} 2^{nN_1} \| t_{j+1} \gamma_{j+1} \|_{\mathcal{U}} \cdot \prod_{j=1}^{J+1} \rho_\delta(t_j) \, dt_j \right|
\leq 2^{nN_1} \delta_{kn} \| \gamma_{J+1} \|_{\mathcal{U}} \leq \| \gamma_{J+1} \|_{\mathcal{U}}
\]
(\( \delta \) defined in (16)), so by (14) the limit in (17) is uniform. Thus \( R_{\text{krnw}} \) is well defined. Similarly (19) can be deduced from (23): By the local equicontinuity of \( dV_{N_L}(\varphi) \) noted in the definition of \( R_{\text{krnw}} \), we get using the mean value theorem
\[
\left| V_{N_L} \left( \varphi + \sum_{j=1}^{J} t_j \gamma_j \right) - V_{N_L}(\varphi) \right| \leq \left\| \sum_{j=1}^{J} t_j \gamma_j \right\|_{\mathcal{U}} \leq \delta = 2^{-n(k+1)N_k} \leq 2^{-nN_L-1}.
\]

From the hypothesis in (2°) \( V_{N_L}(\varphi) < 2^{-nN_L-1} \), we obtain \( \left| V_{N_L} \left( \varphi + \sum_{j=1}^{J} t_j \gamma_j \right) \right| \leq 2^{-nN_L} \) that is the hypothesis in §3, Property (4°). Thus, by §3, (4°) (for \( \ell = 0, 1, \ldots, L - 1 \)) we get from (23):
\[
\left| \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_{J+1}(S_{2-n} \varphi, x) - \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_J(S_{2-n} \varphi, x) \right|
\leq 2^{nN_L} \| \psi_1 \|_{\mathcal{U}} \cdots \| \psi_\ell \|_{\mathcal{U}} \cdot \delta_{kn} \| \gamma_{J+1} \|_{\mathcal{U}} .
\]

As above, thanks to (14), the uniform convergence of the limit in (19) and then the equality (19) follows. \( \square \)

**Proof of (18) and (20):** In all cases where the uniform convergence is already proved, we have
\[
\partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_{\text{krnw}}(S_{2-n} \varphi, x) = \lim_{J \to \infty} \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_J(S_{2-n} \varphi, x)
= \lim_{J \to \infty} \int \cdots \int \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_j \gamma_j \right) , x \right) \cdot \prod_{j=1}^{J} \rho_\delta(t_j) \, dt_j .
\]
It was shown while proving (19) that the hypothesis in $(2^o) \ V_{N_L}(\varphi) < 2^{-nN_L-1}$ implies $V_{N_L}(\varphi + \sum_{j=1}^J t_j \gamma_j) \leq 2^{-nN_L}$, and this is the hypothesis in Properties of $R$ ($§3$) allowing us to use $(3)$ and $(4)$ for estimating the term $\partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R\left(S_{2-n}\left(\varphi + \sum_{j=1}^J t_j \gamma_j\right), x\right)$ in the last integral. By §3, Property $(3^o)$ this hypothesis is not needed for proving (18). Thus (18) and (20) follow from the corresponding properties of $R$.

Proof of (21): (25) holds for $J = 0$ as well with $R_J = R$. Adding the inequalities (25), we get

$$\left|\partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R_{J+1}(S_{2-n}\varphi, x) - \partial^\alpha d^\ell_{\psi_1,...,\psi_\ell} R(S_{2-n}\varphi, x)\right|$$

$$\leq 2^{nN_L} \left\|\psi_1\right\|_U \cdots \left\|\psi_\ell\right\|_U \cdot \delta_{kn} \sum_{j=1}^{J+1} \left\|\gamma_j\right\|_U \leq 2^{nN_L} \left\|\psi_1\right\|_U \cdots \left\|\psi_\ell\right\|_U \cdot \delta_{kn}$$

due to (14). Hence the inequality (21) is proved.

Proof of $3^o$: For $L \in \mathbb{N}$ let $\psi_1, \ldots, \psi_L \in \mathcal{A}(K_r)$ be given functions, let

$$(26) \quad \psi_\ell = \sum_{j=1}^\infty c_{\ell j} \gamma_j, \text{ i.e. } S_{2-n}\psi_\ell = \sum_{j=1}^\infty c_{\ell j} S_{2-n}\gamma_j \quad (\ell = 1, \ldots, L)$$

be their expansions by §4, Notation with $\gamma_j$ fulfilling (14). As $\lim_{j \to \infty} c_{\ell j} = 0$ (rapidly), there is an $A > 0$ for which

$$(27) \quad |c_{\ell j}| \leq A \quad (\forall \ell = 1, \ldots, L, \ j \in \mathbb{N}).$$

In the following calculation, $h_1, \ldots, h_L$ are real variables with

$$(28) \quad |h_\ell| < \frac{1 - \delta}{LA}$$

and we have to put $h_1, \ldots, h_L = 0$ to obtain the following equality:

$$d^L_{\psi_1,...,\psi_L} R_{krn\omega}(S_{2-n} \varphi, x) = \frac{\partial^L}{\partial h_1 \cdots \partial h_L} \lim_{J \to \infty} R_J\left(S_{2-n}\left(\varphi + \sum_{\ell=1}^L h_\ell \psi_\ell\right), x\right)$$

$$= \frac{\partial^L}{\partial h_1 \cdots \partial h_L} \lim_{J \to \infty} \int \cdots \int R\left(S_{2-n}\left(\varphi + \sum_{\ell=1}^L h_\ell \psi_\ell + \sum_{j=1}^J t_j \gamma_j\right), x\right) \prod_{j=1}^J \rho_\delta(t_j) \, dt_j.$$
By (26) this is equal to
\[
\frac{\partial L}{\partial h_1 \ldots \partial h_L} \lim_{J \to \infty} \int \cdots \int R(S_{2-n}(\varphi + \sum_{\ell=1}^{L} h_{\ell} \sum_{j=1}^{\infty} c_{\ell j} \gamma_j + \sum_{j=1}^{J} t_j \gamma_j), x) \cdot \prod_{j=1}^{J} \rho_{\delta}(t_j) \, dt_j
\]
(29)
\[
= \frac{\partial L}{\partial h_1 \ldots \partial h_L} \lim_{J \to \infty} \int \cdots \int R(S_{2-n}(\varphi + \sum_{\ell=1}^{L} h_{\ell} \sum_{j=1}^{J} c_{\ell j} \gamma_j + \sum_{j=1}^{J} t_j \gamma_j), x) \cdot \prod_{j=1}^{J} \rho_{\delta}(t_j) \, dt_j
\]

because by §3, Property (4°), (27) and (28), the difference of both expressions after \( \lim_{J \to \infty} \) is estimated by
\[
2^{nN_1} \sum_{\ell=1}^{L} |h_{\ell}| \sum_{j=J+1}^{\infty} |c_{\ell j}| \cdot \|\gamma_j\|_{\mathcal{U}} \leq 2^{nN_1}(1 - \delta) \sum_{j=J+1}^{\infty} \|\gamma_j\|_{\mathcal{U}}.
\]

This tends to zero thanks to (14), only we have to verify the hypothesis in §3, (4°) that \( \varphi + \sum_{\ell=1}^{L} h_{\ell} \sum_{j=1}^{\infty} c_{\ell j} \gamma_j + \sum_{j=1}^{J} t_j \gamma_j \) and \( \sum_{\ell=1}^{L} h_{\ell} \sum_{j=1}^{J} c_{\ell j} \gamma_j + \sum_{j=1}^{J} t_j \gamma_j \) are elements of \( \omega + 2u \). Indeed, \( \varphi \in \omega + u \) and for the other terms we have by (28), (27) and (14)
\[
\left\| \sum_{\ell=1}^{L} h_{\ell} \sum_{j=1}^{\infty} c_{\ell j} \gamma_j + \sum_{j=1}^{J} t_j \gamma_j \right\|_{\mathcal{U}} \leq \sum_{\ell=1}^{L} \frac{1 - \delta}{LA} \sum_{j=1}^{\infty} A\|\gamma_j\|_{\mathcal{U}} + \sum_{j=1}^{J} \delta\|\gamma_j\|_{\mathcal{U}} \leq 1.
\]

Thus (29) is verified. After a substitution in (29) (and putting \( h_1, \ldots, h_L = 0 \)) we get
\[
\left(30\right) \quad d_{\psi_1, \ldots, \psi_L}^{L} R_{\text{krn} \omega}(S_{2-n} \varphi, x) = \frac{\partial L}{\partial h_1 \ldots \partial h_L} \lim_{J \to \infty} \int \cdots \int R(S_{2-n}(\varphi + \sum_{j=1}^{J} t_j \gamma_j), x) \cdot \prod_{j=1}^{J} \rho_{\delta}(t_j - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) \, dt_j
\]
\[
= \lim_{J \to \infty} \int \cdots \int R(S_{2-n}(\varphi + \sum_{j=1}^{J} t_j \gamma_j), x) \cdot \frac{\partial L}{\partial h_1 \ldots \partial h_L} \prod_{j=1}^{J} \rho_{\delta}(t_j - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) \, dt_j
\]
provided the last limit is uniform with respect to $h_{\ell}$, $|h_{\ell}| < \frac{1-\delta}{L_{A}}$ ($\ell = 1, \ldots, L$).

Now we are going to prove it. Let us denote the last integral by $I_{J}$. Using the Leibniz rule for the derivation of a product:

$$\frac{\partial}{\partial h_{\ell}} \prod_{j=1}^{J} \rho_{\delta}(t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) = \sum_{j_{\ell}=1}^{J} \left( -c_{\ell,j_{\ell}} \frac{\partial}{\partial t_{j_{\ell}}} \right) \prod_{j=1}^{J} \rho_{\delta}(t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}),$$

we obtain

$$I_{J} = \int \cdots \int R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_{j} \gamma_{j} \right), x \right) \cdot \sum_{j_{1}, \ldots, j_{L}=1}^{J} \left( \prod_{\ell=1}^{L} -c_{\ell,j_{\ell}} \prod_{j=1}^{J} \frac{\partial}{\partial t_{j}} \right) \prod_{j=1}^{J} \rho_{\delta}(t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) \, dt_{j}.$$  

Using the Kronecker delta ($\delta_{j}^{j'}$ is the truth value of the statement $j = j'$) we can write

$$I_{J} = \int \cdots \int R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_{j} \gamma_{j} \right), x \right) \cdot \sum_{j_{1}, \ldots, j_{L}=1}^{J+1} \left( \prod_{\ell=1}^{L} -c_{\ell,j_{\ell}} \prod_{j=1}^{J+1} \frac{\partial}{\partial t_{j}} \right) \prod_{j=1}^{J} \delta_{j}^{j_{\ell}} \rho_{\delta}(t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) \, dt_{j}.$$  

and we have as well

$$I_{J} = \int \cdots \int R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_{j} \gamma_{j} \right), x \right) \cdot \sum_{j_{1}, \ldots, j_{L}=1}^{J+1} \left( \prod_{\ell=1}^{L} -c_{\ell,j_{\ell}} \prod_{j=1}^{J+1} \frac{\partial}{\partial t_{j}} \right) \prod_{j=1}^{J} \delta_{j}^{j_{\ell}} \rho_{\delta}(t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) \, dt_{j}.$$  

Indeed, if some $j_{\ell} = J+1$, then the term $\rho_{\delta}(t_{J+1} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell,J+1})$ is differentiated and so its integral is equal to 0; else its integral is is equal to 1. It follows

$$I_{J+1} - I_{J} = \int \cdots \int \left[ R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J+1} t_{j} \gamma_{j} \right), x \right) - R \left( S_{2-n} \left( \varphi + \sum_{j=1}^{J} t_{j} \gamma_{j} \right), x \right) \right] \cdot \sum_{j_{1}, \ldots, j_{L}=1}^{J+1} \left( \prod_{\ell=1}^{L} -c_{\ell,j_{\ell}} \prod_{j=1}^{J+1} \frac{\partial}{\partial t_{j}} \right) \prod_{j=1}^{J+1} \delta_{j}^{j_{\ell}} \rho_{\delta}(t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j}) \, dt_{j}.$$
By (27) and (28) we have \(| \sum_{\ell=1}^{L} h_{\ell} c_{\ell j} | \leq 1 - \delta, \) so \(| t_{j} | \leq 1 \) or \( \rho_{\delta} \left( t_{j} - \sum_{\ell=1}^{L} h_{\ell} c_{\ell j} \right) = 0, \) and we can apply §3, Property of \( R (4^o) \). We get

\[
| I_{J+1} - I_{J} | \leq 2^{nN_{1}} \| \gamma_{J+1} \|_{U} \sum_{j_{1}, \ldots, j_{L}=1}^{J+1} \prod_{\ell=1}^{L} | c_{\ell, j_{\ell}} | \prod_{j=1}^{J+1} \| \delta \sum_{\ell=1}^{L} \delta_{j}^{\ell} \rho_{\delta} \|_{\mathcal{X}^{1}}.
\]

It is \( \| \rho_{\delta} \|_{\mathcal{X}^{1}} = 1 \) (§2, following (2)) and, in the last product, for given \( j_{1}, \ldots, j_{L} \) there are at most \( L \) indices \( j \) for which \( \rho_{\delta} \) is differentiated. Thus this product can be estimated

\[
\prod_{j=1}^{J+1} \| \delta \sum_{\ell=1}^{L} \delta_{j}^{\ell} \rho_{\delta} \|_{\mathcal{X}^{1}} = \prod_{j=1}^{J+1} \delta_{\sum_{\ell=1}^{L} \delta_{j}^{\ell} \rho_{\delta}} \left( \max_{0 \leq \ell \leq L} \| \partial_{\ell} \rho \|_{\mathcal{X}^{1}} \right)^{L} \leq \delta^{-L} \cdot b_{L}
\]

with a constant \( b_{L} \) depending only on \( \rho \) and \( L \). It follows

\[
| I_{J+1} - I_{J} | \leq 2^{nN_{1}} \delta^{-L} b_{L} \| \gamma_{J+1} \|_{U} \sum_{j_{1}, \ldots, j_{L}=1}^{J+1} \prod_{\ell=1}^{L} | c_{\ell, j_{\ell}} |
\]

(31)

\[
= 2^{nN_{1}} \delta^{-L} b_{L} \| \gamma_{J+1} \|_{U} \prod_{\ell=1}^{L} \sum_{j=1}^{J+1} | c_{\ell j} | \leq 2^{nN_{1}} \delta^{-L} b_{L} \| \gamma_{J+1} \|_{U} \prod_{\ell=1}^{L} \sum_{j=1}^{\infty} | c_{\ell j} |
\]

\[
= 2^{nN_{1}} c^{L} \delta^{-L} b_{L} \| \gamma_{J+1} \|_{U}
\]

where the constant \( c = \max_{\ell=1, \ldots, L} \sum_{j=1}^{\infty} | c_{\ell j} | \) depends only on \( \psi_{1}, \ldots, \psi_{L} \) and \( r \).

Now the uniform convergence of (30) can be deduced from (14). The smoothness is verified; it remains to deduce the estimations.

The inequality (31) holds for \( J = 0 \) as well with \( I_{0} = 0 \). Adding these inequalities we get

\[
| I_{J} | \leq 2^{nN_{1}} c^{L} \delta^{-L} b_{L} \sum_{j=1}^{J} \| \gamma_{j} \|_{U} \leq 2^{nN_{1}} c^{L} \delta^{-L} b_{L}.
\]

This is an estimation for the last integral in (30). Hence

\[
| d_{\psi_{1}, \ldots, \psi_{L}}^{L} R_{krn\omega} (S_{2-n} \varphi, x) | \leq b_{L} c^{L} 2^{nN_{1}} \cdot \delta^{-L}.
\]
$b_L$ depends only on $\rho$ and $L$, $c$ is the constant in (31), $c = 1$ if $\psi_1, \ldots, \psi_L$ belong to

$$\mathcal{V} = \{ \psi \in \mathcal{A}(K_r); \psi = \sum c_j \gamma_j, \sum |c_j| \leq 1 \}.$$  

By §4, Result $\gamma_j$ depends on $\mathcal{U}$ not depending on $n$. It remains to prove that $\mathcal{V}$ is a neighbourhood of zero in $\mathcal{A}(K_r)$. It is known that the Fourier coefficients of functions $\psi$ running over a bounded set in $\mathcal{A}(K_r)$ tend uniformly rapidly to zero. Evidently the same holds for the coefficients $c_j$ in §4, Notation. Hence any bounded set is absorbed by $\mathcal{V}$. In a metric vector space such sets $\mathcal{V}$ are neighbourhoods of zero. □

§5. Partition of unity. The space $\mathcal{D}$ has the property of smooth partition of unity expressed by the following

**Theorem.** For $B \subseteq \mathbb{R}^d$, let $\{\omega_s + U_s\}_{s \in S}$ be an open covering of the space $\mathcal{D}(B)$, where $S$ is an arbitrary set of indices, $\omega_s \in \mathcal{D}(B)$ (\(\forall s \in S\)), $U_s$ are open neighbourhoods of zero in $\mathcal{D}(B)$. Then there is a locally finite smooth (i.e. $\mathcal{C}^\infty$) partition of unity on $\mathcal{D}(B)$

$$1 = \sum_{m=1}^{\infty} \Phi_m$$

subordinated to this covering.

This means:

1° The functions $\Phi_m : \mathcal{D}(B) \rightarrow [0, 1]$, fulfilling this equality, are $\mathcal{C}^\infty$ and

$$\forall m \quad \exists s \in S : \text{supp} \Phi_m \subset \omega_s + U_s.$$  

2° For every $\omega \in \mathcal{D}(B)$ there is an absolutely convex open neighbourhood of zero $U \subset \mathcal{D}(B)$ such that $\omega + U$ meets only a finite number of supports of functions $\Phi_m$.

For the proof, we refer to [13, (5.3.8)], where a more general theorem is proved concerning several cathegories of smoothness, not only $\mathcal{C}^\infty$. Hypotheses: $\mathcal{D}$ is a Lindelöf locally convex space and there are sufficiently many $\mathcal{C}^\infty$ functions on $\mathcal{D}$ so that they generate the original topology on $\mathcal{D}$ This is fulfilled, see §2, Remark.

**Corollary.** For $B \subseteq \mathbb{R}^d$, let $\{\omega_s + U_s\}_{s \in S}$ be an open covering of the space $\mathcal{A}_0(B)$, where (\(\forall s \in S\)) $\omega_s \in \mathcal{A}_0(B)$ and $U_s$ is an open neighbourhood of zero in $\mathcal{A}(B)$. Then there is a locally finite smooth partition of unity on $\mathcal{A}_0(B)$

$$1 = \sum_{m=1}^{\infty} \Phi_m$$

subordinated to this covering.
PROOF: We can write $U_s = \tilde{U}_s \cap A(B)$ where $\tilde{U}_s$ are neighbourhoods of zero in the space $\mathcal{D}(B)$. Then we apply Theorem to the covering \{(ω_s + U_s)\}_s \cup \{\mathcal{D}(B) \setminus A_0(B)\} of $\mathcal{D}(B)$.

\section*{6. Notation.} Let us have chosen $k, r \in \mathbb{N}$. Then, for every $\omega \in A_0(K_r)$, we have a neighbourhood of zero $U_\omega \subset A(K_r)$ (independent of $n$) such that $\forall n \in \mathbb{N}$ the function $R_{krn\omega}$ is defined by §4, on $S_{2-n}(\omega + U_\omega) \times \mathbb{R}^d$. Thus we have a covering of $A_0(K_r)$ with the sets $\omega + U_\omega$. We choose a partition of unity $1 = \sum_{m=1}^\infty \Phi_m$ on $A_0(K_r)$ by the above corollary. For every $m$, we choose a test function $\omega_m$ for which supp $\Phi_m \subset \omega_m + U_\omega_m$; we will use the notation $U_m$ rather than $U_\omega_m$.

**Definition of $R_{krn}$**. With the above notation, for $\varphi \in A_0(K_r)$ (so $S_{2-n} \varphi \in A_0(K_r-n)$) we define

\begin{equation}
R_{krn}(S_{2-n} \varphi, x) := \sum_{m=1}^\infty \Phi_m(\varphi) \cdot R_{krn\omega_m}(S_{2-n} \varphi, x).
\end{equation}

If $\varphi$ does not belong to supp $\Phi_m$, the term of this sum is considered to be zero even if $R_{krn\omega_m}(S_{2-n} \varphi, x)$ is not defined.

**Properties of $R_{krn}$**. (1) For every $k, r, n \in \mathbb{N}$, the function $R_{krn}$ is defined on $A_0(K_{r-n}) \times \mathbb{R}^d$.

Moreover, for every $\omega \in A_0(K_r)$ and $L \in \mathbb{N}_0$, there exist an absolutely convex open neighbourhood of zero $U \subset A(K_r)$ and a constant $C > 0$, both independent of $n$, such that for every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ the following hold.

(2) If $1 \leq L \leq k$, $\ell = 0, 1, \ldots, L - 1$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq L$, $\varphi \in \omega + U$, $2^{-nN_L-1} > V_{N_L}(\varphi)$, $\psi_1, \ldots, \psi_\ell \in U$, then $\varphi, x \mapsto d_{\psi_1, \ldots, \psi_\ell}^\ell \partial^\alpha R_{krn}(S_{2-n} \varphi, x)$ is continuous and

\begin{align}
&\left| d_{\psi_1, \ldots, \psi_\ell}^\ell \partial^\alpha R_{krn}(S_{2-n} \varphi, x) \right| \leq C \cdot 2^{nN_L}, \\
&\left| \partial^\alpha d_{\psi_1, \ldots, \psi_\ell}^\ell \left( R_{krn}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x) \right) \right| \leq C 2^{nN_L} \cdot \delta_{kn}
\end{align}

(δ = δ_{kn} by (16)).

(3) $R_{krn}$ is $C^\infty$ with respect to the first variable on its domain $A_0(K_{r-n})$ and if $\varphi \in \omega + U$ and $\psi_\ell \in U$ ($\ell = 1, \ldots, L$), then

\begin{equation}
\left| d_{\psi_1, \ldots, \psi_L}^L R_{krn}(S_{2-n} \varphi, x) \right| \leq C 2^{nN_1} \cdot \delta_{kn}^{-L}.
\end{equation}

**Proof of (2)**: If $\omega \in A_0(K_r)$, we choose a neighbourhood of zero $U \subset A(K_r)$ such that $\omega + U$ meets only a finite number of supports of the functions $\Phi_m$ (by 5.2). So $U$ can be chosen as small as ($\forall m$)

\begin{equation}
\text{either } (\omega + U) \cap \text{supp } \Phi_m = \emptyset \text{ or } (\omega + U) \subset \omega_m + U_m.
\end{equation}

\section*{4. Remarks.}
Furthermore, thanks to [9, §2, Proposition], let \( U \) be as small as

\[
(36) \quad \left| d_{\psi_1, \ldots, \psi_\ell}^\ell \Phi_m(\varphi) \right| \leq 1
\]

whenever \( 1 \leq \ell \leq k - 1, m \in \mathbb{N}, \varphi \in \omega + U, \psi_1, \ldots, \psi_\ell \in U; \) for \( \ell = 0 \) this is fulfilled, too. Now we use the following Leibniz rule for derivation of a product of two functions: if \( F_1, F_2 \) are two smooth functions on (a part of) a locally convex space, then

\[
d^\ell (F_1(\varphi)F_2(\varphi))[\psi_1, \ldots, \psi_\ell] = \sum_{I_1 \cup I_2 = \{1, \ldots, \ell\} \text{disjoint}} d^{#I_1}_{\psi_{I_1}} F_1(\varphi) \cdot d^{#I_2}_{\psi_{I_2}} F_2(\varphi),
\]

where, for \( I = \{i_1, \ldots, i_{#I}\} \subset \{1, \ldots, \ell\}, \psi_I \) denotes the finite sequence \( \psi_{i_1}, \ldots, \psi_{i_{#I}} \) and the summation is extended over all ordered decompositions of multi-index \( (1, \ldots, \ell) \) in two disjoint multi-indices \( I_1, I_2 \) that are written in the increasing order. Unlike the chain rule (theorem on the derivative of the composition), here the multi-index \( I_1 \) or \( I_2 \) can be empty. The proof of the Leibniz formula can be deduced easily from the chain rule if the outer function is \( F(u, v) = uv, u = F_1, v = F_2 \).

We apply the Leibniz rule for differentiating the product to the defining formula (32):

\[
d_{\psi_1, \ldots, \psi_\ell}^\ell \partial^\alpha R_{\kern1.5pt\ker\omega m}(S_{2-n} \varphi, x)
= \sum_{m=1}^{\infty} \sum_{I_1 \cup I_2 = \{1, \ldots, \ell\} \text{disjoint}} d^{#I_1}_{\psi_{I_1}} \Phi_m(\varphi) \cdot d^{#I_2}_{\psi_{I_2}} \partial^\alpha R_{\kern1.5pt\ker\omega m}(S_{2-n} \varphi, x)
\]

and, by (35) as \( \varphi \in \omega + U \), the first sum is extended only over a finite number of \( m \) for which \( (\omega + U) \subset \omega_m + U_m \). Then \( (\omega + U) - (\omega + U) \subset (\omega_m + U_m) - (\omega_m + U_m) \); for absolutely convex sets it follows \( U \subset \mathcal{U}_m \). Hence the functions \( R_{\kern1.5pt\ker\omega m}, \) fulfilling §4, Properties on \( \omega_m + U_m \) fulfil it the more on \( \omega + U \). By the above Leibniz formula, we estimate \( d_{\psi_1, \ldots, \psi_\ell}^\ell \partial^\alpha R_{\kern1.5pt\ker\omega m}(S_{2-n} \varphi, x) \) using (36) for estimating the term \( d^{#I_1}_{\psi_{I_1}} \Phi_m(\varphi) \) and using §4, Properties of \( R_{\kern1.5pt\ker\omega} \) for estimating the term \( d^{#I_2}_{\psi_{I_2}} \partial^\alpha R_{\kern1.5pt\ker\omega m}(S_{2-n} \varphi, x) \). So we deduce (33) from the corresponding inequality (20) in Properties of \( R_{\kern1.5pt\ker\omega} \). The constant \( C \) in (33) depends only on the constants assigned by §4, Properties to the functions \( R_{\kern1.5pt\ker\omega m} \) and on the used finite set of terms of the sum \( \sum_m \), so it is independent of \( n \).
For estimating \( \partial^\alpha d^\ell_{\psi_1,\ldots,\psi_\ell} (R_{krn}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x)) \), we apply the Leibniz formula to the product \( \sum_{m=1}^{\infty} \Phi_m(\varphi) \cdot (R_{krn\omega_m}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x)) \) and proceed similarly.

**Proof of (3°):** Unlike in §4, Properties of \( R_{krn\omega} \), here the neighbourhood \( \mathcal{U} \) depends on \( L \). We chose \( \mathcal{U} \) as small as (36) hold whenever \( 1 \leq \ell \leq L \), \( m \in \mathbb{N} \), \( \varphi \in \omega + \mathcal{U} \), \( \psi_1, \ldots, \psi_\ell \in \mathcal{U} \). Denote by \( \mathcal{V}_m \) the neighbourhood \( \mathcal{V} \) defined by §4 (3°) for the function \( R_{krn\omega_m} \) and chose furthermore \( \mathcal{U} \) such that we have instead of (35):

\[
\text{either } (\omega + \mathcal{U}) \cap \text{supp } \Phi_m = \emptyset \quad \text{or } (\omega + \mathcal{U}) \subset \omega_m + (\mathcal{V}_m \cap \mathcal{U}_m).
\]

Then we follow the above proof and deduce (3°) from the corresponding properties of \( R_{krn\omega} \), §4: (18) and the item (3°). \( \square \)

**§7. Notations.** Choose test functions \( \psi_\alpha \in \mathscr{D}(K_r \setminus K_{r-1}) \) \((\alpha \in \mathbb{N}_0^d, 0 \leq |\alpha| \leq N_k)\), fulfilling (like in [8, (22), (23)])

\[
\int \psi_\alpha(\xi) \cdot \xi^\alpha \, d\xi = 1
\]

\[
\int \psi_\alpha(\xi) \cdot \xi^\beta \, d\xi = 0 \quad \text{for } \beta \neq \alpha, 0 \leq |\beta| \leq N_k.
\]

Let \( \Lambda_r : \mathscr{D}(K_{r+1}) \to \mathscr{D}(K_r) \) be a continuous (hence smooth) linear mapping defined (see end of §2, \( \nabla_r \) means \( \nabla^\otimes d \)):

\[
\Lambda_r \varphi := \varphi \cdot \nabla_{r-1} + \sum_{0 \leq |\alpha| \leq N_k} c_\alpha \psi_\alpha
\]

with such constants \( c_\alpha \) depending on \( \varphi \) that

\[
\forall \beta \in \mathbb{N}_0^d, 0 \leq |\beta| \leq N_k : \quad \int \Lambda_r \varphi(\xi) \xi^\beta \, d\xi = \int \varphi(\xi) \xi^\beta \, d\xi.
\]

This means \( \int \varphi(\xi) \xi^\beta \, d\xi = \int \nabla_{r-1}(\xi) \varphi(\xi) \xi^\beta \, d\xi + c_\beta \), hence \( c_\beta \) are well determined; \( \Lambda_r \) maps \( A_0(K_{r+1}) \) into \( A_0(K_r) \) and \( A(K_{r+1}) \) into \( A(K_r) \). \( \Lambda_r \) is identical on \( \mathscr{D}(K_{r-1}) \); for \( \varphi \in A_0(K_{r+1}) \) and \( N \leq N_k \) we have \( V_N(\Lambda_r \varphi) = V_N(\varphi) \).

**Definition of \( R'_{krn} \).** Let \( k, n \in \mathbb{N} \) be given. For \( r \in \mathbb{N} \), we define the functions \( R'_{krn} \) on \( A_0(K_{r-n}) \times \mathbb{R}^d \) by induction as follows. \( R'_{k1n} = R_{k1n} \). If \( R'_{krn} \) is already defined on \( A_0(K_{r-n}) \times \mathbb{R}^d \), we define

\[
R'_{k,r+1,n}(S_{2-n} \varphi, x) := R_{k,r+1,n}(S_{2-n} \varphi, x) + R'_{krn}(S_{2-n}(\Lambda_r \varphi), x) - R_{k,r+1,n}(S_{2-n}(\Lambda_r \varphi), x)
\]

for \( \varphi \in A_0(K_{r+1}), x \in \mathbb{R}^d \).
Properties of $R'_{krn}$: (1°) For every $k, r, n \in \mathbb{N}$ the function $R'_{krn}$ is defined on $\mathcal{A}_0(K_{r-n}) \times \mathbb{R}^d$ and for $\varphi \in \mathcal{A}_0(K_{r-n-1})$ it is $R'_{k,r,n}(\varphi, x) = R'_{k,r+1,n}(\varphi, x)$.

Moreover, for every $\omega' \in \mathcal{A}_0(K_r)$ and $L \in \mathbb{N}_0$ there is an absolutely convex open neighbourhood of zero $\mathcal{U}' \subset \mathcal{A}(K_r)$ and a constant $C'_{kr} > 0$, both independent of $n$, such that for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ the following hold.

(2°) If $1 \leq L \leq k, \ell = 0, 1, \ldots, L - 1, \alpha \in \mathbb{N}_0^d$, $|\alpha| \leq L$, $\varphi' \in \omega' + \mathcal{U}'$, $2^{-nN_L-1} > V_{N_L}(\varphi')$, $\psi'_1, \ldots, \psi'_\ell \in \mathcal{U}'$, then $\varphi', x \mapsto d_{\psi'_1, \ldots, \psi'_\ell}^\ell R'_{krn}(S_{2-n}\varphi', x)$ is continuous and

\begin{align*}
(38) \quad |d_{\psi'_1, \ldots, \psi'_\ell}^\ell \partial^\alpha R'_{krn}(S_{2-n}\varphi', x)| & \leq C'_{kr} 2^{nN_L}, \\
(39) \quad |\partial^\alpha d_{\psi'_1, \ldots, \psi'_\ell}^\ell (R'_{krn}(S_{2-n}\varphi', x) - R(S_{2-n}\varphi', x))| & \leq C'_{kr} 2^{nN_L} \cdot \delta_{kn}.
\end{align*}

(3°) $R'_{krn}$ is $\mathcal{C}^\infty$ with respect to the first variable on its domain $\mathcal{A}_0(K_{r-n})$ and $\forall \varphi' \in \omega' + \mathcal{U}'$, $\psi'_\ell \in \mathcal{U}'$ ($\ell = 1, \ldots, L$), it is

$$
|d_{\psi'_1, \ldots, \psi'_L} L R'_{krn}(S_{2-n}\varphi', x)| \leq C'_{kr} 2^{nN_1} \cdot \delta_{kn}^{-L}.
$$

Proof of (38): (1°) follows easily from the fact that $\Lambda_r$ is identical on $\mathcal{A}_0(K_{r-1})$. The other properties will be proved by induction. For $r = 1$ this is affirmed by §6, Properties of $R_{krn}$. Assuming that (38) is satisfied for uncertain $r$, we have to prove it for $r + 1$. So we have to prove that every one of the three terms on the right-hand side of the defining equality (37) satisfies (38). This is clear for $R_{k,r+1,n}(S_{2-n}\varphi, x)$ due to Properties of $R_{krn}$, (33). We are going to prove it for $R'_{krn}(S_{2-n}(\Lambda_r\varphi), x)$. Let, by the hypothesis, $\omega \in \mathcal{A}_0(K_{r+1})$. Then $\omega' := \Lambda_r(\omega) \in \mathcal{A}_0(K_r)$ and by the induction assumption we have $\mathcal{U}' \subset \mathcal{A}(K_r)$ and $C'_{kr} > 0$, both independent of $n$, fulfilling (2°). Now, $\mathcal{U} := \Lambda_r^{-1}\mathcal{U}'$ is a neighbourhood of zero in $\mathcal{A}(K_{r+1})$. For $\psi_j \in \mathcal{U}$ ($j = 1, \ldots, \ell$) and $\varphi \in \omega + \mathcal{U}$ we have $\psi'_j := \Lambda_r \psi_j \in \mathcal{U}'$, $\varphi' := \Lambda_r \varphi \in \omega' + \mathcal{U}'$, hence (chain rule with the inner function $\Lambda_r$ linear)

$$
|d_{\psi'_1, \ldots, \psi'_\ell}^\ell \partial^\alpha R'_{krn}(S_{2-n}(\Lambda_r \varphi), x)| = |d_{\psi'_1, \ldots, \psi'_\ell}^\ell \partial^\alpha R'_{krn}(S_{2-n}\varphi', x)| \leq C'_{kr} 2^{nN_L}.
$$

Exactly by the same way we deduce the same estimation for the last term in (37), i.e.

$$
|d_{\psi'_1, \ldots, \psi'_\ell}^\ell \partial^\alpha R_{k,r+1,n}(S_{2-n}(\Lambda_r \varphi), x)| \leq C 2^{nN_L},
$$

only we have to start with (33) instead of the induction assumption. Thus (38) is proved by induction. $\square$
Proof of (39): Taking (39) as the induction assumption, we get by the recurrent definition (37) of $R'_{kn}$:

$$\left| d^\ell_{\psi_1,\ldots,\psi_\ell} \partial^\alpha (R'_{k,r+1,n}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x)) \right|$$

$$\leq \left| d^\ell_{\psi_1,\ldots,\psi_\ell} \partial^\alpha (R_{k,r+1,n}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x)) \right|$$

$$+ \left| d^\ell_{\psi_1,\ldots,\psi_\ell} \partial^\alpha [R'_{k,r,n}(S_{2-n}(\Lambda_r \varphi), x) - R(S_{2-n}(\Lambda_r \varphi), x)] \right|$$

As above, we estimate every one of these three terms using (34) and the induction assumption. □

Proof of (3°): by §6 (3°) can be the same as the proof of (38). □

§8. Definition of $R'_{kn}$. We define

$$R'_{kn}(S_{2-n} \varphi, x) = \lim_{r \to \infty} R'_{kn}(S_{2-n} \varphi, x)$$

for $\varphi \in A_0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Every $\varphi$ belongs to $A_0(K_{r-1})$ for some $r \in \mathbb{N}$; up from this $r$ the sequence $\{R'_{kn}\}$ is constant thanks to §7, Property 1°, so it is $R'_{kn}(S_{2-n} \varphi, x) = R'_{kn}(S_{2-n} \varphi, x)$.

Properties of $R'_{kn}$, (1°). For every $k, n \in \mathbb{N}$ the function $R'_{kn}$ is defined on $A_0(\mathbb{R}^d) \times \mathbb{R}^d$.

Moreover, if $B \subseteq \mathbb{R}^d$, $\mathcal{B}$ a bounded set $\subset A_0(B)$ and $L \in \mathbb{N}_0$, then there is an absolutely convex open neighbourhood of zero $U' = U'_{kn} \subset A(B)$ and a constant $C_k' > 0$, both independent of $n$, such that for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ the following hold.

(2°) If $1 \leq L \leq k, \ell = 0, 1, \ldots, L - 1, \alpha \in \mathbb{N}_0^d$, $|\alpha| \leq L$, $\varphi \in \mathcal{B}$, $2^{-nN_L-1} > V_N(\varphi)$, $\psi_1, \ldots, \psi_\ell \in U'$, then $\varphi, x \mapsto d^\ell_{\psi_1,\ldots,\psi_\ell} \partial^\alpha R'_{kn}(S_{2-n} \varphi, x)$ is continuous and

$$\left| d^\ell_{\psi_1,\ldots,\psi_\ell} \partial^\alpha R'_{kn}(S_{2-n} \varphi, x) \right| \leq C_k' 2^{nN_L},$$

$$\left| \partial^\alpha d^\ell_{\psi_1,\ldots,\psi_\ell} (R'_{kn}(S_{2-n} \varphi, x) - R(S_{2-n} \varphi, x)) \right| \leq C_k' 2^{nN_L} \cdot \delta_{kn}.$$  

(3°) $R'_{kn}$ is $\mathcal{C}^\infty$ with respect to the first variable and if $\varphi \in \mathcal{B}$, $\psi_\ell \in U'$ ($\ell = 1, \ldots, L$), then

$$\left| d^L_{\psi_1,\ldots,\psi_L} R'_{kn}(S_{2-n} \varphi, x) \right| \leq C_k' 2^{nN_1} \cdot \delta_{kn}^{-L}.$$
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PROOF: We have $R'_{kn}(S_{2-n}\varphi, x) = R'_{kn}(S_{2-n}\varphi, x)$ for $\varphi \in \mathcal{A}_0(K_{r-1})$ so by §7, Property (3°), $R'_{kn}$ is smooth with respect to the first variable on every $\mathcal{A}_0(K_{r-1})$

As smoothness depends only on the behaviour of $R'_{kn}$ on bounded sets, $R'_{kn}$ is smooth on $\mathcal{A}_0(\mathbb{R}^d)$. For proving the estimations, we can assume without loss of generality that $B = K_{r-1}$ for some $r \in \mathbb{N}$. It is known that the bounded sets in $\mathcal{B}$ are relatively compact; thus, for a given $L$, the set $\mathcal{B}$ can be covered with a finite number of sets $\omega_m' + \mathcal{U}_m$ where $\mathcal{U}_m$ is assigned to $\omega_m'$ by §7, Properties of $R'_{kn}$. Putting $\mathcal{U}' = \bigcap \mathcal{U}_m'$, we get the properties of $R'_{kn}$ from Properties of $R'_{kn}$. □

§9. Up to now, we have constructed functions that were $C^\infty$ with respect to the first variable. Now we are going to regularize the function $R'_{kn}$ by convolution with respect to the second variable to obtain a simultaneously $C^\infty$ function.

Notation. The function $\rho_\delta$ is introduced in §2, Notation. If $k, n$ are chosen, we have still $\delta = \delta_{kn} = 2^{-n(k+1)N_k}$. Denote furthermore

$$\rho^{\otimes d}(x) := \rho(x_1) \ldots \rho(x_d), \quad \rho_\delta^{\otimes d}(x) := \rho_\delta(x_1) \ldots \rho_\delta(x_d)$$

for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Definition. We define a function $\tilde{R}_{kn}$ on $\mathcal{A}_0(\mathbb{R}^d) \times \mathbb{R}^d$ by convolution as follows.

$$\tilde{R}_{kn}(\varphi, x) := R'_{kn}(\varphi, x) * \rho_\delta^{\otimes d}(x) = \int R'_{kn}(\varphi, y) \rho_\delta^{\otimes d}(x - y) \, dy.$$ 

Properties of $\tilde{R}_{kn}$. For every $k, n \in \mathbb{N}$, $\tilde{R}_{kn}$ is a $C^\infty$ function on $\mathcal{A}_0(\mathbb{R}^d) \times \mathbb{R}^d$. That is: $\tilde{R}_{kn} \in \mathcal{E}(\mathbb{R}^d)$. Moreover, if $B \in \mathbb{R}^d$, $\mathcal{B}$ a bounded set $\subset \mathcal{A}_0(B)$ and $L \in \mathbb{N}$, then there is an absolutely convex open neighbourhood of zero $\mathcal{U} = \mathcal{U}_k \subset \mathcal{A}(B)$ and a constant $\tilde{C}_k > 0$, both independent of $n$, such that for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, $\ell = 0, 1, \ldots, L, \psi_1, \ldots, \psi_\ell \in \mathcal{U}$, $\varphi \in \mathcal{B}$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq L$, we have

$$\left| d_{\psi_1, \ldots, \psi_\ell}^\ell \partial^\alpha \tilde{R}_{kn}(S_{2-n}\varphi, x) \right| \leq \tilde{C}_k 2^{nN_L} \cdot \delta_{kn}^{-2L}. \tag{43}$$

If in addition $L \leq k$, $2^{-nN_L} > V_{N_L}(\varphi)$ and $\ell \leq L - 1$, then

$$\left| d_{\psi_1, \ldots, \psi_\ell}^\ell \partial^\alpha \tilde{R}_{kn}(S_{2-n}\varphi, x) \right| \leq \tilde{C}_k 2^{nN_L}. \tag{44}$$

If in addition $|\alpha| \leq L - 1$, then

$$\left| \partial^\alpha d_{\psi_1, \ldots, \psi_\ell}^\ell \left( \tilde{R}_{kn}(S_{2-n}\varphi, x) - R(S_{2-n}\varphi, x) \right) \right| \leq \tilde{C}_k 2^{nN_L} \cdot \delta_{kn}. \tag{45}$$
Proof of (43):

\[
\left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} \tilde{R}_{kn}(S_{2-n} \varphi, x) \right| = \left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} (R'_{kn}(S_{2-n} \varphi, x) * \rho^\otimes_d(x)) \right|
\]

\[
= \left| (d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, x)) * \partial^\alpha \rho^\otimes_d(x) \right| .
\]

By (42), this is

\[
\leq C'_k 2^{nN_1} \cdot \delta^{-L} \| \partial^\alpha \rho^\otimes_d \|_{\varphi^1} = C'_k 2^{nN_1} \cdot \delta^{-L} \| \partial^\alpha \rho^\otimes_d \|_{\varphi^1} .
\]

As \( |\alpha| \leq L \), we obtain (43).

We see that for given \( k, n \) the derivatives \( d^\ell_{\psi_1, \ldots, \psi_\ell} \partial^\alpha \tilde{R}_{kn}(S_{2-n} \varphi, x) \) are equi-bounded if \( \varphi \in \mathcal{B}, x \in \mathbb{R}^d \), hence they are continuous on \( \mathcal{B} \times \mathbb{R}^d \) for any bounded \( \mathcal{B} \subset A_0(B) \times \mathbb{R}^d \) (\( B \in \mathbb{R}^d \)). They are continuous on \( A_0(B) \times \mathbb{R}^d \) because they are continuous on convergent sequences in a metric space. Thus the order of taking derivatives does not matter and \( \tilde{R}_{kn} \) is smooth ([13, 1.11.5.(2^o)]). \( \square \)

Proof of (44):

\[
\left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} \tilde{R}_{kn}(S_{2-n} \varphi, x) \right| = \left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} (R'_{kn}(S_{2-n} \varphi, x) * \rho^\otimes_d(x)) \right|
\]

\[
= \left| (\partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, x)) * \rho^\otimes_d(x) \right| .
\]

Then we deduce easily (44) from (40). \( \square \)

Proof of (45): We first estimate

\[
\left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} \tilde{R}_{kn}(S_{2-n} \varphi, x) - \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, x) \right|
\]

\[
= \left| \int \left( \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, y) - \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, x) \right) \rho^\otimes_d(x - y) \, dy \right|
\]

\[
\leq \sup \left\{ \left| \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, y) - \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, x) \right| ;
\]

\[
| x_1 - y_1 | \leq \delta, \ldots, | x_d - y_d | \leq \delta \right\} \leq C'_k 2^{nN_1 L} \cdot d \delta
\]

because the function \( x \mapsto \partial^\alpha d^\ell_{\psi_1, \ldots, \psi_\ell} R'_{kn}(S_{2-n} \varphi, x) \) has its derivatives of order 1 estimated by (40). Then (45) follows from (41). \( \square \)

§10. Lemma. Let \( \mathcal{B} \) be a bounded set in \( \mathcal{D}(\mathbb{R}^d) \). Then there is a natural number \( k \) such that \( V'_{N_k}(\varphi) \geq 2^{-k} \) for all \( \varphi \in \mathcal{B} \) (\( V'_N \) defined by §2, Notation).

As \( V'_N \) is non-decreasing, this inequality holds for all sufficiently large \( k \).
Proof: If not, there would be a sequence \( \{ \varphi_k \}_{k=1}^{\infty} \subset \mathcal{B} \) such that \( V'_{N_k}(\varphi_k) < 2^{-k} \) \( (\forall k) \). As \( \mathcal{B} \) is a metrizable compact, a subsequence \( \{ \varphi_{k_n} \}_{n=1}^{\infty} \) is convergent in \( \mathcal{A}_0 \), \( \lim \varphi_{k_n} = \varphi \in \mathcal{A}_0 \). We have \( V'_{N}(\varphi_k) \leq V'_{N_k}(\varphi_k) < 2^{-k} \) for \( N \leq N_k \). As \( N_k \nearrow \infty \) (see §3, Properties of \( \mathcal{R} \)), we have \( 0 \leq V'_{N}(\varphi) \leq \lim 2^{-k} = 0 \) for all \( N \in \mathbb{N} \), that is impossible. (Proof: If \( \varphi \) has all moments of order \( \geq 1 \) equal to \( 0 \), then its Fourier transform has zero derivatives at origin; being holomorphic, it must be constant). □

Notation. Denote \( \vartheta_{kn}(\varphi) := \vartheta(2^{nN_k+k+1} \cdot V'_{N_k}(\varphi)) \) (\( \vartheta \) by §2, Notation). \( \vartheta_{kn} \) is \( \mathcal{C}^{\infty} \) on \( \mathcal{A}_0 \) and

\[
\vartheta_{kn}(\varphi) = 0 \quad \text{if} \quad 2^{nN_k+k+1} \cdot V'_{N_k}(\varphi) \geq 2,
\]

\[
\vartheta_{kn}(\varphi) = 1 \quad \text{if} \quad 2^{nN_k+k+1} \cdot V'_{N_k}(\varphi) \leq 1.
\]

Definition. We define

\[
\tilde{R}_n(\varphi, x) := \sum_{k=1}^{\infty} \left( \vartheta_{kn}(\varphi) - \vartheta_{k+1,n}(\varphi) \right) \cdot \tilde{R}_{kn}(\varphi, x)
\]

(46)

\[
= \sum_{k=1}^{\infty} \vartheta_{kn}(\varphi) \cdot \left( \tilde{R}_{kn}(\varphi, x) - \tilde{R}_{k-1,n}(\varphi, x) \right)
\]

(47)

if we set \( \tilde{R}_{0,n} = 0 \).

Remark. For a given \( n \) and \( \varphi \), at most two terms of the sum (46) are nonzero. Only a finite number of terms of the sum (47) are nonzero: if \( k \) satisfies the above lemma, then \( \vartheta_{kn}(\varphi) = 0 \).

\[
\sum_{k=1}^{\infty} (\vartheta_{kn}(\varphi) - \vartheta_{k+1,n}(\varphi)) = 1
\]

(48)

is a smooth partition of unity on \( \{ \varphi \in \mathcal{A}_0; \ V'_{N_1}(\varphi) < 2^{-nN_1-2} \} \). Indeed, the sequence \( \left\{ 2^{nN_k+k+1}V'_{N_k}(\varphi) \right\}_k \) is non-decreasing and its first member is \( 2^{nN_1+2}V'_{N_1}(\varphi) < 1 \). Thanks to Lemma, there is the greatest index \( k' \) for which \( 2^{nN_{k'}+k'+1}V'_{N_{k'}}(\varphi) \leq 1 \). Then \( 2^{nN_{k'+1}+k'+2}V'_{N_{k'+1}}(\varphi) > 1 \), \( 2^{nN_{k'+2}+k'+3}V'_{N_{k'+2}}(\varphi) > 2 \), hence

\[
(\vartheta_{k',n}(\varphi) - \vartheta_{k'+1,n}(\varphi)) + (\vartheta_{k'+1,n}(\varphi) - \vartheta_{k'+2,n}(\varphi)) = 1
\]

and the other terms of (48) are zero.
Properties of $\tilde{R}_n$. $\tilde{R}_n \in \mathcal{E}(\mathbb{R}^d)$. If $B \subseteq \mathbb{R}^d$, $\mathcal{B} \subset A_0(B)$ is a bounded set and $L \in \mathbb{N}$, then there is an absolutely convex open neighbourhood of zero $\tilde{U} \subset A(\tilde{B})$ and a constant $\tilde{C} > 0$, both independent of $n$, such that for every $n \in \mathbb{N}, x \in \mathbb{R}^d$, the following hold:

If $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq L - 1$, $\ell \in \mathbb{N}_0$, $\ell \leq L - 1$, $\varphi \in \mathcal{B}$ with $\|\varphi\|_{L^2} \geq 1$, $\tilde{\psi}_1, \ldots, \tilde{\psi}_\ell \in \tilde{U}$, then

$$ (49) \quad \left| \frac{d^\ell}{\psi_1, \ldots, \psi_\ell} \partial^\alpha \tilde{R}_n(S_2-n\varphi, x) \right| \leq \tilde{C} 2^{nN_L(L+1)(2L+1)}. $$

If $(\forall q \in \mathbb{N}) \varphi \in \mathcal{B} \cap A_{N_q}$ with $\|\varphi\|_{L^2} \geq 1 q \in \mathbb{N}$, then

$$ (50) \quad \left| \tilde{R}_n(S_2-n\varphi, x) - R(S_2-n\varphi, x) \right| \leq \tilde{C} 2^{-nq}. $$

**Proof of (49):** For a nonzero term of (46) or (47), we have $2^{nN_k+k+1} \cdot V'_N (\varphi) < 2$, i.e. $V'_N (\varphi) < 2^{-nN_k-k}$. If $\|\varphi\|_{L^2} > 1$, then (§2, Notation) $V_N (\varphi) \leq V'_N (\varphi) < 2^{-nN_k-k}$, so the hypothesis $V_{N_L} (\varphi) < 2^{-nN_{L-1}}$ ($L \leq k$) in §9, Properties of $\tilde{R}_{kn}$ for (44) and (45) is always satisfied. By the Leibniz rule (formulated in the proof of §6, Property (2$^\circ$)) applied to the definition (47) of $\tilde{R}_n$ (recall that $V'_N (\varphi) = V_N (S_2-n\varphi)$), we have

$$ (51) \quad \frac{d^\ell}{\psi_1, \ldots, \psi_\ell} \partial^\alpha \tilde{R}_n(S_2-n\varphi, x) $$

$$ = \sum_{k=1}^{\infty} \sum_{\substack{I_1 \cup I_2 = \{1, \ldots, \ell\} \text{ disjoint}}} \frac{d^{\# I_1}}{\psi_{I_1}} \partial_{kn}(\varphi) \cdot \frac{d^{\# I_2}}{\psi_{I_2}} \partial^\alpha \left( \tilde{R}_{kn}(S_2-n\varphi, x) - \tilde{R}_{k-1,n}(S_2-n\varphi, x) \right) $$

(If $I = \{i_1, \ldots, i_{\# I}\}$, then $\tilde{\psi}_I$ denotes $(\tilde{\psi}_{i_1}, \ldots, \tilde{\psi}_{i_{\# I}})$). Due to Lemma, the sum

$\sum_{k=1}^{\infty} \sum_{k_0}^{k_0}$

can be replaced with $\sum_{k=1}^{k_0}$ with a number $k_0$ depending on $\mathcal{B}$ but not on $\varphi \in \mathcal{B}$.

First we estimate

$$ \frac{d^{\# I}}{\psi_{I}} (\partial_{kn}(\varphi)) = \frac{d^{\# I}}{\psi_{I}} (2^{nN_k+k+1} \cdot V'_N (\varphi)) $$

$$ = \sum_{M=1}^{\# I} \sum_{\substack{I = I_1 \cup \cdots \cup I_M \neq \emptyset, \text{ disjoint}}} (\partial^M \psi) (2^{nN_k+k+1} \cdot V'_N (\varphi)) \prod_{m=1}^{M} \frac{d^{\# I_m}}{\psi_{I_m}} (2^{nN_k+k+1} \cdot V'_N (\varphi)) $$

(chain rule [13, 1.8.3] or [8, Theorem 12], the summation is extended over all decompositions $I = I_1 \cup \cdots \cup I_M$ on non-empty disjoint parts). Let us choose
(by [9, §2, Proposition]) an absolutely convex open neighbourhood of zero $\tilde{U} \subset A(\tilde{B})$ such that for every $k = 1, \ldots, k_0$, $\ell = 1, \ldots, L$, $\tilde{\varphi} \in \tilde{\mathcal{B}}$, $\tilde{\psi}_1, \ldots, \tilde{\psi}_\ell \in \tilde{U}$, we have $|d^\ell_{\tilde{\psi}_1, \ldots, \tilde{\psi}_\ell} V'_{N_k}(\tilde{\varphi})| \leq 1$. We obtain:

\begin{equation}
(52) \quad |d_{\tilde{\psi}_1}^\#(\partial_{kn}(\tilde{\varphi}))| \leq C_k 2^{nN_k} \leq C_k 2^{nN_k} L
\end{equation}

with a constant $C_k$ depending only on $\vartheta$, $L$ and $k$, not on $n$.

We are going to estimate the second term in (51)

\[d_{\tilde{\psi}_2}^\# \partial^\alpha \left( \tilde{R}_{kn}(S_{2-n}\tilde{\varphi}, x) - \tilde{R}_{k-1,n}(S_{2-n}\tilde{\varphi}, x) \right)\]

We distinguish two cases. If $L \leq k - 1$ then $\#_2 \leq k - 1$ and we can use the estimation (45) as follows:

\[
\begin{align*}
|d_{\tilde{\psi}_2}^\# & \partial^\alpha \left( \tilde{R}_{kn}(S_{2-n}\tilde{\varphi}, x) - \tilde{R}_{k-1,n}(S_{2-n}\tilde{\varphi}, x) \right)| \\
& = |d_{\tilde{\psi}_2}^\# \partial^\alpha \left( \left( \tilde{R}_{kn}(S_{2-n}\tilde{\varphi}, x) - R(S_{2-n}\tilde{\varphi}, x) \right) \\
& - \left( \tilde{R}_{k-1,n}(S_{2-n}\tilde{\varphi}, x) - R(S_{2-n}\tilde{\varphi}, x) \right) \right)| \\
& \leq 2^{nN_L} \cdot (\tilde{C}_k \delta_{kn} + \tilde{C}_{k-1} \delta_{k-1,n}) \leq 2^{nN_L} \cdot (\tilde{C}_k + \tilde{C}_{k-1}) \delta_{k-1,n}.
\end{align*}
\]

Together with (52), a term of the sum in (51) for $L \leq k - 1$ fulfills

\[
|d_{\tilde{\psi}_1}^\#(\tilde{\varphi}) \cdot d_{\tilde{\psi}_2}^\# \partial^\alpha \left( \tilde{R}_{kn}(S_{2-n}\tilde{\varphi}, x) - \tilde{R}_{k-1,n}(S_{2-n}\tilde{\varphi}, x) \right) | \\
\leq C_k 2^{nN_L} L \cdot 2^{nN_L} (\tilde{C}_k + \tilde{C}_{k-1}) \delta_{k-1,n} \leq C_k (\tilde{C}_k + \tilde{C}_{k-1}) \delta_{k-1,n}
\]

($\delta$ is defined in §9, Notation), that is a constant independent of $n$, however it depends on $\mathcal{B}$ and the number of nonzero terms of the sum in (51) depends on $\mathcal{B}$, too.

If $L \geq k$, we use the estimations (43) valid for all $L$ and we obtain:

\[
d_{\tilde{\psi}_2}^\# \partial^\alpha \tilde{R}_{kn}(S_{2-n}\tilde{\varphi}, x) \leq \tilde{C}_k 2^{nN_1} \cdot \delta_{kn}^{2L} \leq \tilde{C}_k 2^{nN_1} \cdot \delta_{L,n}^{2L}.
\]

Together with (52), a term of the sum in (51) for $L \geq k$ fulfills

\[
|d_{\tilde{\psi}_1}^\#(\tilde{\varphi}) \cdot d_{\tilde{\psi}_2}^\# \partial^\alpha \left( \tilde{R}_{kn}(S_{2-n}\tilde{\varphi}, x) - \tilde{R}_{k-1,n}(S_{2-n}\tilde{\varphi}, x) \right) | \\
\leq C_k 2^{nN_L} L(\tilde{C}_k + \tilde{C}_{k-1}) 2^{nN_1} \cdot \delta_{L,n}^{2L} \leq C_k (\tilde{C}_k + \tilde{C}_{k-1}) 2^{nN_L}(2L+1)(L+1).
\]
So in both cases we can use the last estimation and (49) follows. □

Proof of (50): If \( \tilde{\varphi} \in \mathcal{A}_{N_q} \) then \( \nabla_{N_q}^r(\tilde{\varphi}) = 0 \) and \( k \geq q \) for all nonzero terms of the sum in (46). In that case, it follows from the definition and (48)

\[
\left| \tilde{R}_n(S_{\varepsilon}\tilde{\varphi}, x) - R(S_{\varepsilon}\tilde{\varphi}, x) \right|
\]

\[
= \sum_{k=q}^{k_0} (\vartheta_{kn}(\tilde{\varphi}) - \vartheta_{kn+1}(\tilde{\varphi})) \cdot \tilde{R}_{kn}(S_{\varepsilon}\tilde{\varphi}, x) - \sum_{k=q}^{k_0} (\vartheta_{kn}(\tilde{\varphi}) - \vartheta_{kn+1}(\tilde{\varphi})) \cdot R(S_{\varepsilon}\tilde{\varphi}, x)
\]

\[
\leq \sum_{k=q}^{k_0} (\vartheta_{kn}(\tilde{\varphi}) - \vartheta_{kn+1}(\tilde{\varphi})) \cdot \left| \tilde{R}_{kn}(S_{\varepsilon}\tilde{\varphi}, x) - R(S_{\varepsilon}\tilde{\varphi}, x) \right|
\]

\[
\leq \sum_{k=q}^{k_0} \tilde{C}_k 2^{nN_1} \delta_{kn} = \sum_{k=q}^{k_0} \tilde{C}_k 2^{nN_1-n(k+1)N_k} \leq \tilde{C} 2^{-nq}
\]

by (45), as \( V_{N_1}(\tilde{\varphi}) = 0 \). □

§ 11. Now we have all tools for defining the desired representative \( \tilde{R} \).

Definition of \( \tilde{R} \). We define

\[
\tilde{R}(\varphi, x) := \sum_{n=1}^{\infty} \left( \vartheta_{n+1} \left( \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} \right) - \vartheta_n \left( \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} \right) \right) \tilde{R}_n(\varphi, x)
\]

for \( \varphi \in \mathcal{A}_0(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \) (\( \vartheta \) by §2, Notation).

Remark. Note that \( \vartheta_{n+1} \left( \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} \right) - \vartheta_n \left( \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} \right) \neq 0 \) iff \( 2^n < \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} < 2^{n+2} \) and, for a given \( \varphi \), at most 2 terms of this sum are \( \neq 0 \).

\[
\sum_{n=1}^{\infty} \left( \vartheta_{n+1} \left( \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} \right) - \vartheta_n \left( \|\varphi\|_{\mathcal{L}^{2/d}}^{2/d} \right) \right) = 1
\]

is a smooth partition of unity on \( \{ \varphi \in \mathcal{D}; \|\varphi\|_{\mathcal{L}^{2/d}} > 4 \} \).

Properties of \( \tilde{R} \). \( \tilde{R} \in \mathcal{E}(\mathbb{R}^d) \). If \( B \in \mathbb{R}^d \), \( \mathcal{B} \) a bounded set \( \subset \mathcal{A}_0(B) \) and \( L \in \mathbb{N} \),

then there is an absolutely convex open neighbourhood of zero \( \mathcal{U} \subset \mathcal{A}(B) \) and a constant \( C > 0 \), such that for every \( x \in \mathbb{R}^d \) the following hold.

(1°) If \( \varepsilon \in [0, 1], \alpha \in \mathbb{N}_0^d, |\alpha| \leq L - 1, \ell \in \mathbb{N}_0, \ell \leq L - 1, \varphi \in \mathcal{B}, \psi_1, \ldots, \psi_\ell \in \mathcal{U} \),

then

\[
|d_{\psi_1,\ldots,\psi_\ell}^{\ell} \vartheta^\alpha \tilde{R}(S_{\varepsilon}\varphi, x)| \leq C \varepsilon^{-N_L(2L+1)(L+1)}
\]
Consequently, $\tilde{R} \in \mathcal{E}^d_M$.

(2°) $\forall q \in \mathbb{N}$ it holds: If $\varphi \in \mathcal{B} \cap A_{N_q}$ and $0 < \varepsilon < \min \left\{ 1, \frac{1}{4} \| \chi_B \|_{\mathcal{F}^2}^{-2/d} \right\}$, then

$$\left| \tilde{R}(S_{\varepsilon} \varphi, x) - R(S_{\varepsilon} \varphi, x) \right| \leq C \cdot \varepsilon^q.$$  \hspace{1cm} (54)

Due to [9, §8, (0°)], we have $\tilde{R} - R \in \mathcal{N}$.

**Proof of (53):** We write simply $\| \cdot \|$ instead of $\| \cdot \|_{\mathcal{F}^2}$ and we assume $B$ to be convex and balanced. For proving (53), we calculate (see (2)):

$$\tilde{R}(S_{\varepsilon} \varphi, x) = \sum_{n=1}^{\infty} \left( \vartheta_{n+1}(\| S_{\varepsilon} \varphi \|^2/d) - \vartheta_n(\| S_{\varepsilon} \varphi \|^2/d) \right) \tilde{R}_n(S_{\varepsilon} \varphi, x)$$

$$= \sum_{n=1}^{\infty} \left( \vartheta_1(2^{-n}\| S_{\varepsilon} \varphi \|^2/d) - \vartheta_0(2^{-n}\| S_{\varepsilon} \varphi \|^2/d) \right) \tilde{R}_n(S_{2^{-n}}(S_{2^{n_{\varepsilon}}} \varphi), x)$$

$$= \sum_{n=1}^{\infty} \left( \vartheta_1(\| S_{2^{n_{\varepsilon}}} \varphi \|^2/d) - \vartheta_0(\| S_{2^{n_{\varepsilon}}} \varphi \|^2/d) \right) \tilde{R}_n(S_{2^{-n}}(S_{2^{n_{\varepsilon}}} \varphi), x).$$  \hspace{1cm} (55)

By the definition of $\vartheta$, a term of this sum can be nonzero only if

$$1 < \| S_{2^{n_{\varepsilon}}} \varphi \|^2/d < 4, \text{ i.e. } 1 < \frac{1}{2^n \varepsilon} \| \varphi \|^2/d < 4.$$  \hspace{1cm} (56)

$\mathcal{B} \subseteq A_0(B)$, hence there are constants $c_1, c_2 > 0$ such that

$$c_2 \leq \| \varphi \|^2/d \leq c_1 \quad (\forall \varphi \in \mathcal{B}).$$

Due to (56), it follows that for nonzero terms of the sum in (55), we have

$$\frac{1}{4} c_2 < 2^n \varepsilon < c_1.$$  \hspace{1cm} (57)

By the Leibniz rule (formulated in the proof of §6, Property 2°) applied to (55), we have

$$d_{\psi_1, \ldots, \psi_\ell} \partial^\alpha \tilde{R}(S_{\varepsilon} \varphi, x)$$

$$= \sum_{n=1}^{\infty} \sum_{I_1 \cup I_2 = \{1, \ldots, \ell\} \text{ disjoint}} d_{\psi_{I_1}}^{#I_1} \left( \vartheta_1(\| S_{2^{n_{\varepsilon}}} \varphi \|^2/d) - \vartheta_0(\| S_{2^{n_{\varepsilon}}} \varphi \|^2/d) \right) \cdot d_{\psi_{I_2}}^{#I_2} \partial^\alpha \tilde{R}_n(S_{2^{-n}}(S_{2^{n_{\varepsilon}}} \varphi), x).$$  \hspace{1cm} (58)

The function \( \varphi \mapsto \vartheta_1(\|S_{2n} \varphi\|^2/d) - \vartheta_0(\|S_{2n} \varphi\|^2/d) \) is composed of functions and

\[
[t \mapsto \vartheta_1(t^{1/d}) - \vartheta_0(t^{1/d})] \in \mathcal{D}((1, 4^d])
\]

\[
\varphi \mapsto \|S_{2n} \varphi\|^2.
\]

Hence, for proving that the term \( d^{\#I_1}_{\psi I_1} \left( \vartheta_1(\|S_{2n} \varphi\|^2/d) - \vartheta_0(\|S_{2n} \varphi\|^2/d) \right) \) in (58) is equi-bounded under the hypotheses (1°) (for a fixed \( L \)), it is sufficient to prove the same for derivatives of \( \|S_{2n} \varphi\|^2 \) up to a certain order. We have

\[
d_{\psi} \|S_{2n} \varphi\|^2 = d_{\psi}(S_{2n} \varphi, S_{2n} \varphi) = 2\Re(S_{2n} \varphi, S_{2n} \psi),
\]

\[
d^2_{\psi_1, \psi_2} \|S_{2n} \varphi\|^2 = 2\Re(S_{2n} \psi_1, S_{2n} \psi_2)
\]

and the higher derivatives are zero. So we have using the Hölder inequality

\[
\left| d_{\psi} \|S_{2n} \varphi\|^2 \right| = \left| 2\Re \int S_{2n} \varphi \cdot S_{2n} \overline{\psi} \right| \leq \|S_{2n} \varphi\| \|S_{2n} \psi\| = \frac{1}{(2n)^d} \|\varphi\| \|\psi\|.
\]

Thanks to (57), this will be equi-bounded if \( \mathcal{U} \subset \{ \psi; \|\psi\| < 1 \} \), \( \psi \in \mathcal{U} \), \( \varphi \in \mathcal{B} \), as the bounded set \( \mathcal{B} \) is absorbed by \( \mathcal{U} \). The same can be deduced for the second derivative.

Now, we have to estimate the term \( d^{\#I_2}_{\psi I_2} \partial^\alpha \tilde{R}_n(S_{2-n}(S_{2n} \varphi), x) \) in (58). We apply \( \S 10, \) Properties of \( \tilde{R}_n \), namely the estimation (49), to the bounded set

\[
\tilde{\mathcal{B}} := \{ S_\eta \varphi; \varphi \in \mathcal{B}, \frac{1}{4} c_2 \leq \eta \leq c_1 \}.
\]

The supports of the functions \( \tilde{\varphi} \in \tilde{\mathcal{B}} \) are contained in \( \tilde{B} := c_1 B \) (for \( B \) convex and balanced). Thanks to (57) we have \( S_{2n} \varphi \in \mathcal{B} \) for \( \varphi \in \mathcal{B} \). Thus we get \( \tilde{\mathcal{U}} \) and \( \tilde{C} \) by \( \S 10, \) Properties of \( \tilde{R}_n \). Let

\[
\mathcal{U} := \left\{ \varphi \in A(B); \ S_\eta \varphi \in \tilde{\mathcal{U}} \quad \forall \eta \quad \text{with} \quad \frac{1}{4} c_2 \leq \eta \leq c_1 \right\}.
\]

\( \mathcal{U} \) is a neighbourhood of zero in \( A(B) \) because it absorbs bounded sets in a metric vector space. If \( \varphi \in \mathcal{B} \), \( \psi_1, \ldots, \psi_\ell \in \mathcal{U} \), then \( \tilde{\varphi} := S_{2n} \varphi \in \tilde{\mathcal{B}} \) and \( \tilde{\psi}_j := S_{2n} \psi_j \in \tilde{\mathcal{U}} \) \( (j = 1, \ldots, \ell) \) due to (57). If \( \|\tilde{\varphi}\| = \|S_{2n} \varphi\| < 1 \), then (by (56)) the term \( d^{\#I_2}_{\psi I_2} \partial^\alpha \tilde{R}_n(S_{2-n}(S_{2n} \varphi), x) \) in (58) is multiplied by zero, so we have to estimate this term only if \( \|\tilde{\varphi}\| \geq 1 \), hence we can use (49). It follows using the chain rule for the inner function \( S_{2n} \) linear:

\[
\left| d^{\#I_2}_{\psi I_2} \partial^\alpha \tilde{R}_n(S_{2-n}(S_{2n} \varphi), x) \right| = \left| d^{\#I_2}_{\psi I_2} \partial^\alpha \tilde{R}_n(S_{2-n} \tilde{\varphi}, x) \right|
\]

\[
\leq \tilde{C} \cdot 2^{\varepsilon N_L(2L+1)(L+1)} \leq C c_1^{N_L(2L+1)(L+1)} \varepsilon^{-N_L(2L+1)(L+1)}
\]
due to (57). From (58) we deduce (53).

Proof of (54): By (9), $\|\varphi\| \geq \|\chi_B\|^{-1}$ (norms in $L^2$). If $\varepsilon < \frac{1}{4}\|\chi_B\|^{-2/d}$, then

$$\|S_\varepsilon \varphi\|^{2/d} = \|\varphi\|^{2/d} \cdot \varepsilon^{-1} > \|\varphi\|^{2/d} \cdot 4 \|\chi_B\|^{2/d} \geq 4.$$ Under this hypothesis we have by §11, Remark similarly to (55):

$$R(S_\varepsilon \varphi, x) = \sum_{n=1}^{\infty} \left( \vartheta_{n+1} \left( \|S_\varepsilon \varphi\|^{2/d} \right) - \vartheta_n \left( \|S_\varepsilon \varphi\|^{2/d} \right) \right) R(S_\varepsilon \varphi, x)$$

$$= \sum_{n=1}^{\infty} \left( \vartheta_1 \left( \|S_{2n_\varepsilon} \varphi\|^{2/d} \right) - \vartheta_0 \left( \|S_{2n_\varepsilon} \varphi\|^{2/d} \right) \right) R(S_{2^{-n}}(S_{2n_\varepsilon} \varphi), x).$$

With (55) it gives

$$\left| \tilde{R}(S_\varepsilon \varphi, x) - R(S_\varepsilon \varphi, x) \right|$$

$$\leq \sum_{n=1}^{\infty} \left( \vartheta_1 \left( \|S_{2n_\varepsilon} \varphi\|^{2/d} \right) - \vartheta_0 \left( \|S_{2n_\varepsilon} \varphi\|^{2/d} \right) \right)$$

$$\cdot \left| \tilde{R}_n(S_{2^{-n}}(S_{2n_\varepsilon} \varphi), x) - R(S_{2^{-n}}(S_{2n_\varepsilon} \varphi), x) \right|.$$ If $\varphi \in \mathcal{B} \cap A_{Nq}$ then $\tilde{\varphi} := S_{2n_\varepsilon} \varphi \in \widetilde{\mathcal{B}} \cap A_{Nq}$. By (50) this is $\leq \tilde{C} \cdot 2^{-nq}$ and by (57) this is $\leq \tilde{C} \cdot \left( \frac{4}{e^2} \right)^q \varepsilon^q$. By the above lemma, $\mathcal{B} \cap A_{Nq} = \emptyset$ for sufficiently large $q$, so the constant in our estimation can be independent of $q$. Hence, (54) is proved.

References


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