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The maximal regular ideal of some commutative rings

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Abstract. In 1950 in volume 1 of Proc. Amer. Math. Soc., B. Brown and N. McCoy showed that every (not necessarily commutative) ring $R$ has an ideal $\mathfrak{M}(R)$ consisting of elements $a$ for which there is an $x$ such that $axa = a$, and maximal with respect to this property. Considering only the case when $R$ is commutative and has an identity element, it is often not easy to determine when $\mathfrak{M}(R)$ is not just the zero ideal. We determine when this happens in a number of cases: Namely when at least one of $a$ or $1 - a$ has a von Neumann inverse, when $R$ is a product of local rings (e.g., when $R$ is $\mathbb{Z}_n$ or $\mathbb{Z}_n[i]$), when $R$ is a polynomial or a power series ring, and when $R$ is the ring of all real-valued continuous functions on a topological space.

Keywords: commutative rings, von Neumann regular rings, von Neumann local rings, Gelfand rings, polynomial rings, power series rings, rings of Gaussian integers (mod $n$), prime and maximal ideals, maximal regular ideals, pure ideals, quadratic residues, Stone-Czech compactification, $C(X)$, zero sets, cozero sets, $P$-spaces

Classification: 13A, 13FXX, 54G10, 10A10

1. Introduction

Throughout $R$ will denote a commutative ring with identity element 1 unless the contrary is stated explicitly, and the notation of [AHA04] will be followed.

1.1 Definition. An element $a \in R$ is called regular if there is a $b \in R$ such that $a = a^2 b$. Let $\mathfrak{v}(R) = \{ a \in R : a$ is regular $\}$ and $\mathfrak{n}(R) = R \setminus \mathfrak{v}(R)$. An ideal $I$ of $R$ is called a regular ideal if $I \subseteq \mathfrak{v}(R)$. The element $a$ is called $m$-regular if the ideal generated by $a$ is a regular ideal. Let $\mathfrak{M}(R) = \{ a \in R : a$ is $m$-regular $\}$. A ring $R$ is called von Neumann regular ring (VNR ring) if $R = \mathfrak{v}(R)$.

This terminology is motivated in part by a theorem of Brown and McCoy in which they show that $\mathfrak{M}(R)$ is a regular ideal. Indeed it is the largest regular ideal of $R$. See [BM50]. $R$ may contain regular elements which are not $m$-regular, as one can see easily that $3 \in \mathfrak{v}(\mathbb{Z}_4) \setminus \mathfrak{M}(\mathbb{Z}_4)$. (As usual, $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}$ of integers mod $n$ for a positive integer $n$.)

If $S \subset R$, then $\text{Ann}(S)$ denotes $\{ a \in R : aS = \{0\} \}$, the set of maximal ideals of $R$ is denoted by $\text{Max}(R)$, and their intersection $J(R)$ is the Jacobson radical of $R$. In [BM50], the following is also established.
1.2 Lemma.

\[ \mathcal{M}(R) \cap J(R) = \{0\}. \]
\[ \mathcal{M}(R) \cap \text{Ann}(J(R)) = \{0\}. \]
\[ \mathcal{M}(R) \subset \text{Ann}(J(R)). \]
\[ \mathcal{M}(R) \cap \text{Ann}(\mathcal{M}(R)) = \{0\}. \]

If \( R/J(R) \) is VNR-ring, then \( \mathcal{M}(R) = \{0\} \) if and only if \( \text{Ann}(J(R)) \subset J(R) \).

If \( R \) satisfies the descending chain condition on ideals, then \( R = \mathcal{M}(R) + \text{Ann}(\mathcal{M}(R)) \).

For each ideal \( I \) of \( R \), let \( mI = \{a \in I : a \in aI\} = \{a \in R : I + \text{Ann}(a) = R\} \). Then \( mI \) is called the pure part of \( I \). An ideal \( I \) is called a pure ideal if \( I = mI \). It is clear that \( a \in mM \) for an \( M \in \text{Max}(R) \), if and only if \( \text{Ann}(a) \) is not contained in \( M \).

The following description of \( \mathcal{M}(R) \) will be used frequently below.

1.3 Theorem. If \( R \) is not a von Neumann regular ring, then \( \mathcal{M}(R) = \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\} \) is the intersection of the pure parts of those maximal ideals \( M \) of \( R \) that are not pure.

Proof: If \( a \notin \mathcal{M}(R) \), then there is an \( x \in R \) such that \( ax \notin \text{vr}(R) \). So by Theorem 2.4 of [AHA04], there is an \( N \in \text{Max}(R) \) such that \( ax \in N \setminus mN \). It follows that \( N \) is not pure and \( a \notin \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\} \). Thus \( \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\} \subset \mathcal{M}(R) \).

If instead \( a \in \mathcal{M}(R) \) and there is an \( M \in \text{Max}(R) \) and an \( x \in M \setminus mM \), then \( ax \in mM \) and so as noted above, there is a \( b \notin M \) such that \( bx = 0 \). So \( ba \in \text{Ann}(x) \) which is contained in \( M \) because this maximal ideal is not pure. But \( M \) is a prime ideal, so \( a \in M \). Thus \( \mathcal{M}(R) \subset mM \). Hence \( \mathcal{M}(R) \subset \bigcap \{mM : M \in \text{Max}(R) \text{ and } M \neq mM\} \). \( \square \)

In this article, we determine when \( \mathcal{M}(R) \) is not the zero ideal for a number of classes of rings. In Section 2, we study rings in which at least one of \( a \) or \( 1 - a \) has a von Neumann inverse. Section 3 is devoted to the study of products of local rings (e.g., the ring \( Z_n \) of integers modulo an integer \( n \geq 2 \) and to \( Z_n[i] \)). The complicated conditions needed to describe when \( \mathcal{M}(Z_n[i]) \neq \{0\} \) hint at why it may be quite difficult to describe when the maximal regular ideal of a finite ring is nonzero. In Section 4, it is shown that the maximal regular ideal of a polynomial or powers series ring is the zero ideal, and in Section 5, it is determined when the maximal regular ideal of the ring of all continuous functions on a topological space is nonzero.

2. Von Neumann local and strong von Neumann local rings

Recall from [AHA04] that \( R \) is called a von Neumann local (VNL) ring if \( a \in \text{vr}(R) \) or \( 1 - a \in \text{vr}(R) \) for each \( a \in R \). It is easy to see that VNR rings and local rings are VNL rings. \( R \) is called a strong von Neumann local (SVNL) ring if
whenever the ideal \( \langle S \rangle \) generated by a subset \( S \) of \( R \) is all of \( R \), then some element of \( S \) is in \( \text{vr}(R) \), or equivalently if \( (\text{vr}(R)) \) \( \neq R \). Clearly every SVNL ring is a VNL ring, but the validity of the converse remains an open problem. \( R \) is called a Gelfand ring or a PM ring if each of its proper prime ideals is contained in a unique maximal ideal. If \( M \) is a maximal ideal of \( R \), then \( O_M \) denotes intersection of all of the (minimal) prime ideals of \( R \) that are contained in \( M \).

2.1 Lemma. Every VNL ring \( R \) is a Gelfand ring and if \( R \) is also reduced, then \( mm = O_M \) whenever \( M \in \text{Max}(R) \).

Proof: The first assertion is shown in [C84]. (Combine in that paper Proposition 4.4, Theorems 3.2 and 2.4 with Proposition 1.1.) The second assertion is shown in Proposition 3 of [H77].

See also [D071].

Next, we make use of Theorem 1.1 above.

In Theorem 2.6 of [AHA04] it is shown that \( R \) is an SVNL ring that is not a VNR ring if and only if it has exactly one maximal ideal that fails to be pure. Combining this with Theorem 1.3 yields:

2.2 Theorem. If \( R \) is an SVNL ring that is not a VNR ring, then it has a unique maximal \( N \) that is not pure. Moreover \( \mathfrak{M}(R) = mN = O_M \).

Proof: The first assertion is part of Theorem 2.6 of [AHA04], and the second is immediate from Theorem 1.3 and Lemma 2.1.

Next we begin to exhibit a class of rings whose maximal regular ideal is not the zero ideal.

2.3 Lemma. If \( R \) and \( S \) are commutative rings with identity whose direct sum \( R \oplus S \) is a VNL ring, then at least one of \( R \) and \( S \) is a VNR ring.

Proof: Suppose instead that there are \( r \in R \) and \( s \in S \) that are not von Neumann regular. Then neither \( (r,1-s) \) nor \( (1,1-(r,1-s)) = (1-r,s) \) are von Neumann regular in \( R \oplus S \), so the conclusion follows.

2.4 Theorem. If \( R \) is a VNL ring that is neither local nor a VNR ring, then \( \mathfrak{M}(R) \) contains \( fR \) for some idempotent \( f \) not in \( \{0,1\} \) and hence is not the zero ideal.

Proof: By Theorem 4.6 of [AHA04], a nonlocal VNL ring has an idempotent \( e \notin \{0,1\} \), so \( R = eR \oplus (1-e)R \). Thus by Lemma 2.3, exactly one of these two summands must be a VNR ring, which is a nonzero ideal included in \( \mathfrak{M}(R) \).

3. Products of local rings

In this section, it will be determined when a direct product of local rings has a nonzero maximal regular ideal.
It is an exercise to show that a local VNR ring is a field. Moreover, if \( M \) is
the unique maximal ideal of \( R \), and \( a = am \in mM \) for some \( m \in M \), then \( a = 0 \)
since \( 1 - m \) in invertible. Because each element of \( \mathcal{M}(R) \) is in \( mM \), we conclude
from Theorem 1.3 that:

3.1 Lemma. If \( R \) is a local ring, then \( R \) is a field or \( \mathcal{M}(R) = \{ 0 \} \).

3.2 Lemma. If \( R = \prod_{i \in I} R_i \) is the direct product of rings \( R_i \) with identity, then
\begin{enumerate}
  \item \( (r_i)_{i \in I} \in \text{vr}(R) \) if and only if \( r_i \in \text{vr}(R_i) \) for each \( i \in I \), and
  \item \( (r_i)_{i \in I} \in \mathcal{M}(R) \) if and only if \( r_i \in \mathcal{M}(R_i) \) for each \( i \in I \).
\end{enumerate}

Proof: (1) \( (r_i)_{i \in I} \in \text{vr}(R) \) if and only if there exists \( (x_i)_{i \in I} \in R \) such that
\( (r_i)_{i \in I} = ((r_i)_{i \in I})(x_i)_{i \in I} = (r_i^2x_i)_{i \in I} \) if and only if \( r_i = r_i^2x_i \) for each \( i \in I \) if
and only if \( r_i \in \text{vr}(R_i) \) for each \( i \in I \).

(2) Suppose that \( (r_i)_{i \in I} \in \mathcal{M}(R) \). Pick \( r_k \in R_k \) and let \( x \in R_k \).
Define \( x_i = \begin{cases} x & i = k \\ 0 & i \neq k \end{cases} \).

Now, \( (r_i)_{i \in I} \in \text{vr}(R) \), so there exists \( (y_i)_{i \in I} \in R \) such that
\( (r_i)_{i \in I} = ((r_i)_{i \in I})(x_i)_{i \in I} = ((r_i^2x_i)_{i \in I} \). In particular \( r_kx = (r_kx)^2y_k \). Thus
\( r_k \in \mathcal{M}(R_k) \). Conversely, suppose that \( r_i \in \mathcal{M}(R_i) \) for each \( i \in I \). Let \( (x_i)_{i \in I} \in R \). Then
\( r_i \in \text{vr}(R_i) \) for each \( i \in I \), which implies that there exists \( y_i \in R_i \) such that
\( r_i = r_i^2y_i \) for each \( i \in I \). Hence \( (r_i)_{i \in I} = ((r_i^2y_i)_{i \in I} = ((r_i)_{i \in I}(x_i)_{i \in I})^2(y_i)_{i \in I} \) which implies that \( (r_i)_{i \in I} \in \mathcal{M}(R) \). \( \square \)

It follows that:

3.3 Theorem. If \( R = \prod_{i \in I} R_i \) is the direct product of rings \( R_i \) with identity, then
\( \mathcal{M}(R) = \prod_{i \in I} \mathcal{M}(R_i) \).

Because a local VNR ring is a field and if \( R \) is a field, then \( R = \mathcal{M}(R) \), it follows that:

3.4 Corollary. If \( R = \prod_{i \in I} R_i \) is the direct product of local rings \( R_i \) with
identity, then \( \mathcal{M}(R) \neq \{ 0 \} \) if and only if \( R_j \) is a field for at least one \( j \in I \).

In Chapter VI of [M74], it is shown that every finite commutative ring with
identity element is a direct product of local rings. Hence we have

3.5 Theorem. If \( R \) is finite, then \( \mathcal{M}(R) \neq \{ 0 \} \) if and only if \( R \) is a direct
product of local rings at least one of which is a field.

Much more is said about finite local rings in [M74]. If \( R \) is such a ring then
its unique maximal ideal \( M \) is nilpotent and \( \mathcal{M}(R) = \{ 0 \} \) by Lemma 3.1. Indeed,
every element of \( R \) is either nilpotent or invertible.

Next, some examples are considered.

It is well known that if \( n > 1 \) is in \( \mathbb{Z} \), then \( \mathbb{Z}_n \) is local if and only if \( n = p^k \)
for some prime \( p \) and positive integer \( k \), and is a field if and only if \( k = 1 \).
3.6 Corollary. If \( n = \prod_{i=1}^{s} p_i^{k_i} \) is the prime power decomposition of the positive integer \( n \), then \( \mathbb{Z}_n \) is the direct product of the local rings \( \mathbb{Z}_{p_i^{k_i}} \) and \( \mathfrak{M}(R) \neq \{0\} \) if and only if \( k_j = 1 \) for at least one \( j \in \{1, \ldots, s\} \).

3.7 Definition. If \( i^2 = -1 \) and \( \mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\} \) is the ring of Gaussian integers, then for any integer \( n > 1 \), \( \mathbb{Z}_n[i] = \mathbb{Z}[i]/n\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}_n\} \) denotes the ring of Gaussian integers mod \( n \).

3.8 Lemma. (a) If an element \( a + ib \) of \( \mathbb{Z}_n[i] \) is nilpotent [resp. idempotent] then \( a^2 + b^2 \) is nilpotent [resp. idempotent] in \( \mathbb{Z}_n \).
(b) \( a + ib \) is a unit in \( \mathbb{Z}_n[i] \) if and only if \( a^2 + b^2 \) is a unit of \( \mathbb{Z}_n \).
(c) \( (a + ib)^2 = a + ib \) is a nontrivial idempotent if and only if \( a^2 - b^2 = a \) and \( 2ab = b \) in \( \mathbb{Z}_n \) and neither \( a \) nor \( b \) is zero in \( \mathbb{Z}_n \).

Proof: (a) If \( a + ib \) is nilpotent, then so is \( (a - ib)(a + ib) = a^2 + b^2 \) because complex conjugation is an automorphism of \( \mathbb{Z}_n[i] \). The proof for idempotents is similar.
(b) follows because \( (a - ib)(a + ib) = a^2 + b^2 \) and any divisor of a unit is a unit.
(c) is an exercise. \( \Box \)

As in Corollary 3.6, if \( n = \prod_{i=1}^{s} p_i^{k_i} \) is the prime power decomposition of the positive integer \( n \), then \( \mathbb{Z}_n[i] \) is the direct product of the rings \( \mathbb{Z}_{p_i^{k_i}}[i] \). So by Theorem 3.3, \( \mathfrak{M}(\mathbb{Z}_n[i]) = \prod_{i=1}^{s} \mathfrak{M}(\mathbb{Z}_{p_i^{k_i}}[i]) \neq \{0\} \) if and only if at least one of the ideals in this latter product is nonzero. This motivates the question:

(*) If \( p \) and \( k \) are positive integers and \( p \) is prime, when is \( \mathfrak{M}(\mathbb{Z}_{p^k}[i]) \neq \{0\} \)?

While it is true that \( \mathbb{Z}_n \) is a local ring whenever \( n \) is a power of a prime, this is not the case for \( \mathbb{Z}_n[i] \) as will be shown next. Recall that if a ring \( R \) is finite, then \( R \) is local if and only if its only idempotents are 0 and 1 (which are called trivial idempotents).

3.9 Theorem. If \( m = p^k \) for some prime \( p \) and positive integer \( k \), then \( \mathbb{Z}_m[i] \) is local if and only if \( p = 2 \) or \( p \equiv -1(\mod 4) \).

Proof: We will show that if \( a + ib \) is a nontrivial idempotent of \( \mathbb{Z}_m[i] \), then
(i) \( 2a \equiv 1(\mod p^k) \), and
(ii) there is a \( c \) such that \( c^2 \equiv -1(\mod p^k) \).

To see (i), recall from Lemma 3.8 that if \( a + ib \) is an idempotent, then \( a^2 - b^2 = a \) and \( 2ab = b \) in \( \mathbb{Z}_m \) and neither \( a \) nor \( b \) is \( 0(\mod p^k) \). This latter equation says \( b(2a - 1) \equiv 0(\mod p^k) \). By Lemma 3.8, \( a^2 + b^2 \) is an idempotent in \( \mathbb{Z}_m \) and hence is congruent to 0, so if \( p \mid b \), then \( p \mid a \). It follows that \( p^2 \mid b \) because \( 2ab = b \). A routine induction yields \( p^k \mid b \) and hence that \( b \equiv 0(\mod p^k) \); contrary to the assumption that \( a + ib \) is a nontrivial idempotent. Hence \( p \) is not a divisor of \( b \), i.e. \( b \) is a unit in \( \mathbb{Z}_m \), but \( b(2a - 1) \equiv 0(\mod p^k) \). So (i) holds.
This shows that there are no nontrivial idempotents in \( \mathbb{Z}_{2^k} [i] \). So this ring is local and is never a field because it contains the nonzero nilpotent ideal \((1 + i)\mathbb{Z}_{2^k} [i]\). Thus \( \mathfrak{M}(\mathbb{Z}_{2^k}) = \{0\} \) for all \( k \).

Assume next that \( p \) is odd and note that by (i) and its proof \((2b)^2 = 4(a^2 - a) \equiv (2a)^2 - 2(2a) = (p^k + 1)^2 - 2(p^k + 1) \equiv -1 (\text{mod } p^k)\). So \( c = 2b \) is the solution of the equation in (ii). Thus \( \mathbb{Z}_m[i] \) has a nontrivial idempotent exactly when the equation in (ii) has a solution in which case \( \frac{1}{2} + i\frac{c}{2} \) is such an idempotent.

It is noted in Chapter 5 of [L58] that for \( p \) odd, the congruence \( c^2 \equiv -1 (\text{mod } p^k) \) has a solution, i.e. \(-1\) is a quadratic residue mod \( p^k \), when \( p \) is odd if and only if it has one for \( k = 1 \). It is shown that \(-1\) is a quadratic residue mod \( p \) if and only if \( p \equiv 1 (\text{mod } 4) \). This completes the proof of the theorem. \( \square \)

For a more thorough discussion of the topic of the last paragraph, see Section 5.8 of [L58].

**3.10 Corollary.** If \( p \) is an odd prime, then \( \mathbb{Z}_p[i] \) is a VNR ring.

**Proof:** If \( p \equiv -1 (\text{mod } 4) \), then \( \mathbb{Z}_p[i] \) is a field because by Theorem 7.2 of [L58], the congruence \( a^2 + b^2 \equiv 0 (\text{mod } p) \) has no solution.

Assume next that \( p \equiv 1 (\text{mod } 4) \). It follows by Theorem 3.9 that \( \mathbb{Z}_p[i] \) is not local, thus \( \mathbb{Z}_p[i] \) (which has \( p^2 \) elements) is product of exactly two local rings, each isomorphic to \( \mathbb{Z}_p \). Hence \( \mathbb{Z}_p[i] \) is isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \) a product of two VNR rings. \( \square \)

**3.11 Corollary.** If \( m = p^k \) for some odd prime \( p \) and positive integer \( k \), then \( \mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\} \) if and only if \( k = 1 \).

**Proof:** As noted in the proof of Theorem 3.9, \( \mathfrak{M}(\mathbb{Z}_{p^k}[i]) = \{0\} \) for all \( k \). By the last corollary, if \( p \) is an odd prime and \( k = 1 \), then \( \mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\} \).

Now if \( k > 1 \) and \( p \equiv -1 (\text{mod } 4) \) or if \( p = 2 \), then by Theorem 3.9, \( \mathbb{Z}_m[i] \) is a local ring which is not a field. So \( \mathfrak{M}(\mathbb{Z}_m[i]) = \{0\} \) by Lemma 3.1.

If \( k > 1 \), \( p \equiv 1 (\text{mod } 4) \), and \( a + ib \) is a nonunit of \( \mathbb{Z}_m[i] \), then \( a^2 + b^2 \equiv 0 (\text{mod } p) \). If \( p \mid a \), or \( p \mid b \), then \( p \) divides the other, so \( p \mid (a + ib) \). Thus \( a + ib \) is a nonzero nilpotent element of \( \mathbb{Z}_m[i] \) since \( k > 1 \). If, instead \( p \) fails to divide \( a \) or \( b \), then it is easy to verify that \( p(a + ib) \) is a nonzero nilpotent in \( \mathbb{Z}_m[i] \). Thus no nonzero nonunit of \( R \) can be \( m \)-regular, and the existence of the nonzero nilpotent ideal \( pR \) shows that no unit of \( \mathbb{Z}_m[i] \) can be \( m \)-regular. Hence \( \mathfrak{M}(\mathbb{Z}_m[i]) = \{0\} \) and the proof is complete. \( \square \)

In summary we have using Theorem 3.3 and the above:

**3.12 Corollary.** If \( n = \prod_{i=1}^{s} p_i^{k_i} \) is the prime power decomposition of the positive integer \( n \), then \( \mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\} \) if and only if \( p_j \) is an odd prime and \( k_j = 1 \) for at least one \( j \in \{1, \ldots, s\} \).
4. Polynomial and power series rings

For each ring $R$, we write the polynomial ring as $R[x] = \{ \sum_{i=0}^{n} a_i x^i : a_i \in R \}$ and the power series ring by $R[[x]] = \{ \sum_{i=0}^{\infty} a_i x^i : a_i \in R \}$ where addition is coefficientwise, and in each case $(\sum a_i x^i)(\sum b_j x^j) = \sum c_k x^k$, where $c_k = \sum_{i+j=k} a_i b_j$. The coefficient of $x^k$ in $c(x) = \sum c_k x^k$ is denoted by $c_k$. Both of these rings are commutative and have an identity. The next lemma is well known. See the first set of exercises in [AM69] and Section 1 of [B81].

4.1 Lemma. (a) $u(x)$ is invertible in $R[x]$ if and only if $u_0$ is invertible and the coefficient of each nonzero power of $x$ is nilpotent.

(b) $u(x)$ is invertible in $R[[x]]$ if and only if $u_0$ is invertible in $R$.

Note that if $e^2 = e$ is an idempotent, then $(1 - 2e)^2 = 1$, so:

4.2 Lemma. If $e$ is an idempotent, then $(1 - 2e)$ is a unit of $R$.

We combine these two lemmas to obtain:

4.3 Lemma. If $a(x)$ is an idempotent in $R[x]$ or $R[[x]]$, then $a(x) = a_0 \in R$.

Proof: If $a(x) = \sum_{i=0}^{\infty} a_i x^i$ and $a(x) = (a(x))^2$, then $\sum_{i+j=n} a_i a_j = a_n$ for $n = 0, 1, 2, \ldots$. If $n = 0$, then $a_0 = a_0^2$, so $(1 - 2a_0)$ is a unit by the last lemma. Equating coefficients of $x$ yields $a_1(1 - 2a_0) = 0$, which implies that $a_1 = 0$. Doing the same with the coefficients of $x^2$ yields $a_2(1 - 2a_0) = -a_1 a_1 = 0$, which implies that $a_2 = 0$. Proceeding inductively, if $a_1 = a_2 = \cdots = a_{n-1} = 0$, then $a_n(1 - 2a_0) = -\sum_{i+j=n} a_i a_j = 0$. Thus $a_n = 0$ for each $n \geq 1$ and hence $a(x) = a_0 \in R$. 

We now characterize von Neumann regular elements in $R[x]$ and $R[[x]]$. In the proof of the next theorem, we need the fact that if $a$ is a von Neumann regular element of a commutative ring, then there is unit $u$ such that $a^2 u = a$, and hence that $au$ is an idempotent. See, for example [AHA04].

4.4 Theorem. Let $a(x) = \sum_{i=0}^{n} a_i x^i$. Then $a(x)$ is von Neumann regular in $R[x]$ if and only if $a(x)$ is a product of a von Neumann regular element in $R$ and a unit in $R[x]$.

Proof: If $a(x) \in vR(R[x])$, then there exists a unit $u(x) = \sum_{i=0}^{m} u_i x^i \in R[x]$ such that $a(x) = (a(x))^2 u(x)$. Hence by Lemmas 4.1 and 4.3, we have

(iii) $a(x) u(x) = a_0 u_0 = (a_0 u_0)^2$ and
(iv) $\sum_{i+j=k} a_i u_j = 0$ for $k = 1, 2, 3, \ldots, n$.

By Lemma 4.1, $u_j$ is nilpotent if $j \geq 1$ and by the equation in (iv) for $k = 1$, $a_1 = -u_0^{-1} a_0 u_1$, which implies that $a_1$ is nilpotent. Similarly, $a_2 = -u_0^{-1} (a_0 u_2 + a_1 u_1)$, which implies that $a_2$ is nilpotent. Proceeding inductively, if $a_1, a_2, \ldots, a_{n-1}$ are nilpotents, then $a_n = -u_0^{-1} \sum_{i+j=n} a_i u_j$. So $a_k$ is nilpotent.
for each $k \geq 1$, while $a_0 \in \text{vr}(R)$ and $a(x) = a(x)a(x)u(x) = a(x)a_0 u_0$. Let $v(x) = u_0 + a_1 a_0 u_0 x + a_2 a_0 u_0 x^2 + \cdots$ and note that it is a unit of $R[x]$ by Lemma 4.1. Then:

$$
 a(x) = \sum_{i=0}^{n} a_i a_0 u_0 x^i = a_0^2 u_0 + a_1 a_0 u_0 x + a_2 a_0 u_0 x^2 + \cdots \\
= a_0^2 u_0 + a_1 a_0 u_0^2 x + a_2 a_0 u_0^2 x^2 + \cdots = a_0^2 v(x)
$$

is the product of an element of $\text{vr}(R)$ and a unit of $R[x]$.

The converse is clear. \hfill \Box

A similar argument will establish:

4.5 Theorem. If $a(x) = \sum_{i=0}^{\infty} a_i x^i$, then $a(x)$ is von Neumann regular in $R[[x]]$ if and only if $a(x)$ is a product of a von Neumann regular element in $R$ and a unit in $R[[x]]$.

By the last two theorems, $xa(x) \in \text{vr}(R[x])$ implies $a(x) = 0$, so we conclude this section with:

4.6 Corollary. For each ring $R$, $\mathfrak{M}(R[x]) = \{0\}$ and $\mathfrak{M}(R[[x]]) = \{0\}$.

5. The ring $C(X)$

All topological spaces $X$ are assumed to be Tychonoff spaces, $\beta X$ the Stone-Čech compactification of $X$ and $C(X)$ will denote the algebra of continuous real-valued functions under the usual pointwise operations. For each $f \in C(X)$, we denote the zeroset of $f$ by $Z(f) = \{ x \in X : f(x) = 0 \}$, and the cozeroset $\text{coz}(f) = X - Z(f)$. A point $p \in X$ such that for every $f \in C(X)$, $f(p) = 0$ implies $p \in \text{int} Z(f)$ is called a $P$-point, and $X$ is called a $P$-space if each of its points is a $P$-point. If $x \in \beta X$, let $M^x = \{ f \in C(X) : x \in \text{cl}_{\beta X} Z(f) \}$ and $O^x = \{ f \in C(X) : x \in \text{int}_{\beta X} [\text{cl}_{\beta X} Z(f)] \}$. The notation and terminology of [GJ76] is used. In this section we will characterize $m$-regular elements in $C(X)$, we will find for what spaces $X$, $\mathfrak{M}(C(X))$ contains non zero elements.

Recall from Section 2 that $R$ is a VNL ring if for each $a \in R$, one of $a$ or $1-a$ is von Neumann regular.

The next proposition is established in [AHA04] and in [GJ76].

5.1 Proposition. (a) $C(X)$ is a VNR ring if and only if $X$ is a $P$-space if and only if every $G_\delta$-set of $X$ is open.

(b) $C(X)$ is VNL ring if and only if at most one point of $X$ is not a $P$-point (in which case $X$ is said to be essentially a $P$-space).

The next simple lemma will be used below.
5.2 Lemma. If $f \in \operatorname{vr}(C(X))$, then $Z(f)$ is clopen.

Proof: As is noted just above Theorem 4.4, there is a unit $u$ in $C(X)$ such that $f = f(fu)$ and $fu$ is idempotent. Because the zero set of an idempotent is clopen, the conclusion follows. \[\square\]

Thus we obtain:

5.3 Theorem. A function $f$ is in $\mathcal{M}(C(X)) \setminus \{0\}$ if and only if $\operatorname{coz}(f)$ is a nonempty clopen $P$-space.

Proof: Suppose that $f \in \mathcal{M}(C(X)) \setminus \{0\}$, then $f \in \operatorname{vr}(C(X))$ and so $\operatorname{coz}(f)$ is a nonempty clopen set by Lemma 5.2. Let $G = \bigcap_{n=1}^{\infty} G_n$ be a $G_\delta$-set of $X$ contained in $\operatorname{coz}(f)$ and suppose $x \in G$. For each $n$ there exists $g_n \in C(X)$ such that $g_n(x) = 0$ and $g_n(X \setminus G_n) = 1$. Let $g = \sum_{n=1}^{\infty} (|g_n|/2^n)$, then $g \in C(X)$ and $Z(g) = G \in \operatorname{coz}(f)$. Since $fg \in \operatorname{vr}(C(X))$, its zero set is clopen by Lemma 5.2. So, because $Z(fg) = Z(f) \cup Z(g)$, $Z(f) \cap Z(g) = \emptyset$, and $Z(f)$ is clopen, it follows that $Z(g)$ and hence $\operatorname{coz}(g)$ is clopen. Thus, by Proposition 5.1, $\operatorname{coz}(f)$ is a $P$-space.

Suppose conversely that $\operatorname{coz}(f)$ is a nonempty clopen $P$-space. Then $C(X)$ is the direct product of $C(\operatorname{coz}(f))$ and $C(Z(f))$, so $f \in \mathcal{M}(C(X)) \setminus \{0\}$. \[\square\]

5.4 Corollary. $\mathcal{M}(C(X)) \neq \{0\}$ if and only if $X$ contains a nonempty clopen $P$-space.

By making use of Theorem 1.3, we can describe $\mathcal{M}(C(X))$ more precisely.

If $Y$ is a subset of $X$, we let $O^Y = \bigcap_{y \notin Y} O^y$. Let $P(X)$ be the set of all $P$-points in $X$, then it is clear that $O^{X - P(X)} = \bigcap_{y \notin P(X)} O^y \subseteq \operatorname{vr}(C(X))$ and so, $O^{X - P(X)} \subseteq \mathcal{M}(C(X))$. For each $x \in \beta X$, $mM^x = O^x$, using this together with Theorem 1.3 above we conclude that:

5.5 Corollary. $\mathcal{M}(C(X)) = O^{X - P(X)}$ for any space $X$.

We conclude with an interesting example.

5.6 Example. Let $X_1 = (0, 1)$ with its usual topology and $X_2 = \mathbb{N}$ with its discrete topology. Let $X = X_1 \bigoplus X_2$ and define $f(x) = \begin{cases} 0 & x \in X_1 \\ 1 & x \in X_2 \end{cases}$, then $f \in \mathcal{M}(C(X)) \setminus \{0\}$, while $C(X)$ is not a VNR ring.

References


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