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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 46 (2005), No. 1, 131--135

Persistent URL: <http://dml.cz/dmlcz/119514>

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## On the cardinality of Hausdorff spaces and Pol-Šapirovsĭii technique

ALEJANDRO RAMÍREZ-PÁRAMO

*Abstract.* In this paper we make use of the Pol-Šapirovsĭii technique to prove three cardinal inequalities. The first two results are due to Fedeli [2] and the third theorem of this paper is a common generalization to: (a) (Arhangel'skii [1]) If  $X$  is a  $T_1$  space such that (i)  $L(X)t(X) \leq \kappa$ , (ii)  $\psi(X) \leq 2^\kappa$ , and (iii) for all  $A \in [X]^{\leq 2^\kappa}$ ,  $|\overline{A}| \leq 2^\kappa$ , then  $|X| \leq 2^\kappa$ ; and (b) (Fedeli [2]) If  $X$  is a  $T_2$ -space then  $|X| \leq 2^{\text{ac}(X)t(X)\psi_c(X)}$ .

*Keywords:* cardinal functions, cardinal inequalities, Hausdorff space

*Classification:* 54A25

In [2], Fedeli proved, using the language of elementary submodels, two cardinal inequalities which state (1) “if  $X \in \mathcal{T}_2$ , then  $|X| \leq 2^{\text{ac}(X)H\psi(X)}$ ” and (2) “if  $X \in \mathcal{T}_2$ , then  $|X| \leq 2^{\text{lc}(X)\pi\chi(X)\psi_c(X)}$ ”. Each of these inequalities improve the well known Hajnal-Juhász’s inequality: “for  $X \in \mathcal{T}_2$ ,  $|X| \leq 2^{c(X)\chi(X)}$ ”. In the first part of this paper we give a proof of the inequalities (1) and (2) without using elementary submodels. Our proof makes use of the Pol-Šapirovsĭii technique. This technique provides a unified approach to the difficult inequalities in the theory of cardinal functions. The reader is referred to [4] and [3] for a detailed discussion like for additional inequalities in cardinal functions which can be proved using the Pol-Šapirovsĭii technique.

We refer the reader to [3], [2] and [5] for definitions and terminology not explicitly given. Let  $L, c, \chi, \psi, \psi_c, \pi\chi, t$ , denote the following standard cardinal functions: Lindelöf degree, cellularity, character, pseudocharacter, closed pseudocharacter,  $\pi$ -character and tightness, respectively.

Let  $X$  be a Hausdorff space. The Hausdorff pseudocharacter, denoted  $H\psi(X)$ , is the smallest infinite cardinal  $\kappa$  such that for every  $x \in X$  there is a collection  $\mathcal{U}_x$  of open neighborhoods of  $x$  with  $|\mathcal{U}_x| \leq \kappa$  and such that (\*) if  $x \neq y$ , there exist  $U \in \mathcal{U}_x$  and  $V \in \mathcal{U}_y$  with  $U \cap V = \emptyset$ . If  $\mathcal{U}_x$  is a collection of open neighborhoods of  $x$  which satisfies (\*), we say that  $\mathcal{U}_x$  is a  $H$ -pseudobase of  $x$ .

**Definition 1.** Let  $X$  be a topological space:

(a)  $\text{ac}(X)$  is the smallest infinite cardinal  $\kappa$  such that there is a subset  $S$  of  $X$  with  $|S| \leq 2^\kappa$  and for every open collection  $\mathcal{U}$  in  $X$ , there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ , with  $\bigcup \mathcal{U} \subseteq S \cup \bigcup \{\overline{V} : V \in \mathcal{V}\}$ .

(b)  $lc(X)$  is the smallest infinite cardinal  $\kappa$  such that there is a closed subset  $F$  of  $X$  with  $|F| \leq 2^\kappa$  and for every open collection  $\mathcal{U}$  in  $X$ , there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ , with  $\bigcup \mathcal{U} \subseteq F \cup \overline{\bigcup \{ \bar{V} : V \in \mathcal{V} \}}$ .

(c)  $aql(X)$  is the smallest infinite cardinal  $\kappa$  such that there is a subset  $S$  of  $X$  such that  $|S| \leq 2^\kappa$  and for every open cover  $\mathcal{U}$  of  $X$  there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$  with  $X = S \cup (\bigcup \mathcal{V})$ .

Clearly  $ac(X) \leq lc(X) \leq c(X)$ , and  $aql(X) \leq L(X)$  for every topological space.

**Theorem 2.** *If  $X$  is a  $T_2$ -space then  $|X| \leq 2^{ac(X)H\psi(X)}$ .*

PROOF: Let  $\kappa = ac(X)H\psi(X)$ , and let  $S$  be a subset of  $X$  with  $|S| \leq 2^\kappa$  and witnessing that  $ac(X) \leq \kappa$ . For each  $x \in X$ , let  $\mathcal{B}_x$  an  $H$ -pseudobase of  $x$  in  $X$ , with  $|\mathcal{B}_x| \leq \kappa$ .

Construct a sequence  $\{A_\alpha : 0 \leq \alpha < \kappa^+\}$  of sets in  $X$  and a sequence  $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa^+\}$  of open collections in  $X$  such that

- (1)  $|A_\alpha| \leq 2^\kappa$ ;  $0 \leq \alpha < \kappa^+$ ;
- (2)  $\mathcal{V}_\alpha = \bigcup \{ \mathcal{B}_x : x \in \bigcup_{\beta < \alpha} A_\beta \}$ ;  $0 < \alpha < \kappa^+$ ;
- (3) if  $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$  is a collection ( $\lambda \leq \kappa$ ) of closed sets in  $X$  such that each  $C_\gamma$  has the form  $\overline{\bigcup \{ \bar{V} : V \in \mathcal{U}_\gamma \}}$ , where  $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq \kappa}$ , and if  $X - (S \cup \bigcup \mathcal{C}) \neq \emptyset$ , then  $A_\alpha - (S \cup \bigcup \mathcal{C}) \neq \emptyset$ .

The construction is by transfinite induction. Let  $0 < \alpha < \kappa^+$ , and assume that  $A_\beta$  and  $\mathcal{V}_\beta$  have been constructed for each  $\beta < \alpha$ . Note that  $\mathcal{V}_\alpha$  is defined by (2). For each collection  $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$  with  $\lambda \leq \kappa$  of closed sets in  $X$  such that each  $C_\gamma$  has the form  $\overline{\bigcup \{ \bar{V} : V \in \mathcal{U}_\gamma \}}$ , where  $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq \kappa}$ , and such that  $X \neq S \cup \bigcup \{C_\gamma : \gamma \in \lambda\}$ , choose one point in  $X - (S \cup \bigcup \{C_\gamma : \gamma \in \lambda\})$ . Let  $A_\alpha$  be the set of points chosen in this way. To show that  $|A_\alpha| \leq 2^\kappa$ , let  $F = \bigcup_{\beta < \alpha} A_\beta$ ; then  $\mathcal{V}_\alpha = \bigcup_{x \in F} \mathcal{B}_x$ , hence  $|\mathcal{V}_\alpha| \leq \sum_{x \in F} |\mathcal{B}_x| \leq \kappa \cdot |F| \leq \kappa \cdot \sum_{\beta \in \alpha} |A_\beta| = \kappa \cdot |\alpha| \cdot 2^\kappa = 2^\kappa$ . Since  $|A_\alpha| \leq |[\mathcal{V}_\alpha]^\kappa| \leq (2^\kappa)^\kappa = 2^\kappa$ , we have  $|A_\alpha| \leq 2^\kappa$ . This completes the construction.

Now let  $A = \bigcup_{\alpha < \kappa^+} A_\alpha$  and let  $\mathcal{U} = \bigcup \{ \mathcal{V}_\alpha : \alpha \in \kappa^+ \}$ ; clearly,  $|A| \leq 2^\kappa$ .

The proof is complete if  $X = (S \cup A)$ . Suppose not, and let  $p \in X - (S \cup A)$ . Let  $\mathcal{B} = \{B_\gamma : \gamma \in \lambda\}$  be a family of open neighbourhoods of  $p$  in  $X$ , such that  $\bigcap \{ \bar{B}_\gamma : \gamma \in \lambda \} = \{p\}$  with  $\lambda \leq \kappa$ . For each  $\gamma \in \lambda$ , let  $V_\gamma = X - \bar{B}_\gamma$  and let  $\mathcal{W}_\gamma = \{V \in \mathcal{U} : V \subseteq V_\gamma\}$ . Since  $ac(X) \leq \kappa$ , for each  $\gamma \in \lambda$  there exists  $\mathcal{U}_\gamma \in [\mathcal{W}_\gamma]^{\leq \kappa}$  such that  $\bigcup \mathcal{W}_\gamma \subseteq S \cup \overline{\bigcup \{ \bar{V} : V \in \mathcal{U}_\gamma \}}$ . Note that for each  $\gamma \in \lambda$ ,  $p \notin S \cup \overline{\bigcup \{ \bar{V} : V \in \mathcal{U}_\gamma \}}$ . Finally, let  $C_\gamma = \overline{\bigcup \{ \bar{V} : V \in \mathcal{U}_\gamma \}}$  for each  $\gamma \in \lambda$ . Since  $\mathcal{U}_\gamma \subseteq \mathcal{U}$  and  $|\mathcal{U}_\gamma| \leq \kappa$ , for all  $\gamma \in \lambda$ , by the regularity of  $\kappa^+$  there is an  $\alpha \in \kappa^+$  such that  $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$  is a collection of  $\leq \kappa$  closed sets in  $X$ ,

such that each  $C_\gamma$  has the form  $\overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ , where  $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq \kappa}$ . Moreover  $X - (S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}) \neq \emptyset$ , therefore, by (3),  $A_\alpha - (S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}) \neq \emptyset$ . Since  $A_\alpha \subseteq A \subseteq S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$ , we reach a contradiction. Thus  $X = S \cup A$  and  $|X| = |S \cup A| \leq 2^\kappa$ .  $\square$

**Theorem 3.** *If  $X$  is a  $T_2$ -space then  $|X| \leq 2^{\text{lc}(X)\pi\chi(X)\psi_c(X)}$ .*

PROOF: Let  $\kappa = \text{lc}(X)\pi\chi(X)\psi_c(X)$ , and let  $F$  be a closed set in  $X$  with  $|F| \leq 2^\kappa$  and witnessing that  $\text{lc}(X) \leq \kappa$ . For each  $x \in X$ , let  $\mathcal{V}_x$  a  $\pi$ -base local of  $x$  in  $X$  such that  $|\mathcal{B}_x| \leq \kappa$ .

Construct a sequence  $\{A_\alpha : \alpha \in \kappa^+\}$  of sets in  $X$  and a sequence  $\{\mathcal{B}_\alpha : \alpha \in \kappa^+\}$  of open collections in  $X$  such that:

- (1)  $\alpha \in \kappa^+$ ,  $|A_\alpha| \leq 2^\kappa$ ;  $0 \leq \alpha \leq \kappa^+$ ;
- (2)  $\mathcal{V}_\alpha = \bigcup\{\mathcal{B}_x : x \in \bigcup_{\beta < \alpha} A_\beta\}$ ;  $0 < \alpha < \kappa^+$ ;
- (3) if  $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$ , with  $\lambda \leq \kappa$ , is a collection of closed sets in  $X$ , where each  $C_\gamma$  has the form  $\overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ , where  $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq \kappa}$  and  $X - (F \cup \bigcup \mathcal{C}) \neq \emptyset$ , then  $A_\alpha - (F \cup \bigcup \mathcal{C}) \neq \emptyset$ .

The construction is by transfinite induction. Let  $0 < \alpha < \kappa^+$ , and assume that  $A_\beta$  and  $\mathcal{V}_\beta$  have been constructed for each  $\beta < \alpha$ . Note that  $\mathcal{V}_\alpha$  is defined by (2). Let  $P_\alpha = \bigcup_{\beta < \alpha} A_\beta$ ; we have  $\mathcal{V}_\alpha = \bigcup\{\mathcal{B}_x : x \in P_\alpha\}$ . Now, for each collection  $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$ ,  $\lambda \leq \kappa$ , of closed sets in  $X$  such that each  $C_\gamma$  has the form  $\overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ , where  $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq \kappa}$  and  $X \neq F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$ , choose one point in  $X - (F \cup \bigcup\{C_\gamma : \gamma \in \lambda\})$ . Let  $A_\alpha$  be the set of points chosen in this way. Observe that  $|A_\alpha| \leq |[[\mathcal{V}_\alpha]^{\leq \kappa}]^{\leq \kappa}| \leq 2^\kappa$ . This completes the construction.

Let  $A = \bigcup\{A_\alpha : \alpha \in \kappa^+\}$  and let  $\mathcal{U} = \bigcup\{\mathcal{V}_\alpha : \alpha \in \kappa^+\}$ . It is clear that  $|A| \leq 2^\kappa$ . The proof is complete if  $X = F \cup A$ . Assume, on the contrary, that  $p \in X - (F \cup A)$ , and consider  $\mathcal{V} = \{B_\gamma : \gamma \in \lambda\}$ , where  $\lambda \leq \kappa$ , a family of neighbourhoods of  $p$  in  $X$  such that  $\bigcap\{\bar{B}_\gamma : \gamma \in \lambda\} = \{p\}$ . For each  $\gamma \in \lambda$ , let  $V_\gamma = X - \bar{B}_\gamma$  and let  $\mathcal{W}_\gamma = \{V \subseteq V_\lambda : V \in \mathcal{U}\}$ . Since  $\text{lc}(X) \leq \kappa$  for each  $\gamma \in \lambda$ , there exists  $\mathcal{U}_\gamma \in [\mathcal{W}_\gamma]^{\leq \kappa}$  such that  $\bigcup \mathcal{W}_\gamma \subseteq F \cup \overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ . Observe that, for each  $\gamma \in \lambda$ ,  $p \notin F \cup \overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ . Let  $\mathcal{W} = \bigcup\{\mathcal{W}_\gamma : \gamma \in \lambda\}$ . Finally, for each  $\gamma \in \lambda$ , let  $C_\gamma = \overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ . Since  $\mathcal{U}_\gamma \subseteq \mathcal{U}$  and  $|\mathcal{U}_\gamma| \leq \kappa$  for all  $\gamma \in \lambda$ , then by the regularity of  $\kappa^+$  there exists  $\alpha \in \kappa^+$  such that  $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$  is a collection of  $\leq \kappa$  closed sets in  $X$  and each  $C_\gamma$  has the form  $\overline{\bigcup\{\bar{V} : V \in \mathcal{U}_\gamma\}}$ , where  $\mathcal{U}_\gamma \in [\bigcup\{\mathcal{V}_x : x \in A_\alpha\}]^{\leq \kappa}$ . Moreover,  $X - (F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}) \neq \emptyset$ , hence by (3),  $A_\alpha - (F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}) \neq \emptyset$ . Since  $A_\alpha \subseteq A \subseteq \bigcup \mathcal{W} \subseteq F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$ , we reach a contradiction. Thus  $X = F \cup A$ ; therefore  $|X| \leq 2^\kappa$ .  $\square$

Now we turn to the second part of this paper. Another well known cardinal inequality is due to Arhangel'skii [3]: "For  $X \in \mathcal{T}_2$ ,  $|X| \leq 2^{L(X)t(X)\psi(X)}$ ". Fedeli

[2] proved, making use of elementary submodels, that: if  $X$  is a  $T_2$ -space then  $|X| \leq 2^{\text{aql}(X)t(X)\psi_c(X)}$ . This result generalizes the Arhangel'skii's inequality. On the other hand, in [1], Arhangel'skii proved that: (a) "If  $X$  is a  $T_1$  space such that (i)  $L(X)t(X) \leq \kappa$ , (ii)  $\psi(X) \leq 2^\kappa$ , and (iii) for all  $A \in [X]^{\leq 2^\kappa}$ ,  $|\overline{A}| \leq 2^\kappa$ , then  $|X| \leq 2^{\kappa^\omega}$ ". From this result one easily obtains the Arhangel'skii's inequality mentioned above.

Since  $\text{aql}(X) \leq L(X)$  for every topological space  $X$ , it is natural to ask if  $L$  can be replaced by  $\text{aql}$  in the inequality (a). The next theorem gives an affirmative answer to this question. Our proof makes use of the Pol-Šapirovskii technique.

**Theorem 4.** *Let  $X$  be a  $T_1$ -space such that (i)  $\text{aql}(X)t(X) \leq \kappa$ , (ii)  $\psi(X) \leq 2^\kappa$ , and (iii) if  $A \in [X]^{\leq 2^\kappa}$  then  $|\overline{A}| \leq 2^\kappa$ . Then  $|X| \leq 2^\kappa$ .*

PROOF: Let  $S$  be an element of  $[X]^{\leq 2^\kappa}$  witnessing that  $\text{aql}(X) \leq \kappa$ . For each  $x \in X$ , let  $\mathcal{B}_x$  be a pseudobase of  $x$  in  $X$  such that  $|\mathcal{B}_x| \leq \kappa$ .

Construct an increasing sequence  $\{A_\alpha : \alpha \in \kappa^+\}$  of closed sets in  $X$  and a sequence  $\{\mathcal{V}_\alpha : \alpha \in \kappa^+\}$  of open collections in  $X$  such that

- (1)  $|A_\alpha| \leq 2^\kappa$ ,  $0 \leq \alpha < \kappa^+$ ;
- (2)  $\mathcal{V}_\alpha = \bigcup \{\mathcal{B}_x : x \in A_\alpha\}$ ;
- (3) if  $\mathcal{U} \subseteq \bigcup \{\mathcal{B}_x : x \in \text{cl}_X(\bigcup_{\beta < \alpha} A_\beta)\}$  with  $|\mathcal{U}| \leq \kappa$  and  $X - (S \cup \bigcup \mathcal{U}) \neq \emptyset$ , then  $A_\alpha - (S \cup \bigcup \mathcal{U}) \neq \emptyset$ .

The construction is by transfinite induction. Let  $0 < \alpha < \kappa^+$  and assume that  $A_\beta$  and  $\mathcal{V}_\beta$  have been constructed for each  $\beta \in \alpha$ . Note that  $\mathcal{V}_\alpha$  is defined by (2). Let  $P_\alpha = \text{cl}_X(\bigcup_{\beta < \alpha} A_\beta)$  and let  $\mathcal{C}_\alpha = \bigcup \{\mathcal{B}_x : x \in P_\alpha\}$ . Since  $|\bigcup_{\beta < \alpha} A_\beta| \leq 2^\kappa$ , it follows by (iii) that  $|P_\alpha| \leq 2^\kappa$ , hence,  $|\mathcal{C}_\alpha| \leq 2^\kappa$ . For each  $\mathcal{U} \subseteq \mathcal{C}_\alpha$  with  $|\mathcal{U}| \leq \kappa$  and  $X - (S \cup \bigcup \mathcal{U}) \neq \emptyset$ , choose one point in  $X - (S \cup \bigcup \mathcal{U})$ . Let  $L_\alpha$  be the set of points chosen in this way. Clearly  $|L_\alpha| \leq 2^\kappa$ . Let  $A_\alpha = \overline{P_\alpha \cup L_\alpha}$ . This completes the construction.

Let  $A = \bigcup \{A_\alpha : \alpha \in \kappa^+\}$  and note that  $A$  is closed in  $X$ ; moreover, clearly  $|A| \leq 2^\kappa$ . Let  $\mathcal{V} = \bigcup \{\mathcal{V}_\alpha : \alpha \in \kappa^+\}$ . The proof is complete if  $X = S \cup A$ . Suppose not, let  $p \in X - (S \cup A)$  and for each  $x \in A$ , choose  $V_x \in \mathcal{B}_x$  such that  $p \notin V_x$ . Then  $\{V_x : x \in A\}$  together with  $\{X - A\}$  cover  $X$ ; hence, there exists  $B \subseteq [A]^{\leq \kappa}$  such that  $X = S \cup (\bigcup \{V_x : x \in B\}) \cup (X - A)$ . Let  $U = \bigcup \{V_x : x \in B\}$ . Since  $|B| \leq \kappa$ , by the regularity of  $\kappa^+$  there exists  $\alpha \in \kappa^+$  such that  $\{V_x : x \in B\} \subseteq \bigcup \{\mathcal{B}_x : x \in \text{cl}_X(\bigcup_{\beta < \alpha} A_\beta)\}$ , that is  $U$  is the union of  $\leq \kappa$  elements of  $\bigcup \{\mathcal{B}_x : x \in \text{cl}_X(\bigcup_{\beta < \alpha} A_\beta)\}$  and  $X - (S \cup U) \neq \emptyset$ . Hence by (3),  $A_\alpha - (S \cup U) \neq \emptyset$ . Since  $A_\alpha \subseteq A \subseteq S \cup U$ , we reach a contradiction. Thus  $X = S \cup A$ .  $\square$

Now we have the inequality (a), as a consequence of our theorem.

**Corollary 5** (Arhangel'skii). *Let  $X$  be a  $T_1$ -space such that: (i)  $L(X)t(X) \leq \kappa$ , (ii)  $\psi(X) \leq 2^\kappa$ , and (iii) for all  $A \in [X]^{\leq 2^\kappa}$ ,  $|\overline{A}| \leq 2^\kappa$ . Then  $|X| \leq 2^\kappa$ .*

Another consequence of Theorem 5 is the next theorem due to Fedeli.

**Corollary 6.** *If  $X$  is a  $T_2$ -space then  $|X| \leq 2^{\text{aq}l(X)\psi_c(X)t(X)}$ .*

PROOF: Let  $\kappa = \text{aq}l(X)\psi_c(X)t(X)$ . It is enough to note that for all  $A \in [X]^{\leq 2^\kappa}$ ,  $|\overline{A}| \leq 2^\kappa$ .  $\square$

**Acknowledgments.** The author would like to thank the referee for many valuable suggestions and very careful corrections. I also want to thank my wife Homaira Athenea Ramírez Gutiérrez.

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(Received April 29, 2004, revised November 2, 2004)