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Ordinary selfdistributive rings

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Abstract. Left selfdistributive rings (i.e., xyz = xyxz) which are semidirect sums of boolean rings and rings nilpotent of index at most 3 are studied.

Keywords: ring, selfdistributive, ordinary Classification: 16399

1. Introduction

The present short note is an immediate continuation of [3] and the reader is referred to [3] as concerns terminology, notation, prerequisities, comments, further references, etc.

2. Preliminaries

In what follows, all rings are associative, possibly non–commutative and with or without unity.

If R is a ring, then Id(R) is the set of idempotent elements of R and Nl(R) that of nilpotent elements of R. The ring R will be called *id-generated*, if R is generated by the set Id(R) (as a ring). If A is a subset of R, then $(0:A)_l = \{r \in R \mid rA = 0\}$ and $(0:A)_r = \{r \in R \mid Ar = 0\}$. The subset A will be called *reduced*, if $A \cap Nl(R) \subseteq \{0\}$.

A ring R is called *left selfdistributive* (an LD-ring) if it satisfies the equation xyz = xyxz. An id-generated LD-ring will be called an ILD-ring in the sequel.

2.1 Proposition ([2]). Let R be an LD-ring, I = Id(R) and N = Nl(R). Then:

- (i) $a^3 = a^n \in I$ for all $a \in R$ and $n \ge 4$;
- (ii) 2abc = 0 for all $a, b, c \in R$ (i.e., $2R^3 = 0$);
- (iii) N is an ideal of R, $N^3 = 0$ and R/N is a boolean ring;
- (iv) $RN \cup NR \subseteq (0:R)_{l} \subseteq N$ and $NR \subseteq (0:R)_{r} \subseteq N$;
- (v) $S = R/(0:R)_1$ is a commutative ring satisfying $2S^2 = 0$ and $w^2 = w^3$, $w \in S$;
- (vi) $(0:I)_l = N;$

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- (vii) if R contains a right unity, then R is a boolean ring;
- (viii) $(0:R)_{\mathbf{r}}R \subseteq (0:R)_{\mathbf{l}} \cap (0:R)_{\mathbf{r}};$
- (ix) if $R = R^2$, then $(0:R)_r \subseteq (0:R)_l$.

2.2 Corollary. A ring R is an LD-ring if and only if R satisfies the equations xyz = yxz and $xyz = x^2yz$ (or $(x - x^2)yz = 0$).

2.3 Lemma. Let A be a generator set of an LD-ring R. Denote by K the ideal generated by $\{a - a^2 \mid a \in A\}$ and by L the left ideal generated by $\{ab - ba \mid a, b \in A\}$. Then K + L = Nl(R).

PROOF: Clearly, $J = K + L \subseteq Nl(R)$ and J is an ideal. On the other hand, the factor-ring S = R/J is generated by a set of pair-wise commuting idempotents. Consequently, S is a boolean ring and $Nl(R) \subseteq K$.

2.4 Remark. According to [2, 2.6], a subdirectly irreducible LD-ring is either nilpotent of index at most 3 or a two-element field or is isomorphic to the semigroup ring $\mathbb{Z}_2(T)$, T being a two-element semigroup of left units.

2.5 Lemma. Let R be an LD-ring. Then:

- (i) $(a a^2)^2 = a^2 a^3 = (a a^3)^2$ for every $a \in R$; (ii) $(a - a^2)^3 = (a - a^3)^2 = (a^2 - a^3)^2 = 0$ for every $a \in R$;
- (iii) $(a a^2)bc = 0 = (a a^3)bc$ for all $a, b, c \in R$;
- (iv) $(a^2 a^3)b = 0$ for all $a, b \in R$.

PROOF: Use 2.1(i) and 2.2.

2.6 Lemma. The following conditions are equivalent for an LD-ring R:

- (i) $\operatorname{Nl}(R) \subseteq (0:R)_{l};$
- (ii) $Nl(R) = (0:R)_l;$
- (iii) $(a + a^2)b = 0$ for all $a, b \in R$;
- (iv) $(a + a^3)b = 0$ for all $a, b \in R$.

PROOF: Clearly, (i) is equivalent to (ii) and (ii) implies (iii) by 2.5(ii).

If (iii) is true, then $0 = (a + a^3)b = (a + a^2)b + (a^2 + a^3)b = (a + a^2)b$, since $a^2b = a^3b$.

Finally, if (iv) is true, and $a \in Nl(R)$, then $a^3 = 0$ by 2.1(iii), and hence ab = 0 for every $b \in R$.

An LD-ring satisfying the equations $2x = 0 = (x + x^2)y$ will be called an SILD-ring.

2.7 Proposition. Let R be an LD-ring.

- (i) If $R = R^2$, then R is an SILD-ring.
- (ii) If R is an SILD-ring, then $Nl(R) = (0 : R)_l$ (i.e., Nl(R)R = 0) and $Nl(R)^2 = 0$.

PROOF: (i) Firstly, 2R = 0 follows from 2.1(ii). Next, by 2.2, $(a + a^2)bc = 0$ for all $a, b, c \in R$, and hence $(a + a^2)R = 0$. (ii) Use 2.6.

2.8 Corollary ([3]). Every ILD-ring is an SILD-ring.

2.9 Remark. It follows easily from [2, 2.6] that, up to isomorphism, the only subdirectly irreducible SILD-rings are the two-element field \mathbb{Z}_2 , the semigroup ring $\mathbb{Z}_2(T)$ and the zero-multiplication ring \mathbb{Z}_2^0 defined on $\mathbb{Z}_2(+)$. The field \mathbb{Z}_2 has a unity, the ring $\mathbb{Z}_2(T)$ has just two left unities and the ring \mathbb{Z}_2^0 is isomorphic to a subring of $\mathbb{Z}_2(T)$.

2.10 Proposition. Let an LD-ring R contain a left unity e. Then R is an ILD-ring.

PROOF: Denote by S the subring of R generated by the set Id(R). Clearly, R is an SILD-ring and, if $a \in Nl(R)$, then $(a + e)^2 = a^2 + ae + ea + e = a + e$ (use also 2.1(vi)), $a + e \in Id(R)$ and a = (a + e) + e. It follows that $Nl(R) \subseteq S$. Finally, if $b \in R$, then $b = b^2 + (b + b^2)$ and we have $b^2 \in Id(R) \subseteq S$ and $b + b^2 \in Nl(R) \subseteq Id(R)$.

2.11 Theorem. The following conditions are equivalent for an LD-ring R:

- (i) R is an SILD-ring;
- (ii) R is a subring of an LD-ring with left unity;
- (iii) R is a subring of an ILD-ring;
- (iv) R is a subring of an LD-ring S with $S = S^2$.

PROOF: (i) implies (ii) by 2.9, (ii) implies (iii) by 2.10, (iii) implies (iv) trivially and (iv) implies (i) by 2.8. \Box

3. Semidirect decompositions of LD-rings

3.1 Proposition ([3]). Let R be an LD-ring.

- (i) A subset A of R is a maximal reduced left ideal of R if and only if A is a maximal set of commuting idempotents.
- (ii) If A is a maximal reduced left ideal of R, then $(0: A)_1 = Nl(R)$, A + Nl(R) is an ideal and A contains every reduced right ideal of R.

3.2 Lemma ([3]). The following conditions are equivalent for a subset A of an LD-ring R:

- (i) A is a reduced subring of R and every reduced left ideal is in A + N, where N = Nl(R);
- (ii) A is a set of commuting idempotents and R = A + N;
- (iii) A is a maximal reduced left ideal of R and R = A + N.

Let R be an LD-ring. We will say that R is ordinary if R = A + Nl(R) for a (maximal) reduced left ideal A of R (see Lemma 3.2). Any such left ideal A will be called *critical*.

3.3 Theorem ([3]). Let R be an LD-ring and S = R/Nl(R). Then R is ordinary in each of the following cases:

- (i) R is countable;
- (ii) the (boolean) factor-ring S is countable;
- (iii) S is a ring direct sum of copies of the two-element field \mathbb{Z}_2 ;
- (iv) R possesses a left (or right) unity;
- (v) S possesses a unity;
- (vi) the subring of R generated by the set Id(R) is ordinary.

3.4 Remark. An example of a non-ordinary LD-ring R is given in [4] (see also [3, 5.2]). The ring R enjoys the properties $R = R^2$, 2R = 0, $Nl(R)^2 = 0$ and Nl(R)R = 0.

3.5 Example ([2]). Let A denote the set of sequences $\alpha = (\alpha(0), \alpha(1), \dots) \in \mathbb{Z}_2^{\omega}$ such that at least one of the sets $\operatorname{supp}(\alpha) = \{i \mid i < \omega, \alpha(i) \neq 0\}$ and $\omega \setminus \operatorname{supp}(\alpha)$ is finite. Then A is a boolean ring with unit and R is an LD-ring, where $R = A \times B$, $B = \{\alpha \mid |\operatorname{supp}(\alpha)| < \omega\}, (a, b) + (c, d) = (a + c, b + d)$ and (a, b)(c, d) = (ac, ad) for all $a, c \in A$ and $b, d \in B$. Moreover $I = \{(b, b) \mid b \in B\}$ is a maximal reduced left ideal of R and I is not critical.

3.6 Example ([3]). Let A be an uncountable set and let S' denote the set of ordered pairs (F, f), where $F \subseteq A$, $|F| \leq 2$ and $f \in F$. Put $S = S' \cup \{o\}$, $o \notin S'$, and define a multiplication on S by $(F, f)(G, g) = (F \cup G, g)$ if $|F \cup G| \leq 2$, (F, f)(G, g) = o otherwise and $\alpha o = o = o\alpha$ for every $\alpha \in S$. Then S becomes an idempotent semigroup satisfying xyz = yxz, o is an absorbing element of S and the corresponding contracted semigroup ring R of S over the two-element field \mathbb{Z}_2 is a non-ordinary LD-ring.

3.7 Remark. (i) The class of ordinary LD-rings is closed under homomorphic images.

(ii) According to [2], every subdirectly irreducible LD-ring is ordinary. Consequently, every LD-ring is a subring of an ordinary LD-ring.

PROOF: (i) Obvious.

(ii) The assertion follows immediately from [2, 2.6].

3.8 Corollary. Every LD-ring is a subring of an ordinary LD-ring.

3.9 Proposition. Let R be an LD-ring with a left unity e. Then R is ordinary and Re is a critical left ideal of R.

PROOF: For every $a \in R$, we have a = ae + (a + ae), where $ae \in Re$ and $a + ae \in Nl(R)$. Since $Re \subseteq Id(R)$, Re is a reduced left ideal which is critical. \Box

3.10 Theorem (cf. 2.11). The following conditions are equivalent for an LD-ring:

- (i) *R* is an ordinary SILD-ring;
- (ii) R is an ideal of an LD-ring S with left unity such that $S/R \simeq \mathbb{Z}_2$;
- (iii) R is a right ideal of an LD-ring with left unity.

PROOF: (i) implies (ii). We have R = A + N, where N = Nl(R) and A is a (maximal) reduced left ideal of R. Then $A \subseteq Id(R)$, $A \cap N = 0$ and we put f(a + w) = a for all $a \in A$ and $w \in N$. Now, d(a + w)f(b + v) = ab = f(ab + av) = f(a + w(b + w)) and $f(a + w) + a + w = 2a + w = w \in N = (0 : R)_1$ (2.7(ii)). It follows that f is a (ring) endomorphism of R such that Ker(f) = N, $Im(f) \subseteq Id(R)$, f(R) is a boolean ring, $f^2 = f$ and $f(x) + x \in (0 : R)_1$ for every $x \in R$.

Put $S = R \times \mathbb{Z}_2$ and define an addition and a multiplication on S by (a, i) + (b, j) = (a + b, i + j) and (a, i)(b, j) = (ab + jf(a) + ib, ij) for all $a, b \in R$ and $i, j \in \mathbb{Z}_2$. One verifies readily that S becomes an LD-ring and that the element (0,1) is a left unity of S.

(ii) implies (iii). This implication is trivial.

(iii) implies (i). It follows from 2.10 that R is an SILD-ring. By 3.7(i), R is ordinary.

4. Construction of ordinary selfdistributive rings

4.1 Construction (cf. [1]). Let A be a boolean ring and M an (associative) A-algebra nilpotent of index at most 3 and such that $AM^2 = 0$. Put $R = A \times M$ and define an addition and a multiplication on R by (a, u) + (b, v) = (a + b, u + v)and (a, u)(b, v) = (ab, av + uv). Then $R = \mathbb{R}(A, M)$ becomes an ordinary LD-ring, $Id(R) = \{(a, u) \mid au = u\}, NI(R) = \{(0, u)\} \simeq M$ (as A-algebras), $A_1 = \{(a, 0)\} \simeq$ A (as rings), $A \simeq R/NI(R)$ and $R = A_1 + NI(R)$. Put $M_1 = (0 : M)_{M,r} = \{u \in$ $M \mid Mu = 0\}$; clearly, M_1 is an ideal of M and $AM_1 \subseteq M_1$.

(i) Let B be an ideal of A and $\rho : B \to M_1$ an A-module homomorphism. Then the set $\{(b, \rho(b)) | b \in B\}$ is a reduced left ideal of R and every reduced left ideal of R is of this type.

(ii) Every maximal reduced left ideal of R is critical if and only if the A-module ${}_{A}M_{1}$ is injective with respect to all imbeddings $B \subseteq A, B$ being an ideal of A.

4.2 Theorem. (i) If A is a boolean ring and M an A-algebra with $M^3 = 0 = AM^2$, then $R = \mathbb{R}(A, M)$ is an ordinary LD-ring.

- (ii) Critical left ideals of R are just left ideals of the form $\{(a, \phi(a)) | a \in A\}, \phi:_A A \to_A M_1$ being a module homomorphism, $M_1 = (0:M)_{M,r}$.
- (iii) Every maximal reduced left ideal of R is critical if and only if the A-module ${}_AM_1$ is injective with respect to all imbeddings $B \subseteq A, B$ being an ideal of A.
- (iv) The A-algebras M and Nl(R) are isomorphic and the boolean rings A and R/Nl(R) are isomorphic.

(v) R is generated by Id(R) (as a ring) if and only if AM = M.

Proof: See 4.1.

4.3 Proposition. $\mathbb{R}(A, M) \simeq \mathbb{R}(B, N)$ if and only if there exist ring isomorphisms $\rho : A \to B$ and $\lambda : M \to N$ such that $\lambda(au) = \rho(a)\lambda(u)$ for all $a \in A$ and $u \in M$.

PROOF: An easy exercise.

4.4 Theorem. Let R be an ordinary LD-ring, N = Nl(R) and S = R/N. Then:

- (i) for every critical left ideal A, the mapping a → ā = a + N, a ∈ A, is a ring isomorphism of A onto S and N becomes both an A-algebra and S-algebra;
- (ii) $AN^{2} = 0 = N^{3}$ and $SN^{2} = 0 = N^{3}$;
- (iii) $R \simeq \mathbb{R}(A, N)$ and $R \simeq \mathbb{R}(S, N)$;
- (iv) every maximal reduced left ideal of R is critical if and only if the A-module (S-module, respectively) $_{A}(0:N)_{N,r}$ ($_{S}(0:N_{N,r})$, resp.) is injective with respect to all imbeddings $B \subseteq A$ ($T \subseteq S$, resp.), B being an ideal of A (T an ideal of S, resp.);
- (v) R is generated by Id(R) if and only if AN = N (or SN = N).

PROOF: Combine 2.1 and 4.2.

4.5 Corollary. There exists a one-to-one correspondence between ordinary LDrings and ordered pairs (A, M), A being a boolean ring and M an A-algebra with $M^3 = 0 = AM^2$.

4.6 Remark. Let A be a boolean ring with unit. An A-module M is injective with respect to all the inclusions $B \subseteq A$, B being an ideal of A, if and only if the submodule $M_1 = AM$ is an injective A-module. Notice also that M_1 is a unitary module.

5. Free left selfdistributive rings

5.1 Construction. Let X be a non-empty set and let E be the set of ordered triples (U, u, i), where U is a finite subset of X, $u \in X$ and either i = 0 or |U| = 1 and i = 1. Now, define a multiplication on E by $(U, u, i)(V, v, j) = (U \cup V \cup \{u\}, v, k)$, where k = 0 for $U \cup V \neq \{u\}$ and k = 1 for $U \cup V = \{u\}$. Then E becomes a free left permutable LD-semigroup freely generated by the set $\{(\emptyset, x, 0) | x \in X\}$.

5.2 Construction. Let F be a free left permutable LD-semigroup freely generated by a non-empty set X and let $S = \mathbb{Z}[F]$ be the corresponding semigroup ring of S over the ring \mathbb{Z} of integers. Clearly, S is a left permutable ring. Now, put $G = F \setminus \{x, xy \mid x, y \in X\}$, denote by I the ideal of S generated by 2G (notice

 \square

that G is a subsemigroup of F) and by R the corresponding factor-ring S/I; let $\phi: S \to R$ be the natural projection. Clearly, $\phi \mid F$ is injective and we will identify F with its image $\phi(F)$. Now, it is easy to check that R is a free LD-ring over X. Every element of R can be written in a unique way as a sum

$$\sum_{x \in X} n_x + \sum_{x,y \in X} n_{(x,y)} + \sum_{g \in G} k_g$$

where $n_x, n_{(x,y)} \in \mathbb{Z}, k_g \in \mathbb{Z}_2 = \{0, 1\}$ and only finitely many of these coefficients are non-zero. It follows from 3.3(i) and [3, 5.2] that R is ordinary if and only if X is countable.

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