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## The maximal regular ideal of some commutative rings

EMAD ABU OSBA, MELVIN HENRIKSEN, OSAMA ALKAM, F.A. SMITH

*Abstract.* In 1950 in volume 1 of Proc. Amer. Math. Soc., B. Brown and N. McCoy showed that every (not necessarily commutative) ring  $R$  has an ideal  $\mathfrak{M}(R)$  consisting of elements  $a$  for which there is an  $x$  such that  $axa = a$ , and maximal with respect to this property. Considering only the case when  $R$  is commutative and has an identity element, it is often not easy to determine when  $\mathfrak{M}(R)$  is not just the zero ideal. We determine when this happens in a number of cases: Namely when at least one of  $a$  or  $1 - a$  has a von Neumann inverse, when  $R$  is a product of local rings (e.g., when  $R$  is  $\mathbb{Z}_n$  or  $\mathbb{Z}_n[i]$ ), when  $R$  is a polynomial or a power series ring, and when  $R$  is the ring of all real-valued continuous functions on a topological space.

*Keywords:* commutative rings, von Neumann regular rings, von Neumann local rings, Gelfand rings, polynomial rings, power series rings, rings of Gaussian integers (mod  $n$ ), prime and maximal ideals, maximal regular ideals, pure ideals, quadratic residues, Stone-Ćech compactification,  $C(X)$ , zerosets, cozerosets,  $P$ -spaces

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### 1. Introduction

Throughout  $R$  will denote a commutative ring with identity element 1 unless the contrary is stated explicitly, and the notation of [AHA04] will be followed.

**1.1 Definition.** An element  $a \in R$  is called *regular* if there is a  $b \in R$  such that  $a = a^2b$ . Let  $\text{vr}(R) = \{a \in R : a \text{ is regular}\}$  and  $\text{nvr}(R) = R \setminus \text{vr}(R)$ . An ideal  $I$  of  $R$  is called a *regular ideal* if  $I \subset \text{vr}(R)$ . The element  $a$  is called *m-regular* if the ideal generated by  $a$  is a regular ideal. Let  $\mathfrak{M}(R) = \{a \in R : a \text{ is m-regular}\}$ . A ring  $R$  is called *von Neumann regular ring* (VNR ring) if  $R = \text{vr}(R)$ .

This terminology is motivated in part by a theorem of Brown and McCoy in which they show that  $\mathfrak{M}(R)$  is a regular ideal. Indeed it is the largest regular ideal or  $R$ . See [BM50].  $R$  may contain regular elements which are not m-regular, as one can see easily that  $3 \in \text{vr}(\mathbb{Z}_4) \setminus \mathfrak{M}(\mathbb{Z}_4)$ . (As usual,  $\mathbb{Z}_n$  denotes the ring  $\mathbb{Z}$  of integers mod  $n$  for a positive integer  $n$ .)

If  $S \subset R$ , then  $\text{Ann}(S)$  denotes  $\{a \in R : aS = \{0\}\}$ , the set of maximal ideals of  $R$  is denoted by  $\text{Max}(R)$ , and their intersection  $J(R)$  is the Jacobson radical of  $R$ . In [BM50], the following is also established.

**1.2 Lemma.**

$$\begin{aligned} \mathfrak{M}(R/\mathfrak{M}(R)) &= \{0\}. \\ \mathfrak{M}(R) \cap J(R) &= \{0\}. \\ \mathfrak{M}(R) &\subset \text{Ann}(J(R)). \\ \mathfrak{M}(R) \cap \text{Ann}(\mathfrak{M}(R)) &= \{0\}. \end{aligned}$$

If  $R/J(R)$  is VNR-ring, then  $\mathfrak{M}(R) = \{0\}$  if and only if  $\text{Ann}(J(R)) \subset J(R)$ .

If  $R$  satisfies the descending chain condition on ideals, then  $R = \mathfrak{M}(R) + \text{Ann}(\mathfrak{M}(R))$ .

For each ideal  $I$  of  $R$ , let  $mI = \{a \in I : a \in aI\} = \{a \in R : I + \text{Ann}(a) = R\}$ . Then  $mI$  is called the *pure part* of  $I$ . An ideal  $I$  is called a *pure ideal* if  $I = mI$ . It is clear that  $a \in mM$  for an  $M \in \text{Max}(R)$ , if and only if  $\text{Ann}(a)$  is not contained in  $M$ .

The following description of  $\mathfrak{M}(R)$  will be used frequently below.

**1.3 Theorem.** *If  $R$  is not a von Neumann regular ring, then  $\mathfrak{M}(R) = \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\}$  is the intersection of the pure parts of those maximal ideals  $M$  of  $R$  that are not pure.*

PROOF: If  $a \notin \mathfrak{M}(R)$ , then there is an  $x \in R$  such that  $ax \notin \text{vr}(R)$ . So by Theorem 2.4 of [AHA04], there is an  $N \in \text{Max}(R)$  such that  $ax \in N \setminus mN$ . It follows that  $N$  is not pure and  $a \notin \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\}$ . Thus  $\bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\} \subset \mathfrak{M}(R)$ .

If instead  $a \in \mathfrak{M}(R)$  and there is an  $M \in \text{Max}(R)$  and an  $x \in M \setminus mM$ , then  $ax \in mM$  and so as noted above, there is a  $b \notin M$  such that  $ba = 0$ . So  $ba \in \text{Ann}(x)$  which is contained in  $M$  because this maximal ideal is not pure. But  $M$  is a prime ideal, so  $a \in M$ . Thus  $\mathfrak{M}(R) \subset mM$ . Hence  $\mathfrak{M}(R) \subset \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\}$ .  $\square$

In this article, we determine when  $\mathfrak{M}(R)$  is not the zero ideal for a number of classes of rings. In Section 2, we study rings in which at least one of  $a$  or  $1 - a$  has a von Neumann inverse. Section 3 is devoted to the study of products of local rings (e.g., the ring  $\mathbb{Z}_n$  of integers modulo an integer  $n \geq 2$  and to  $\mathbb{Z}_n[i]$ ). The complicated conditions needed to describe when  $\mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\}$  hint at why it may be quite difficult to describe when the maximal regular ideal of a finite ring is nonzero. In Section 4, it is shown that the maximal regular ideal of a polynomial or powers series ring is the zero ideal, and in Section 5, it is determined when the maximal regular ideal of the ring of all continuous functions on a topological space is nonzero.

**2. Von Neumann local and strong von Neumann local rings**

Recall from [AHA04] that  $R$  is called a *von Neumann local* (VNL) ring if  $a \in \text{vr}(R)$  or  $1 - a \in \text{vr}(R)$  for each  $a \in R$ . It is easy to see that VNR rings and local rings are VNL rings.  $R$  is called a *strong von Neumann local* (SVNL) ring if

whenever the ideal  $\langle S \rangle$  generated by a subset  $S$  of  $R$  is all of  $R$ , then some element of  $S$  is in  $\text{vr}(R)$ , or equivalently if  $\langle \text{vnr}(R) \rangle \neq R$ . Clearly every SVN $L$  ring is a VNL ring, but the validity of the converse remains an open problem.  $R$  is called a *Gelfand ring* or a *PM ring* if each of its proper prime ideals is contained in a unique maximal ideal. If  $M$  is a maximal ideal of  $R$ , then  $O_M$  denotes intersection of all of the (minimal) prime ideals of  $R$  that are contained in  $M$ .

**2.1 Lemma.** *Every VNL ring  $R$  is a Gelfand ring and if  $R$  is also reduced, then  $mM = O_M$  whenever  $M \in \text{Max}(R)$ .*

PROOF: The first assertion is shown in [C84]. (Combine in that paper Proposition 4.4, Theorems 3.2 and 2.4 with Proposition 1.1.) The second assertion is shown in Proposition 3 of [H77].  $\square$

See also [DO71].

Next, we make use of Theorem 1.1 above.

In Theorem 2.6 of [AHA04] it is shown that  $R$  is an SVN $L$  ring that is not a VNR ring if and only if it has exactly one maximal ideal that fails to be pure. Combining this with Theorem 1.3 yields:

**2.2 Theorem.** *If  $R$  is an SVN $L$  ring that is not a VNR ring, then it has a unique maximal  $N$  that is not pure. Moreover  $\mathfrak{M}(R) = mN = O_M$ .*

PROOF: The first assertion is part of Theorem 2.6 of [AHA04], and the second is immediate from Theorem 1.3 and Lemma 2.1.  $\square$

Next we begin to exhibit a class of rings whose maximal regular ideal is not the zero ideal.

**2.3 Lemma.** *If  $R$  and  $S$  are commutative rings with identity whose direct sum  $R \oplus S$  is a VNL ring, then at least one of  $R$  and  $S$  is a VNR ring.*

PROOF: Suppose instead that there are  $r \in R$  and  $s \in S$  that are not von Neumann regular. Then neither  $(r, 1 - s)$  nor  $(1, 1) - (r, 1 - s) = (1 - r, s)$  are von Neumann regular in  $R \oplus S$ , so the conclusion follows.  $\square$

**2.4 Theorem.** *If  $R$  is a VNL ring that is neither local nor a VNR ring, then  $\mathfrak{M}(R)$  contains  $fR$  for some idempotent  $f$  not in  $\{0, 1\}$  and hence is not the zero ideal.*

PROOF: By Theorem 4.6 of [AHA04], a nonlocal VNL ring has an idempotent  $e \notin \{0, 1\}$ , so  $R = eR \oplus (1 - e)R$ . Thus by Lemma 2.3, exactly one of these two summands must be a VNR ring, which is a nonzero ideal included in  $\mathfrak{M}(R)$ .  $\square$

### 3. Products of local rings

In this section, it will be determined when a direct product of local rings has a nonzero maximal regular ideal.

It is an exercise to show that a local VNR ring is a field. Moreover, if  $M$  is the unique maximal ideal of  $R$ , and  $a = am \in mM$  for some  $m \in M$ , then  $a = 0$  since  $1 - m$  is invertible. Because each element of  $\mathfrak{M}(R)$  is in  $mM$ , we conclude from Theorem 1.3 that:

**3.1 Lemma.** *If  $R$  is a local ring, then  $R$  is a field or  $\mathfrak{M}(R) = \{0\}$ .*

**3.2 Lemma.** *If  $R = \prod_{i \in I} R_i$  is the direct product of rings  $R_i$  with identity, then*

- (1)  $(r_i)_{i \in I} \in \text{vr}(R)$  if and only if  $r_i \in \text{vr}(R_i)$  for each  $i \in I$ , and
- (2)  $(r_i)_{i \in I} \in \mathfrak{M}(R)$  if and only if  $r_i \in \mathfrak{M}(R_i)$  for each  $i \in I$ .

PROOF: (1)  $(r_i)_{i \in I} \in \text{vr}(R)$  if and only if there exists  $(x_i)_{i \in I} \in R$  such that  $(r_i)_{i \in I} = ((r_i)_{i \in I})^2 (x_i)_{i \in I} = (r_i^2 x_i)_{i \in I}$  if and only if  $r_i = r_i^2 x_i$  for each  $i \in I$  and only if  $r_i \in \text{vr}(R_i)$  for each  $i \in I$ .

(2) Suppose that  $(r_i)_{i \in I} \in \mathfrak{M}(R)$ . Pick  $r_k \in R_k$  and let  $x \in R_k$ .

$$\text{Define } x_i = \begin{cases} x & i=k \\ 0 & i \neq k \end{cases}.$$

Now,  $(r_i)_{i \in I} (x_i)_{i \in I} \in \text{vr}(R)$ , so there exists  $(y_i)_{i \in I} \in R$  such that  $(r_i)_{i \in I} (x_i)_{i \in I} = ((r_i)_{i \in I} (x_i)_{i \in I})^2 (y_i)_{i \in I} = ((r_i x_i)^2 y_i)_{i \in I}$ . In particular  $r_k x = (r_k x)^2 y_k$ . Thus  $r_k \in \mathfrak{M}(R_k)$ . Conversely, suppose that  $r_i \in \mathfrak{M}(R_i)$  for each  $i \in I$ . Let  $(x_i)_{i \in I} \in R$ . Then  $r_i x_i \in \text{vr}(R_i)$  for each  $i \in I$ , which implies that there exists  $y_i \in R_i$  such that  $r_i x_i = (r_i x_i)^2 y_i$  for each  $i \in I$ . Hence  $(r_i)_{i \in I} (x_i)_{i \in I} = ((r_i x_i)^2 y_i)_{i \in I} = ((r_i)_{i \in I} (x_i)_{i \in I})^2 (y_i)_{i \in I}$  which implies that  $(r_i)_{i \in I} \in \mathfrak{M}(R)$ .  $\square$

It follows that:

**3.3 Theorem.** *If  $R = \prod_{i \in I} R_i$  is the direct product of rings  $R_i$  with identity, then  $\mathfrak{M}(R) = \prod_{i \in I} \mathfrak{M}(R_i)$ .*

Because a local VNR ring is a field and if  $R$  is a field, then  $R = \mathfrak{M}(R)$ , it follows that:

**3.4 Corollary.** *If  $R = \prod_{i \in I} R_i$  is the direct product of local rings  $R_i$  with identity, then  $\mathfrak{M}(R) \neq \{0\}$  if and only if  $R_j$  is a field for at least one  $j \in I$ .*

In Chapter VI of [M74], it is shown that every finite commutative ring with identity element is a direct product of local rings. Hence we have

**3.5 Theorem.** *If  $R$  is finite, then  $\mathfrak{M}(R) \neq \{0\}$  if and only if  $R$  is a direct product of local rings at least one of which is a field.*

Much more is said about finite local rings in [M74]. If  $R$  is such a ring then its unique maximal ideal  $M$  is nilpotent and  $\mathfrak{M}(R) = \{0\}$  by Lemma 3.1. Indeed, every element of  $R$  is either nilpotent or invertible.

Next, some examples are considered.

It is well known that if  $n > 1$  is in  $\mathbb{Z}$ , then  $\mathbb{Z}_n$  is local if and only if  $n = p^k$  for some prime  $p$  and positive integer  $k$ , and is a field if and only if  $k = 1$ .

**3.6 Corollary.** *If  $n = \prod_{i=1}^s p_i^{k_i}$  is the prime power decomposition of the positive integer  $n$ , then  $\mathbb{Z}_n$  is the direct product of the local rings  $\mathbb{Z}_{p_i^{k_i}}$  and  $\mathfrak{M}(R) \neq \{0\}$  if and only if  $k_j = 1$  for at least one  $j \in \{1, \dots, s\}$ .*

**3.7 Definition.** If  $i^2 = -1$  and  $Z[i] = \{a + ib : a, b \in Z\}$  is the ring of Gaussian integers, then for any integer  $n > 1$ ,  $\mathbb{Z}_n[i] = \mathbb{Z}[i]/n\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}_n\}$  denotes the ring of *Gaussian integers mod  $n$* .

**3.8 Lemma.** (a) *If an element  $a + ib$  of  $\mathbb{Z}_n[i]$  is nilpotent [resp. idempotent] then  $a^2 + b^2$  is nilpotent [resp. idempotent] in  $\mathbb{Z}_n$ .*  
 (b)  *$a + ib$  is a unit in  $\mathbb{Z}_n[i]$  if and only if  $a^2 + b^2$  is a unit of  $\mathbb{Z}_n$ .*  
 (c)  *$(a + ib)^2 = a + ib$  is a nontrivial idempotent if and only if  $a^2 - b^2 = a$  and  $2ab = b$  in  $\mathbb{Z}_n$  and neither  $a$  nor  $b$  is zero in  $\mathbb{Z}_n$ .*

PROOF: (a) If  $a + ib$  is nilpotent, then so is  $(a - ib)(a + ib) = a^2 + b^2$  because complex conjugation is an automorphism of  $\mathbb{Z}_n[i]$ . The proof for idempotents is similar.

(b) follows because  $(a - ib)(a + ib) = a^2 + b^2$  and any divisor of a unit is a unit.  
 (c) is an exercise. □

As in Corollary 3.6, if  $n = \prod_{i=1}^s p_i^{k_i}$  is the prime power decomposition of the positive integer  $n$ , then  $\mathbb{Z}_n[i]$  is the direct product of the rings  $\mathbb{Z}_{p_i^{k_i}}[i]$ . So by Theorem 3.3,  $\mathfrak{M}(\mathbb{Z}_n[i]) = \prod_{i=1}^s \mathfrak{M}(\mathbb{Z}_{p_i^{k_i}}[i]) \neq \{0\}$  if and only if at least one of the ideals in this latter product is nonzero. This motivates the question:

(\*) If  $p$  and  $k$  are positive integers and  $p$  is prime, when is  $\mathfrak{M}(\mathbb{Z}_{p^k}[i]) \neq \{0\}$ ?

While it is true that  $\mathbb{Z}_n$  is a local ring whenever  $n$  is a power of a prime, this is not the case for  $\mathbb{Z}_n[i]$  as will be shown next. Recall that if a ring  $R$  is finite, then  $R$  is local if and only if its only idempotents are 0 and 1 (which are called *trivial idempotents*).

**3.9 Theorem.** *If  $m = p^k$  for some prime  $p$  and positive integer  $k$ , then  $\mathbb{Z}_m[i]$  is local if and only if  $p = 2$  or  $p \equiv -1 \pmod{4}$ .*

PROOF: We will show that if  $a + ib$  is a nontrivial idempotent of  $\mathbb{Z}_m[i]$ , then

- (i)  $2a \equiv 1 \pmod{p^k}$ , and
- (ii) there is a  $c$  such that  $c^2 \equiv -1 \pmod{p^k}$ .

To see (i), recall from Lemma 3.8 that if  $a + ib$  is a nontrivial idempotent, then  $a^2 - b^2 = a$  and  $2ab = b$  in  $\mathbb{Z}_m$  and neither  $a$  nor  $b$  is  $0 \pmod{p^k}$ . This latter equation says  $b(2a - 1) \equiv 0 \pmod{p^k}$ . By Lemma 3.8,  $a^2 + b^2$  is an idempotent in  $\mathbb{Z}_m$  and hence is congruent to 0, so if  $p \mid b$ , then  $p \mid a$ . It follows that  $p^2 \mid b$  because  $2ab = b$ . A routine induction yields  $p^k \mid b$  and hence that  $b \equiv 0 \pmod{p^k}$ ; contrary to the assumption that  $a + ib$  is a nontrivial idempotent. Hence  $p$  is not a divisor of  $b$ , i.e.  $b$  is a unit in  $\mathbb{Z}_m$ , but  $b(2a - 1) \equiv 0 \pmod{p^k}$ . So (i) holds.

This shows that there are no nontrivial idempotents in  $\mathbb{Z}_{2^k}[i]$ . So this ring is local and is never a field because it contains the nonzero nilpotent ideal  $(1+i)\mathbb{Z}_{2^k}[i]$ . Thus  $\mathfrak{M}(\mathbb{Z}_{2^k}) = \{0\}$  for all  $k$ .

Assume next that  $p$  is odd and note that by (i) and its proof  $(2b)^2 = 4(a^2 - a) \equiv (2a)^2 - 2(2a) = (p^k + 1)^2 - 2(p^k + 1) \equiv -1 \pmod{p^k}$ . So  $c = 2b$  is the solution of the equation in (ii). Thus  $\mathbb{Z}_m[i]$  has a nontrivial idempotent exactly when the equation in (ii) has a solution in which case  $\frac{1}{2} + i\frac{c}{2}$  is such an idempotent.

It is noted in Chapter 5 of [L58] that for  $p$  odd, the congruence  $c^2 \equiv -1 \pmod{p^k}$  has a solution, i.e.  $-1$  is a quadratic residue mod  $p^k$ , when  $p$  is odd if and only if it has one for  $k = 1$ . It is shown that  $-1$  is a quadratic residue mod  $p$  if and only if  $p \equiv 1 \pmod{4}$ . This completes the proof of the theorem.  $\square$

For a more thorough discussion of the topic of the last paragraph, see Section 5.8 of [L58].

**3.10 Corollary.** *If  $p$  is an odd prime, then  $\mathbb{Z}_p[i]$  is a VNR ring.*

PROOF: If  $p \equiv -1 \pmod{4}$ , then  $\mathbb{Z}_p[i]$  is a field because by Theorem 7.2 of [L58], the congruence  $a^2 + b^2 \equiv 0 \pmod{p}$  has no solution.

Assume next that  $p \equiv 1 \pmod{4}$ . It follows by Theorem 3.9 that  $\mathbb{Z}_p[i]$  is not local, thus  $\mathbb{Z}_p[i]$  (which has  $p^2$  elements) is product of exactly two local rings, each isomorphic to  $\mathbb{Z}_p$ . Hence  $\mathbb{Z}_p[i]$  is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$  a product of two VNR rings.  $\square$

**3.11 Corollary.** *If  $m = p^k$  for some odd prime  $p$  and positive integer  $k$ , then  $\mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\}$  if and only if  $k = 1$ .*

PROOF: As noted in the proof of Theorem 3.9,  $\mathfrak{M}(\mathbb{Z}_{2^k}[i]) = \{0\}$  for all  $k$ . By the last corollary, if  $p$  is an odd prime and  $k = 1$ , then  $\mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\}$ .

Now if  $k > 1$  and  $p \equiv -1 \pmod{4}$  or if  $p = 2$ , then by Theorem 3.9,  $\mathbb{Z}_m[i]$  is a local ring which is not a field. So  $\mathfrak{M}(\mathbb{Z}_m[i]) = \{0\}$  by Lemma 3.1.

If  $k > 1$ ,  $p \equiv 1 \pmod{4}$ , and  $a+ib$  is a nonunit of  $\mathbb{Z}_m[i]$ , then  $a^2 + b^2 \equiv 0 \pmod{p}$ . If  $p \mid a$ , or  $p \mid b$ , then  $p$  divides the other, so  $p \mid (a+ib)$ . Thus  $a+ib$  is a nonzero nilpotent element of  $\mathbb{Z}_m[i]$  since  $k > 1$ . If, instead  $p$  fails to divide  $a$  or  $b$ , then it is easy to verify that  $p(a+ib)$  is a nonzero nilpotent in  $\mathbb{Z}_m[i]$ . Thus no nonzero nonunit of  $R$  can be m-regular, and the existence of the nonzero nilpotent ideal  $pR$  shows that no unit of  $\mathbb{Z}_m[i]$  can be m-regular. Hence  $\mathfrak{M}(\mathbb{Z}_m[i]) = \{0\}$  and the proof is complete.  $\square$

In summary we have using Theorem 3.3 and the above:

**3.12 Corollary.** *If  $n = \prod_{i=1}^s p_i^{k_i}$  is the prime power decomposition of the positive integer  $n$ , then  $\mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\}$  if and only if  $p_j$  is an odd prime and  $k_j = 1$  for at least one  $j \in \{1, \dots, s\}$ .*

#### 4. Polynomial and power series rings

For each ring  $R$ , we write the *polynomial ring* as  $R[x] = \{\sum_{i=0}^n a_i x^i : a_i \in R\}$  and the *power series ring* by  $R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$  where addition is coefficientwise, and in each case  $(\sum a_i x^i)(\sum b_j x^j) = \sum c_k x^k$ , where  $c_k = \sum_{i+j=k} a_i b_j$ . The coefficient of  $x^k$  in  $c(x) = \sum c_k x^k$  is denoted by  $c_k$ . Both of these rings are commutative and have an identity. The next lemma is well known. See the first set of exercises in [AM69] and Section 1 of [B81].

**4.1 Lemma.** (a)  $u(x)$  is invertible in  $R[x]$  if and only if  $u_0$  is invertible and the coefficient of each nonzero power of  $x$  is nilpotent.

(b)  $u(x)$  is invertible in  $R[[x]]$  if and only if  $u_0$  is invertible in  $R$ .

Note that if  $e^2 = e$  is an idempotent, then  $(1 - 2e)^2 = 1$ , so:

**4.2 Lemma.** If  $e$  is an idempotent, then  $(1 - 2e)$  is a unit of  $R$ .

We combine these two lemmas to obtain:

**4.3 Lemma.** If  $a(x)$  is an idempotent in  $R[x]$  or  $R[[x]]$ , then  $a(x) = a_0 \in R$ .

PROOF: If  $a(x) = \sum_{i=0}^{\infty} a_i x^i$  and  $a(x) = (a(x))^2$ , then  $\sum_{i+j=n} a_i a_j = a_n$  for  $n = 0, 1, 2, \dots$ . If  $n = 0$ , then  $a_0 = a_0^2$ , so  $(1 - 2a_0)$  is a unit by the last lemma. Equating coefficients of  $x$  yields  $a_1(1 - 2a_0) = 0$ , which implies that  $a_1 = 0$ . Doing the same with the coefficients of  $x^2$  yields  $a_2(1 - 2a_0) = -a_1 a_1 = 0$ , which implies that  $a_2 = 0$ . Proceeding inductively, if  $a_1 = a_2 = \dots = a_{n-1} = 0$ , then  $a_n(1 - 2a_0) = -\sum_{i+j=n} a_i a_j = 0$ . Thus  $a_n = 0$  for each  $n \geq 1$  and hence  $a(x) = a_0 \in R$ .  $\square$

We now characterize von Neumann regular elements in  $R[x]$  and  $R[[x]]$ . In the proof of the next theorem, we need the fact that if  $a$  is a von Neumann regular element of a commutative ring, then there is unit  $u$  such that  $a^2 u = a$ , and hence that  $au$  is an idempotent. See, for example [AHA04].

**4.4 Theorem.** Let  $a(x) = \sum_{i=0}^n a_i x^i$ . Then  $a(x)$  is von Neumann regular in  $R[x]$  if and only if  $a(x)$  is a product of a von Neumann regular element in  $R$  and a unit in  $R[x]$ .

PROOF: If  $a(x) \in \text{vr}(R[x])$ , then there exists a unit  $u(x) = \sum_{i=0}^m u_i x^i \in R[x]$  such that  $a(x) = (a(x))^2 u(x)$ . Hence by Lemmas 4.1 and 4.3, we have

(iii)  $a(x)u(x) = a_0 u_0 = (a_0 u_0)^2$  and

(iv)  $\sum_{i+j=k} a_i u_j = 0$  for  $k = 1, 2, 3, \dots, n$ .

By Lemma 4.1,  $u_j$  is nilpotent if  $j \geq 1$  and by the equation in (iv) for  $k = 1$ ,  $a_1 = -u_0^{-1} a_0 u_1$ , which implies that  $a_1$  is nilpotent. Similarly,  $a_2 = -u_0^{-1} (a_0 u_2 + a_1 u_1)$ , which implies that  $a_2$  is nilpotent. Proceeding inductively, if  $a_1, a_2, \dots, a_{n-1}$  are nilpotents, then  $a_n = -u_0^{-1} \sum_{i+j=n} a_i u_j$ . So  $a_k$  is nilpotent

for each  $k \geq 1$ , while  $a_0 \in \text{vr}(R)$  and  $a(x) = a(x)a(x)u(x) = a(x)a_0u_0$ . Let  $v(x) = u_0 + a_1u_0^2x + a_2u_0^2x^2 + \dots$  and note that it is a unit of  $R[x]$  by Lemma 4.1. Then:

$$\begin{aligned} a(x) &= \sum_{i=0}^n a_i a_0 u_0 x^i = a_0^2 u_0 + a_1 a_0 u_0 x + a_2 a_0 u_0 x^2 + \dots \\ &= a_0^2 u_0 + a_1 a_0^2 u_0^2 x + a_2 a_0^2 u_0^2 x^2 + \dots = a_0^2 v(x) \end{aligned}$$

is the product of an element of  $\text{vr}(R)$  and a unit of  $R[x]$ .

The converse is clear.  $\square$

A similar argument will establish:

**4.5 Theorem.** *If  $a(x) = \sum_{i=0}^{\infty} a_i x^i$ , then  $a(x)$  is von Neumann regular in  $R[[x]]$  if and only if  $a(x)$  is a product of a von Neumann regular element in  $R$  and a unit in  $R[[x]]$ .*

By the last two theorems,  $xa(x) \in \text{vr}(R[x])$  implies  $a(x) = \mathbf{0}$ , so we conclude this section with:

**4.6 Corollary.** *For each ring  $R$ ,  $\mathfrak{M}(R[x]) = \{0\}$  and  $\mathfrak{M}(R[[x]]) = \{0\}$ .*

## 5. The ring $C(X)$

All topological spaces  $X$  are assumed to be Tychonoff spaces,  $\beta X$  the Stone-Ćech compactification of  $X$  and  $C(X)$  will denote the algebra of continuous real-valued functions under the usual pointwise operations. For each  $f \in C(X)$ , we denote the *zeroset* of  $f$  by  $Z(f) = \{x \in X : f(x) = 0\}$ , and the *cozeroset*  $\text{coz}(f) = X - Z(f)$ . A point  $p \in X$  such that for every  $f \in C(X)$ ,  $f(p) = 0$  implies  $p \in \text{int } Z(f)$  is called a *P-point*, and  $X$  is called a *P-space* if each of its points is a *P-point*. If  $x \in \beta X$ , let  $M^x = \{f \in C(X) : x \in \text{cl}_{\beta X} Z(f)\}$  and  $O^x = \{f \in C(X) : x \in \text{int}_{\beta X} [\text{cl}_{\beta X} Z(f)]\}$ . The notation and terminology of [GJ76] is used. In this section we will characterize m-regular elements in  $C(X)$ , we will find for what spaces  $X$ ,  $\mathfrak{M}(C(X))$  contains non zero elements.

Recall from Section 2 that  $R$  is a VNL ring if for each  $a \in R$ , one of  $a$  or  $1 - a$  is von Neumann regular.

The next proposition is established in [AHA04] and in [GJ76].

**5.1 Proposition.** (a)  *$C(X)$  is a VNR ring if and only if  $X$  is a *P-space* if and only if every  $G_\delta$ -set of  $X$  is open.*

(b)  *$C(X)$  is VNL ring if and only if at most one point of  $X$  is not a *P-point* (in which case  $X$  is said to be essentially a *P-space*).*

The next simple lemma will be used below.

**5.2 Lemma.** *If  $f \in \text{vr}(C(X))$ , then  $Z(f)$  is clopen.*

PROOF: As is noted just above Theorem 4.4, there is a unit  $u$  in  $C(X)$  such that  $f = f(fu)$  and  $fu$  is idempotent. Because the zeroset of an idempotent is clopen, the conclusion follows.  $\square$

Thus we obtain:

**5.3 Theorem.** *A function  $f$  is in  $\mathfrak{M}(C(X)) \setminus \{0\}$  if and only if  $\text{coz}(f)$  is a nonempty clopen  $P$ -space.*

PROOF: Suppose that  $f \in \mathfrak{M}(C(X)) \setminus \{0\}$ , then  $f \in \text{vr}(C(X))$  and so  $\text{coz}(f)$  is a nonempty clopen set by Lemma 5.2. Let  $G = \bigcap_{n=1}^{\infty} G_n$  be a  $G_\delta$ -set of  $X$  contained in  $\text{coz}(f)$  and suppose  $x \in G$ . For each  $n$  there exists  $g_n \in C(X)$  such that  $g_n(x) = 0$  and  $g_n(X \setminus G_n) = 1$ . Let  $g = \sum_{n=1}^{\infty} (|g_n|/2^n)$ , then  $g \in C(X)$  and  $Z(g) = G \subset \text{coz}(f)$ . Since  $fg \in \text{vr}(C(X))$ , its zeroset is clopen by Lemma 5.2. So, because  $Z(fg) = Z(f) \cup Z(g)$ ,  $Z(f) \cap Z(g) = \emptyset$ , and  $Z(f)$  is clopen, it follows that  $Z(g)$  and hence  $\text{coz}(g)$  is clopen. Thus, by Proposition 5.1,  $\text{coz}(f)$  is a  $P$ -space.

Suppose conversely that  $\text{coz}(f)$  is a nonempty clopen  $P$ -space. Then  $C(X)$  is the direct product of  $C(\text{coz}(f))$  and  $C(Z(f))$ , so  $f \in \mathfrak{M}(C(X)) \setminus \{0\}$ .  $\square$

**5.4 Corollary.**  *$\mathfrak{M}(C(X)) \neq \{0\}$  if and only if  $X$  contains a nonempty clopen  $P$ -space.*

By making use of Theorem 1.3, we can describe  $\mathfrak{M}(C(X))$  more precisely.

If  $Y$  is a subset of  $X$ , we let  $O^Y = \bigcap_{y \in Y} O^y$ . Let  $P(X)$  be the set of all  $P$ -points in  $X$ , then it is clear that  $O^{X-P(X)} = \bigcap_{y \notin P(X)} O^y \subseteq \text{vr}(C(X))$  and so,  $O^{X-P(X)} \subseteq \mathfrak{M}(C(X))$ . For each  $x \in \beta X$ ,  $mM^x = O^x$ , using this together with Theorem 1.3 above we conclude that:

**5.5 Corollary.**  *$\mathfrak{M}(C(X)) = O^{X-P(X)}$  for any space  $X$ .*

We conclude with an interesting example.

**5.6 Example.** Let  $X_1 = (0, 1)$  with its usual topology and  $X_2 = \mathbb{N}$  with its discrete topology. Let  $X = X_1 \oplus X_2$  and define  $f(x) = \begin{cases} 0 & x \in X_1 \\ 1 & x \in X_2 \end{cases}$ , then  $f \in \mathfrak{M}(C(X)) \setminus \{0\}$ , while  $C(X)$  is not a VNR ring.

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