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On subsets of Alexandroff duplicates

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Abstract. We characterize the subsets of the Alexandroff duplicate which have a $G_\delta$-diagonal and the subsets which are M-spaces in the sense of Morita.

Keywords: Alexandroff duplicate, resolution

Classification: 54B99, 54E18

1. Introduction

All spaces are assumed to be regular $T_1$, and all mappings to be continuous. We denote all positive integers, real numbers by $\mathbb{N}$, $\mathbb{R}$, respectively.

As it is well known, the Alexandroff duplicate of $\mathbb{R}$ does not have a $G_\delta$-diagonal and the famous Michael line is not an M-space in the sense of Morita, although it is a subspace of the Alexandroff duplicate of $\mathbb{R}$. So, in this paper, we characterize the subspaces of the Alexandroff duplicate $X \times \text{ad}(2)$ which have a $G_\delta$-diagonal, where $X$ has a $G_\delta$-diagonal, and also characterize the subspaces of $Y \times \text{ad}(2)$ which are M-spaces, where $X$ is a metrizable space. The former gives an answer to the problem posed by S. Watson, [3, Problem 3.1.29], where he asks how to characterize the subsets of $[0,1] \times \text{ad}(2)$ which have a $G_\delta$-diagonal.

As for the properties of $G_\delta$-diagonals and M-spaces used here, we refer to Gruenhage [1]. We recall the definition of the Alexandroff duplicate $X \times \text{ad}(2)$ of a space $X$, stated in [3, Definition 3.1.1]. Let $(X, \tau)$ be a space. Define the topology on $Z = X \times 2$ by declaring that each $(x, 1)$ is open and that for each open $U \in \tau, U \times 2 \setminus \{(x, 1)\}$ is open. The space $Z$ so defined is denoted by $X \times \text{ad}(2)$, where $\text{ad}$ stands for Alexandroff duplicate. In the sequel, we write a subspace of $X \times \text{ad}(2)$ in the following form:

$$T(A, B) = A \times \{1\} \cup B \times \{0\},$$

where $A, B \subset X$.

2. On subspaces of Alexandroff duplicates

For a subset $A$ of a space $X$, we denote by $A^d$ the set of all accumulation points of $A$ in $X$. 

**Theorem 2.1.** Assume that a space $X$ has a $G_δ$-diagonal and $T(A, B) \subset X \times_{ad}(2)$. Then $T(A, B)$ has a $G_δ$-diagonal if and only if $A \cap B = \bigcup\{C_i : i \in \mathbb{N}\}$ with $(C_i)^d \cap B = \emptyset$ for each $i$.

**Proof:** Only if part: Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a $G_δ$-diagonal sequence for $T(A, B)$. For each $x \in A \cap B$, there exists $n(x) \in \mathbb{N}$ such that

$$(x, 0) \notin S((x, 1), \mathcal{U}_{n(x)}).$$

Let

$$C_n = \{x \in A \cap B : n(x) = n\}, \quad n \in \mathbb{N}.$$ 

Then $A \cap B = \bigcup_n C_n$. Assume that $(C_n)^d \cap B \neq \emptyset$ for some $n$. For a point $x \in (C_n)^d \cap B$, there exists $U \in \mathcal{U}_n$ such that $(x, 0) \in U$. Since $x$ is an accumulation point of $C_n$, there exists $x' \in C_n$ such that $(x', 0), (x', 1) \in U$, but this is impossible.

If part: Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a $G_δ$-diagonal sequence for $A \cup B$. By the assumption, $A \cap B = \bigcup\{C_n : n \in \mathbb{N}\}$, where $(C_n)^d \cap B = \emptyset$ for each $n$. Since $C_n$ is discrete in $B$, there exists a family $\{V(x) : x \in C_n\}$ of open subsets of $A \cup B$ such that for each $x \in C_n$, $V(x) \cap B = \{x\}$ and $x \in V(x) \subset U$ for some $U \in \mathcal{U}_n$. For each $U \in \mathcal{U}_n$, $n \in \mathbb{N}$, let

$$\widehat{U} = (U \setminus C_n) \times \{0, 1\} \cap T(A, B).$$

For each $x \in C_n$, $n \in \mathbb{N}$, let

$$\widehat{V}(x) = (V(x) \times \{0, 1\} \setminus \{(x, 1)\}) \cap T(A, B).$$

For each $n \in \mathbb{N}$, define an open cover

$$\mathcal{W}(n) = \{\widehat{U} : U \in \mathcal{U}_n\} \cup \{\widehat{V}(x) : x \in C_n\} \cup \{\{(x, 1) : x \in A\}.$$

We show that $(\mathcal{W}(n))_{n \in \mathbb{N}}$ is a $G_δ$-diagonal sequence for $T(A, B)$. To this end, let

$$p = (x, s), \quad q = (y, t)$$

be different points of $T(A, B)$. If $x \neq y$, then there exists $n \in \mathbb{N}$ such that $x \notin S(y, \mathcal{U}_n)$. Then it is easily seen that $p \notin S(q, \mathcal{W}(n))$. If $x = y$, $s = 0$, $t = 1$, then we have $x \in A \cap B$ and $x \in C_n$ for some $n \in \mathbb{N}$. In this case, we easily have

$$p \notin S(q, \mathcal{W}(n)) = \widehat{V}(x).$$

Hence $T(A, B)$ has a $G_δ$-diagonal. \qed

We give a remark to some special cases of $X$:
Remark 2.1. (1) If \( X = \mathbb{R} \), \( T(A, B) \) has a \( G_\delta \)-diagonal if and only if \( A \cap B \) is countable. It is because any uncountable subset of \( \mathbb{R} \) has an accumulation point in \( \mathbb{R} \).

(2) If \( X \) is metrizable, the above condition for \( T(A, B) \) to have a \( G_\delta \)-diagonal is that for \( T(A, B) \) to be submetrizable. This follows from the fact that \( T(A, B) \) is paracompact.

Next, we characterize \( T(A, B) \) which is an M-space in the sense of Morita. A space \( X \) is called an M-space if there exists a sequence \( (U_n)_{n \in \mathbb{N}} \) of open covers of \( X \) such that for each \( n \), \( U_{n+1} \) star-refines \( U_n \) and if \( x_n \in S(x, U_n) \), then \( \{x_n : n \in \mathbb{N}\} \) clusters in \( X \). Such a sequence \( (U_n)_{n \in \mathbb{N}} \) is called an M-sequence for \( X \). On the other hand, in 1963 Arhangel’ski˘ı gave the concept of \( p \)-spaces. As it is well known, M-spaces and \( p \)-spaces are equivalent in the presence of paracompactness [1, Corollary 3.20], and paracompact \( p \)-spaces coincide with pre-images of a metric space under a perfect mapping [1, Corollary 3.7].

Let \( (X, d) \) be a metric space. We denote an open ball with center \( x \) and radius \( r \) by \( B(x, r) \). We note that the projection \( \pi : T(A, B) \to A \cup B \) is continuous.

In connection with the next theorem, the referee informed us about the interesting fact that E.G. Pytkeev wrote a paper in which he proved that if a space \( X \) is a Tychonoff space such that each subspace of \( X \) is a paracompact \( p \)-space, then the structure of \( X \) is very similar to that of the Alexandroff duplicate of a metric space; indeed, then the subspace of all non-isolated points is metrizable.

Theorem 2.2. Let \( T(A, B) \subset X \times_{ad} (2) \), where \( X \) is a metric space. Then \( T(A, B) \) is an M-space if and only if \( B \) is a \( G_\delta \)-set in \( A \cup B \).

Proof: Only if part: Assume that \( B \) were not a \( G_\delta \)-set. Let \( (U_n)_{n \in \mathbb{N}} \) be an M-space for \( T(A, B) \). Since \( X \) is a metric space, without loss of generality we can assume that if \( (x_n, s_n) \in S((x, s), U_n) \), \( n \in \mathbb{N} \), then \( x_n \to x \) as \( n \to \infty \). Let \( n \in \mathbb{N} \) be fixed. For each \( x \in B \), there exists \( U \in U_n \) such that \( (x, 0) \in U \). There exists a basic open neighborhood \( N(x, r(x)) \) of \( (x, 0) \) in \( X \times_{ad} (2) \) such that

\[
N(x, r(x)) = B(x, r(x)) \times \{0, 1\} \setminus \{(x, 1)\},
\]

\[
N(x, r(x)) \cap T(A, B) \subset U.
\]

Let

\[
G_n = \left( \bigcup \{B(x, r(x)) : x \in B\} \right) \cap (A \cup B),
\]

which is open in \( A \cup B \). By the assumption, there exists \( a \in \bigcap_n G_n \setminus B \). Then for each \( n \in \mathbb{N} \), there exists a point

\[
(x_n, 0) \in B \times \{0\} \cap S((a, 1), U_n).
\]

Since \( (U_n) \) is an M-sequence and \( x_n \to a \) as \( n \to \infty \), \( \{(x_n, 0) : n \in \mathbb{N}\} \) clusters at \( (a, 1) \), but this is a contradiction because \( \{(a, 1)\} \) is open.
If part: Let \( B = \bigcap_n G_n, \ G_{n+1} \subset G_n, \ n \in \mathbb{N}, \) where each \( G_n \) is open in \( A \cup B. \) Since \( A \cup B \) is a metric space, there exists a development \((\mathcal{U}_n)_{n \in \mathbb{N}}\) for \( A \cup B \) such that \( \mathcal{U}_n^* < \mathcal{U}_n, \ n \in \mathbb{N}. \) We construct a sequence \((\mathcal{V}_n)_{n \in \mathbb{N}}\) of open covers of \( T(A, B) \) as follows:

\[
\mathcal{V}_n = \pi^{-1}(\mathcal{U}_n | G_n) \cup \{(x, 1) \mid x \in A \setminus G_n\}, \ n \in \mathbb{N}.
\]

Then it is easily checked that each \( \mathcal{V}_{n+1} \) star-refines \( \mathcal{V}_n. \) We show that \((\mathcal{V}_n)_{n \in \mathbb{N}}\) is an M-sequence for \( T(A, B). \) Let

\[
(x_n, r_n) \in S((x, r), \mathcal{V}_n), \ n \in \mathbb{N}.
\]

If \( x \in B, \) then \((x, 0)\) is a cluster point of \( \{(x_n, r_n) \mid n \in \mathbb{N}\}. \) If \( x \in A \setminus B, \) then there exists \( k \in \mathbb{N} \) such that \( x \notin G_k. \) From the construction of \((\mathcal{V}_n), \) it follows that \((x_n, r_n) = (x, 0)\) for \( n \geq k, \) which means that \((x_n, r_n) \rightarrow (x, 0)\) as \( n \rightarrow \infty. \)

\[\square\]

**Corollary 2.1.** Let \( T(A, B) \subset X \times ad (2), \) where \( X \) is a metric space. Then \( T(A, B) \) is metrizable if and only if \( B \) is a \( G_δ\)-set in \( A \cup B \) and \( A \cap B = \bigcup_{i \in \mathbb{N}} C_i, \) where for each \( i, \ (C_i)^d \cap B = \emptyset. \)

Here, we recall the definition of resolutions of spaces. Let \( X \) be a space and for each \( x \in X, \) let \( f_x : X \setminus \{x\} \rightarrow Y_x \) be a mapping. We topologize

\[ Z = \bigcup\{\{x\} \times Y_x : x \in X\} \]

by defining an open set \( U \otimes V \) for each \( x \in X \) and each open subset \( U \) of \( X \) with \( x \in U \) and open subset \( V \) of \( Y_x \) as

\[ U \otimes V = (\{x\} \times V) \cup \bigcup\{\{p\} \times Y_p : p \in U \cap f_x^{-1}(V)\}. \]

We call \( Z \) thus defined the *resolution* of \( X \) at each point \( x \in X \) into \( Y_x \) by \( f_x \) [3, Definition 3.1.32], and we denote it by \( Z = R(X, f_x, Y_x). \) We note that the projection \( \pi : Z \rightarrow X \) defined by \( \pi((x, y)) = x \) for each \((x, y) \in Z\) is continuous.

**Example 2.1.** There exists a resolution \( Z = R(X, f_x, Y_x) \) of a compact space \( X \) into paracompact M-spaces \( Y_x, \ x \in X, \) such that \( Z \) is not an M-space.

**Proof:** Let \( X = \omega_1 + 1 \) with the order topology. For each \( \alpha < \omega_1, \) let \( Y_\alpha \) be the copy of \( \mathbb{R} \) with the usual topology. Let \( f_\alpha : X \setminus \{\alpha\} \rightarrow Y_\alpha \) be a constant mapping such that \( f_\alpha(X \setminus \{\alpha\}) = y_\alpha \in Y_\alpha. \) For \( \alpha = \omega_1, \ Y_\omega_1 = \{\omega_1\} \) and let \( f_{\omega_1} : X \setminus \{\omega_1\} \rightarrow Y_{\omega_1} \) be a natural mapping. Let \( Z = R(X, f_x, Y_x). \) Assume that there exists an M-sequence \((\mathcal{U}_n)_{n \in \mathbb{N}}\) for \( Z. \) For \( p = (\omega_1, \omega_1), \) there exists \( \alpha \in \omega_1 \) such that

\[
\{\alpha\} \times Y_\alpha \subset \bigcap_{n \in \mathbb{N}} S((\omega_1, \omega_1), \mathcal{U}_n).
\]
Since $Y_\alpha$ is not countably compact, this is impossible. □

We say that a subset $\Lambda$ is $F_\sigma$-discrete in $X$ if $\Lambda = \bigcup_{n \in \mathbb{N}} \Lambda_n$, where each $\Lambda_n$ is discrete and closed in $X$. Richardson and Watson showed that if $X$ and each $Y_x$ are metrizable and

$$\lambda = \{x \in X : |Y_x| > 1\}$$

is $F_\sigma$-discrete in $X$, then $R(X, f_x, Y_x)$ is metrizable [2, Proposition 9]. We recall a characterization of paracompact $p$-spaces: a space $X$ is a paracompact $p$-space if and only if there exists a perfect mapping of $X$ onto a metric space.

**Theorem 2.3.** Let $X$ be a metric space and each $Y_x$, $x \in X$, a paracompact $p$-space. If $\Lambda$, defined above, is $F_\sigma$-discrete in $X$, then $Z = R(X, f_x, Y_x)$ is a paracompact $p$-space.

**Proof:** By the above characterization, for each $x \in X$ there exists a perfect mapping $g_x : Y_x \rightarrow M_x$ with $M_x$ metric. By the condition on $\Lambda$, the resolution $Z' = R(X, g_x f_x, M_x)$ is a metric space. So, it suffices to show that the mapping $\Phi : Z \rightarrow Z'$ defined by

$$\Phi(x, y) = (x, g_x(y)), \ (x, y) \in Z,$$

is a perfect mapping. It is easily checked that $\Phi$ is continuous. To see that $\Phi$ is closed, let $W$ be an open set of $Z$ containing $\Phi^{-1}(x, y') = \{x\} \times g_x^{-1}(y')$. There exists a finite open cover $\{U_i \otimes V_i \mid i = 1, \ldots, k\}$ of $\Phi^{-1}(x, y')$ in $Z$ such that

$$\Phi^{-1}(x, y') \subset \bigcup_{i=1}^{k} U_i \otimes V_i \subset W,$$

where each $U_i$ is an open neighborhood of $x$ in $X$. Since $g_x : Y_x \rightarrow M_x$ is a perfect mapping, there exists an open neighborhood $O$ of $y'$ in $M_x$ such that $g_x^{-1}(O) \subset \bigcup_{i=1}^{k} V_i$. Then we can easily see that $(\bigcap_{i=1}^{k} U_i) \otimes O$ is an open neighborhood of $(x, y')$ in $Z'$ such that $\Phi^{-1}((\bigcap_{i=1}^{k} U_i) \otimes O) \subset W$. Hence $\Phi$ is a perfect mapping. □

Since $\pi : R(X, f_x, Y_x) \rightarrow X$ is a perfect mapping if each $Y_x$ is compact [2, Lemma 6], the following is easy to see:

**Theorem 2.4.** Let $X$ be an $M$-space and let each $Y_x$ be compact. Then $Z = R(X, f_x, Y_x)$ is an $M$-space.

**References**


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