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## Direct limit of matricially Riesz normed spaces

J.V. RAMANI, ANIL K. KARN, SUNIL YADAV

*Abstract.* In this paper, the  $\mathcal{F}$ -Riesz norm for ordered  $\mathcal{F}$ -bimodules is introduced and characterized in terms of order theoretic and geometric concepts. Using this notion,  $\mathcal{F}$ -Riesz normed bimodules are introduced and characterized as the inductive limits of matricially Riesz normed spaces.

*Keywords:* Riesz norm, matricially Riesz normed space, positively bounded, absolutely  $\mathcal{F}$ -convex,  $\mathcal{F}$ -Riesz norm

*Classification:* Primary 46L07

### 1. Introduction

Effros and Ruan, as suggested by B.E. Johnson, initiated a study of normed  $\mathcal{F}$ -bimodules as direct limits of matrix normed spaces [2]. In [6] the authors studied the direct limit of matrix ordered spaces. Continuing this line, in this paper we discuss the direct limits of matricially Riesz normed spaces (studied by [4], [5]). As a consequence we introduce the notion of  $\mathcal{F}$ -Riesz normed bimodules.

We recall the following notions discussed in [6] (see also [2]).

#### Matricial notions.

Let  $V$  be a complex vector space. Let  $M_n(V)$  denote the set of all  $n \times n$  matrices with entries from  $V$ . For  $V = \mathcal{C}$ , we denote  $M_n(\mathcal{C})$  by  $M_n$ . For  $\alpha = [\alpha_{ij}] \in M_n$  and  $v = [v_{ij}] \in M_n(V)$  we define

$$\alpha v = \left[ \sum_{j=1}^n \alpha_{ij} v_{jk} \right], \quad v \alpha = \left[ \sum_{j=1}^n v_{ij} \alpha_{jk} \right].$$

Then  $M_n(V)$  is a  $M_n$ -bimodule for all  $n \in \mathbb{N}$ . In particular  $M_n(V)$  is a complex vector space for all  $n \in \mathbb{N}$ . For  $v \in M_n(V)$ ,  $w \in M_m(V)$ , we define

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+m}(V).$$

Next, we consider the family  $\{M_n\}$ . For each  $n, m \in \mathbb{N}$  define  $\sigma_{n,n+m} : M_n \longrightarrow M_{n+m}$  by  $\sigma_{n,n+m}(\alpha) = \alpha \oplus 0_m$ . Then  $\sigma_{n,n+m}$  is a vector space isomorphism with

$$\sigma_{n,n+m}(\alpha\beta) = \sigma_{n,n+m}(\alpha)\sigma_{n,n+m}(\beta).$$

Thus we may “identify”  $M_n$  in  $M_{n+m}$  as a subalgebra for every  $m \in \mathbb{N}$ . More generally, we may identify  $M_n$  in the set  $\mathcal{F}$  of  $\infty \times \infty$  complex matrices, having entries zero after first  $n$  rows and first  $n$  columns. Then  $\mathcal{F}$  may be considered as the direct or inductive limit of the family  $\{M_n\}$ . In this sense

$$\mathcal{F} = \bigcup_{n=1}^{\infty} M_n.$$

Let  $e_{ij}$  denote the  $\infty \times \infty$  matrix with 1 at the  $(i, j)$ th entry and 0 elsewhere. Then the collection  $\{e_{ij}\}$  is called the set of matrix units in  $\mathcal{F}$ . We write  $1_n$  for  $\sum_{i=1}^n e_{ii}$ .

For  $i, j, k, l \in \mathbb{N}$ , we have  $e_{ij}e_{kl} = \delta_{jk}e_{il}$ . Note that for any  $\alpha \in \mathcal{F}$ , there exist complex numbers  $\alpha_{ij}$  such that

$$\alpha = \sum_{i,j} \alpha_{ij}e_{ij} \quad (\text{a finite sum}).$$

Thus  $\mathcal{F}$  is an algebra.

For  $\alpha = \sum_{i,j} \alpha_{ij}e_{ij} \in \mathcal{F}$ , we define  $\alpha^* = \sum_{i,j} \bar{\alpha}_{ji}e_{ij} \in \mathcal{F}$ . Then  $\alpha \mapsto \alpha^*$  is an involution. In other words,  $\mathcal{F}$  is a  $*$ -algebra.

**Definition 1.1.** Let  $V$  be a complex vector space. Consider the family  $\{M_n(V)\}$ . For each  $n, m \in \mathbb{N}$ , define  $T_{n,n+m} : M_n(V) \rightarrow M_{n+m}(V)$  by  $T_{n,n+m}(v) = v \oplus 0_m$ ,  $0_m \in M_m(V)$ . Then  $T_{n,n+m}$  is an injective homomorphism. Let  $\mathcal{V}$  be the inductive limit of the directed family  $\{M_n(V), T_{n,n+m}\}$ . We shall call  $\mathcal{V}$  the *matricial inductive limit* or *direct limit* of  $V$ .

The matricial inductive limit of a complex vector space  $V$  may be characterized in the following sense:

**Theorem 1.2.** Let  $\mathcal{W}$  be a non-degenerate  $\mathcal{F}$ -bimodule. Put  $W = e_{11}\mathcal{W}e_{11}$ . Then  $W$  is a complex vector space and  $W$  is its matricial inductive limit ([2]).

**Definition 1.3** (Matrix normed space). Let  $V$  be a complex vector space. Then  $M_n(V)$ , the space of  $n \times n$  matrices with entries from  $V$ , is an  $M_n$ -bimodule for all  $n \in \mathbb{N}$ . A *matrix norm* on  $V$  is a sequence  $\{\|\cdot\|_n\}$  such that  $\|\cdot\|_n$  is a norm on  $M_n(V)$  for all  $n \in \mathbb{N}$ . We say that  $(V, \{\|\cdot\|_n\})$  is a *matrix normed space* if  $\|v \oplus 0_m\|_{n+m} = \|v\|_n$  and  $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_n \|\beta\|$  for all  $v \in M_n(V)$ ,  $\alpha, \beta \in M_n$  and  $n, m \in \mathbb{N}$  ([7]).

**Definition 1.4** ( $\mathcal{F}$ -bimodule norm). Let  $\mathcal{V}$  be a non-degenerate  $\mathcal{F}$ -bimodule. Let  $\|\cdot\|$  be a norm on  $\mathcal{V}$ . Then we say  $\|\cdot\|$  is an  $\mathcal{F}$ -bimodule norm on  $\mathcal{V}$  if  $\|\alpha v \beta\| \leq \|\alpha\| \|v\| \|\beta\|$ , for any  $\alpha, \beta \in \mathcal{F}$ ,  $v \in \mathcal{V}$ . In this case we say that  $\mathcal{V}$  is a non-degenerate *normed  $\mathcal{F}$ -bimodule*.

**Theorem 1.5.** *Let  $(V, \{\|\cdot\|_n\})$  be a matrix normed space. Let  $\mathcal{V}$  be the matricial inductive limit of  $V$ . For each  $v \in \mathcal{V}$ , we define  $\|v\|$  as follows: let  $n \in \mathbb{N}$  be such that  $1_n v 1_n = v$ . Write  $\|v\| = \|v\|_n$ . Then this definition is independent of the choice of  $n$  and introduces an  $\mathcal{F}$ -bimodule norm on  $\mathcal{V}$  such that  $(\mathcal{V}, \|\cdot\|)$  is a non-degenerate normed  $\mathcal{F}$ -bimodule.*

*Conversely, let  $(\mathcal{W}, \|\cdot\|)$  be a non-degenerate normed  $\mathcal{F}$ -bimodule and let  $W = 1_1 \mathcal{W} 1_1$  and  $\|\cdot\|_n = \|\cdot\|_{M_n(W)}$  for all  $n \in \mathbb{N}$ . Then  $(W, \{\|\cdot\|_n\})$  is a matrix normed space whose matricial inductive limit is  $(\mathcal{W}, \|\cdot\|)$ .*

**Remark.** This characterization can be extended to  $*$  vector spaces as follows: Let  $V$  be a  $*$  vector space and let  $\mathcal{V}$  be the matricial inductive limit of  $V$ , so that  $\mathcal{V}$  is a non-degenerate  $\mathcal{F}$ -bimodule ([6]). Let  $(V, \{\|\cdot\|_n\})$  be a matrix normed space such that for every  $n \in \mathbb{N}$  and  $v \in M_n(V)$ ,  $\|v^*\|_n = \|v\|_n$ . Let  $(\mathcal{V}, \|\cdot\|)$  be the matricial inductive limit of the matrix normed space  $(V, \{\|\cdot\|_n\})$ . Then  $\|v^*\| = \|v\|$  for all  $v \in \mathcal{V}$ .

Next, we recall the definition of an ordered  $\mathcal{F}$ -bimodule and its characterization as a matricial inductive limit space from [6]:

**Definition 1.6** (Ordered  $\mathcal{F}$ -bimodule). Let  $\mathcal{V}$  be a  $*$ - $\mathcal{F}$ -bimodule. Let  $\mathcal{V}^+$  be a bimodule cone in  $\mathcal{V}_{sa}$ . That is

1.  $v_1, v_2 \in \mathcal{V}^+ \Rightarrow v_1 + v_2 \in \mathcal{V}^+$ ,
2.  $v \in \mathcal{V}^+, \alpha \in \mathcal{F} \Rightarrow \alpha^* v \alpha \in \mathcal{V}^+$ .

Then  $(\mathcal{V}, \mathcal{V}^+)$  will be called an *ordered  $\mathcal{F}$ -bimodule*.

The following result is obtained from [6].

**Theorem 1.7.** *Let  $(V, \{M_n(V)^+\})$  be a matrix ordered space. Let  $\mathcal{V}$  be the matricial inductive limit of  $V$ . Then  $(\mathcal{V}, \mathcal{V}^+)$  is a non-degenerate ordered  $\mathcal{F}$ -bimodule, where  $\mathcal{V}^+ = \bigcup_{n=1}^{\infty} M_n(V)^+$ . Conversely, let  $(\mathcal{W}, \mathcal{W}^+)$  be a non-degenerate ordered  $\mathcal{F}$ -bimodule. Put  $W = 1_1 \mathcal{W} 1_1$  and  $M_n(W)^+ = 1_n \mathcal{W}^+ 1_n$  for all  $n \in \mathbb{N}$ . Then  $(W, \{M_n(W)^+\})$  is a matrix ordered space with  $\mathcal{W}^+ = \bigcup_{n=1}^{\infty} M_n(W)^+$ .*

## 2. $\mathcal{F}$ -Riesz norm

We now characterize  $\mathcal{F}$ -bimodule norms.

**Definition 2.1.** Let  $\mathcal{V}$  be a non-degenerate  $\mathcal{F}$ -bimodule. Let  $\mathcal{U} \subset \mathcal{V}$ . We say  $\mathcal{U}$  is *absolutely  $\mathcal{F}$ -convex* if  $\sum_{i=1}^k \alpha_i u_i \beta_i \in \mathcal{U}$  whenever  $u_1, u_2, \dots, u_k \in \mathcal{U}$  and  $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$  with  $\sum_{i=1}^k \|\alpha_i\|^2 \leq 1$  and  $\sum_{i=1}^k \|\beta_i\|^2 \leq 1$ . If the property holds true only for  $k = 1$  then we say  $\mathcal{U}$  is  *$\mathcal{F}$ -circled*.

**Theorem 2.2.** *The open unit ball of a non-degenerate normed  $\mathcal{F}$ -bimodule  $(\mathcal{V}, \|\cdot\|)$  is absolutely  $\mathcal{F}$ -convex and absorbing.*

PROOF: Let  $\mathcal{U}$  denote the open unit ball of  $(\mathcal{V}, \|\cdot\|)$ . Let  $u_1, u_2, \dots, u_k \in \mathcal{U}$  and  $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$  with  $\sum_{i=1}^k \|\alpha_i\|^2 \leq 1$  and  $\sum_{i=1}^k \|\beta_i\|^2 \leq 1$ . Consider  $u = \sum_{i=1}^k \alpha_i u_i \beta_i$ . Then

$$\begin{aligned} \|u\| &= \left\| \sum_{i=1}^k \alpha_i u_i \beta_i \right\| \leq \sum_{i=1}^k \|\alpha_i\| \|u_i\| \|\beta_i\| < \sum_{i=1}^k \|\alpha_i\| \|\beta_i\| \\ &\leq \left( \sum_{i=1}^k \|\alpha_i\|^2 \right)^{1/2} \left( \sum_{i=1}^k \|\beta_i\|^2 \right)^{1/2} \leq 1. \end{aligned}$$

Therefore  $u \in \mathcal{U}$ . Thus  $\mathcal{U}$  is absolutely  $\mathcal{F}$ -convex. To show that  $\mathcal{U}$  is absorbing consider a  $v \in \mathcal{V}$  and  $\epsilon > 0$ . Put  $v_1 = \frac{v}{(\|v\| + \epsilon)}$ . Then  $v_1 \in \mathcal{U}$  and  $v = v_1 (\|v\| + \epsilon)$ . Therefore  $\mathcal{U}$  is absorbing.  $\square$

The following theorem completes the characterization of  $\mathcal{F}$ -bimodule norms among norms on  $\mathcal{V}$ .

**Theorem 2.3.** *Let  $\mathcal{A} \subset \mathcal{V}$  be absolutely  $\mathcal{F}$ -convex and absorbing. Then the gauge of  $\mathcal{A}$ ,*

$$p(v) = \inf \{k > 0 \mid v \in k\mathcal{A}\}$$

*determines an  $\mathcal{F}$ -bimodule semi-norm on  $\mathcal{V}$ .*

PROOF: First we note that  $p(v) \geq 0$  for all  $v \in \mathcal{V}$ . From the definition, we get that  $p(kv) = |k|p(v)$  for all  $k \in \mathcal{C}$ . We now show that  $p(v+w) \leq p(v) + p(w)$  for all  $v, w \in \mathcal{V}$ . Let  $v, w \in \mathcal{V}$  and  $\epsilon > 0$ . Then there exist  $k_1, k_2 > 0$  such that  $k_1 < p(v) + \frac{\epsilon}{2}$  with  $v \in k_1\mathcal{A}$  and  $k_2 < p(w) + \frac{\epsilon}{2}$  with  $w \in k_2\mathcal{A}$ . We show that  $v+w \in (k_1 + k_2)\mathcal{A}$ . We set  $\alpha = \frac{k_1}{k_1 + k_2}$ ,  $\beta = \frac{k_2}{k_1 + k_2}$ . Then  $\alpha + \beta = 1$ . Also  $\frac{\alpha v}{k_1} = \frac{v}{k_1 + k_2}$ ,  $\frac{\beta w}{k_2} = \frac{w}{k_1 + k_2}$ . Thus we get  $\frac{\alpha v}{k_1} + \frac{\beta w}{k_2} = \frac{v+w}{k_1 + k_2}$ . As  $\mathcal{A}$  is absolutely  $\mathcal{F}$ -convex, it is convex. Thus  $v+w \in (k_1 + k_2)\mathcal{A}$ . It follows that

$$p(v+w) \leq k_1 + k_2 < p(v) + p(w) + \epsilon.$$

As  $\epsilon > 0$  is arbitrary we get that  $p(v+w) \leq p(v) + p(w)$ . Next, we show that  $p(\alpha v \beta) \leq \|\alpha\| p(v) \|\beta\|$  for all  $\alpha, \beta \in \mathcal{F}$ ,  $v \in \mathcal{V}$ . First, let  $v \in \mathcal{A}$ . Then  $p(v) \leq 1$ . Let  $\alpha, \beta \in \mathcal{F}$  with  $\|\alpha\| \leq 1$ ,  $\|\beta\| \leq 1$ . Since  $\mathcal{A}$  is absolutely  $\mathcal{F}$ -convex,  $\alpha v \beta \in \mathcal{A}$ . Therefore  $p(\alpha v \beta) \leq 1$ . Now let  $v \in \mathcal{V}$  and  $\alpha, \beta \in \mathcal{F}$ ,  $\epsilon > 0$ . Put  $v_1 = \frac{v}{p(v) + \epsilon}$ . Then  $p(v_1) = \frac{p(v)}{p(v) + \epsilon} < 1$ . That is  $v_1 \in \mathcal{A}$ . Without loss of generality we may take  $\alpha \neq 0$ ,  $\beta \neq 0$ . Let  $\alpha_1 = \frac{\alpha}{\|\alpha\|}$ ,  $\beta_1 = \frac{\beta}{\|\beta\|}$ . Then  $p(\alpha_1 v_1 \beta_1) \leq 1$  so that

$$p(\alpha v \beta) \leq \|\alpha\| (p(v) + \epsilon) \|\beta\|.$$

As  $\epsilon > 0$  is arbitrary we get

$$p(\alpha v \beta) \leq \|\alpha\| (p(v)) \|\beta\|.$$

Hence  $p(\cdot)$  is a  $\mathcal{F}$ -semi-norm on  $\mathcal{V}$ .  $\square$

In the rest of the paper we will be dealing with non-degenerate ordered  $\mathcal{F}$ -bimodules. We introduce some more notations.

We write  $I_n = \sum_{i=1}^n e_{ii}$ ,  $J_n = \sum_{i=1}^n e_{i,n+i}$  for any  $n \in \mathbb{N}$ . Note that  $\|I_n\| = \|J_n\| = 1$  and  $J_n I_n = 0$ ,  $I_n J_n = J_n$ ,  $J_n J_n = 0$ ,  $J_n J_n^* = I_n$ . Let  $(\mathcal{V}, \mathcal{V}^+)$  be a non-degenerate ordered  $\mathcal{F}$ -bimodule ([6]). Let  $u_1, u_2 \in \mathcal{V}^*$  and  $n \in \mathbb{N}$  such that  $1_n u_1 1_n = u_1$ ,  $1_n u_2 1_n = u_2$ . We denote  $u_1 + J_n^* u_2 J_n$  by  $(u_1, u_2)_n^+$ . For any  $v \in \mathcal{V}$  and an  $n \in \mathbb{N}$  with  $1_n v 1_n = v$  we denote  $I_n v J_n + J_n^* v^* I_n$  by  $sa_n(v)$ .

Before we define  $\mathcal{F}$ -Riesz norm, we need the following reformulation of the concept that  $\mathcal{V}^+$  is generating.

**Proposition 2.4.** *Let  $(\mathcal{V}, \mathcal{V}^+)$  be a non-degenerate ordered  $\mathcal{F}$ -bimodule. Then  $\mathcal{V}^+$  is generating if and only if for every  $v \in \mathcal{V}$  there exist  $u_1, u_2 \in \mathcal{V}^+$  such that  $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$ , for a suitable  $n \in \mathbb{N}$ .*

**Note.** In the notation  $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$ , we say that  $n \in \mathbb{N}$  is “suitable” provided  $1_n u_1 1_n = u_1$ ,  $1_n u_2 1_n = u_2$  and  $1_n v 1_n = v$ . This terminology will be used throughout the paper without any further explanation.

PROOF: First, let  $\mathcal{V}^+$  be generating. Let  $v \in \mathcal{V}_{sa}$ . Then by [6, Theorem 3.10] there exist  $v_1, v_2 \in \mathcal{V}^+$  such that  $v = v_1 - v_2$ . Put  $u = v_1 + v_2$ . Then  $u \in \mathcal{V}^+$  and  $u \pm v \in \mathcal{V}^+$ . Next let  $v \in \mathcal{V}$  be arbitrary. Find an  $n \in \mathbb{N}$  such that  $1_n v 1_n = v$ . Consider  $sa_n(v)$ :  $sa_n(v) = I_n v J_n + J_n^* v^* I_n \in \mathcal{V}_{sa}$ . Then as above there exists a  $u \in \mathcal{V}^+$  such that  $u \pm sa_n(v) \in \mathcal{V}^+$ . Let  $u' = I_{2n} u I_{2n} \in \mathcal{V}^+$ . Then  $u' \pm sa_n(v) \in \mathcal{V}^+$  for  $I_{2n} sa_n(v) I_{2n} = sa_n(v)$ . Set  $u_1 = I_n u' I_n$ ,  $u_2 = J_n u' J_n^*$ . Then  $(u_1, u_2)_n^+ = I_n u' I_n + J_n^* (J_n u' J_n^*) J_n$ . We show that  $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$ . Note that

$$\begin{aligned} (1) \quad & I_n u' I_n - I_n u' J_n^* J_n - J_n^* J_n u' I_n + J_n^* J_n u' J_n^* J_n \mp sa_n(v) \\ &= (I_n - J_n^* J_n) (u' \pm sa_n(v)) (I_n - J_n^* J_n) \in \mathcal{V}^+. \end{aligned}$$

Similarly

$$\begin{aligned} (2) \quad & I_n u' I_n + I_n u' J_n^* J_n + J_n^* J_n u' I_n + J_n^* J_n u' J_n^* J_n \pm sa_n(v) \\ &= (I_n + J_n^* J_n) (u' \pm sa_n(v)) (I_n + J_n^* J_n) \in \mathcal{V}^+. \end{aligned}$$

Adding (1) and (2) suitably, we get

$$(u_1, u_2)_n^+ \pm sa_n(v) = I_n u' I_n + J_n^* (J_n u' J_n^*) J_n \pm sa_n(v) \in \mathcal{V}^+.$$

Conversely assume that for every  $v \in \mathcal{V}$  there exist  $u_1, u_2 \in \mathcal{V}^+$  such that  $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$ , for a suitable  $n \in \mathbb{N}$ . We show that  $\mathcal{V}^+$  is generating. Let  $v \in \mathcal{V}$ . Then there exist  $u_1, u_2 \in \mathcal{V}^+$  such that  $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$ , for a suitable  $n \in \mathbb{N}$ . Therefore

$$(I_n + J_n) \left( (u_1, u_2)_n^+ \pm sa_n(v) \right) (I_n + J_n^*) \in \mathcal{V}^+.$$

This gives  $u_1 + u_2 \pm (v + v^*) \in \mathcal{V}^+$ . Similarly

$$(I_n + iJ_n) \left( (u_1, u_2)_n^+ \pm sa_n(v) \right) (I_n - iJ_n^*) \in \mathcal{V}^+$$

which gives  $u_1 + u_2 \pm i(v - v^*) \in \mathcal{V}^+$ . Put

$$\begin{aligned} v_0 &= \frac{1}{4}(u_1 + u_2 + v + v^*), \\ v_1 &= \frac{1}{4}(u_1 + u_2 - i(v - v^*)), \\ v_2 &= \frac{1}{4}(u_1 + u_2 - v - v^*), \\ v_3 &= \frac{1}{4}(u_1 + u_2 + i(v - v^*)). \end{aligned}$$

Then  $v_0, v_1, v_2, v_3 \in \mathcal{V}^+$  and we have

$$v_0 + iv_1 - v_2 - iv_3 = v.$$

Hence  $\mathcal{V}^+$  is generating.  $\square$

**Definition 2.5.** Let  $(\mathcal{V}, \mathcal{V}^+)$  be a positively generated non-degenerate ordered  $\mathcal{F}$ -bimodule. Let  $\|\cdot\|$  be an  $\mathcal{F}$ -bimodule norm on  $\mathcal{V}$ . We say  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$  if for any  $v \in \mathcal{V}$ ,

$$\begin{aligned} \|v\| &= \inf\{\max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ &\quad \text{for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}. \end{aligned}$$

In what follows we characterize  $\mathcal{F}$ -Riesz norms on a non-degenerate positively ordered  $\mathcal{F}$ -bimodule in the lines of Theorem 2.2.

**Definition 2.6.** Let  $(\mathcal{V}, \mathcal{V}^+)$  be an ordered  $\mathcal{F}$ -bimodule and  $\mathcal{A} \subset \mathcal{V}^+$ . We define  $\mathcal{S}(\mathcal{A})$  as follows:

$$\begin{aligned} \mathcal{S}(\mathcal{A}) &= \{v \in \mathcal{V} \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ &\quad \text{for some } u_1, u_2 \in \mathcal{A} \text{ and a suitable } N \in \mathbb{N}\}. \end{aligned}$$

**Remarks.**

- (a)  $\mathcal{A} \subset \mathcal{S}(\mathcal{A})$ .
- (b)  $v^* \in \mathcal{S}(\mathcal{A})$  whenever  $v \in \mathcal{S}(\mathcal{A})$ .

**Definition 2.7.** Let  $\mathcal{A} \subset \mathcal{V}^+$ . Then we say that  $\mathcal{A}$  is *order absolutely  $\mathcal{F}$ -convex* if  $\sum_{i=1}^k \alpha_i^* u_i \alpha_i \in \mathcal{A}$  whenever  $u_1, u_2, \dots, u_k \in \mathcal{A}$  and  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathcal{F}$  with  $\sum_{i=1}^k \|\alpha_i^* \alpha_i\| \leq 1$ .

If the above condition holds only for  $k = 1$  for some  $\mathcal{A} \subset \mathcal{V}^+$ , then we say  $\mathcal{A}$  is *order  $\mathcal{F}$ -circled*.

**Definition 2.8.**  $\mathcal{S} \subset \mathcal{V}^+$  is called  *$\mathcal{F}$ -absorbing* if for each  $v \in \mathcal{V}$  there exist  $\alpha, \beta \in \mathcal{F}$  such that  $\alpha v \beta \in \mathcal{S}$ .

**Definition 2.9.**  $\mathcal{S} \subset \mathcal{V}^+$  is called *positively  $\mathcal{F}$ -absorbing* if for each  $u \in \mathcal{V}^+$  there exists a  $\alpha \in \mathcal{F}$  such that  $\alpha^* u \alpha \in \mathcal{S}$ .

**Lemma 2.10.** Let  $\mathcal{A} \subset \mathcal{V}^+$  be order absolutely  $\mathcal{F}$ -convex. Then  $\mathcal{S}(\mathcal{A})$  is absolutely  $\mathcal{F}$ -convex.

PROOF: Let  $v_1, v_2, \dots, v_k \in \mathcal{S}(\mathcal{A})$  and let  $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$  with  $\sum_{i=1}^k \|\alpha_i\|^2 \leq 1$  and  $\sum_{i=1}^k \|\beta_i\|^2 \leq 1$ . Then for each  $i = 1, 2, \dots, k$  there exist  $N_i \in \mathbb{N}$ ,  $u_1^i, u_2^i \in \mathcal{A}$  with  $1_{N_i} v_i 1_{N_i} = v_i$ ,  $1_{N_i} u_1^i 1_{N_i} = u_1^i$ ,  $1_{N_i} u_2^i 1_{N_i} = u_2^i$  with  $(u_1^i, u_2^i)_{N_i}^+ \pm sa_{N_i}(v_i) \in \mathcal{V}^+$ . Now  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathcal{F}$ . Therefore there exist  $M_1, M_2, \dots, M_k \in \mathbb{N}$  such that  $1_{M_i} \alpha_i 1_{M_i} = \alpha_i$ ,  $i = 1, 2, \dots, k$ . Also  $\beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$ . Therefore there exist  $P_1, P_2, \dots, P_k \in \mathbb{N}$  such that  $1_{P_i} \beta_i 1_{P_i} = \beta_i$ ,  $i = 1, 2, \dots, k$ . Let  $N = \max\{N_1, N_2, \dots, N_k, M_1, \dots, M_k, P_1, \dots, P_k\}$ . Then for each  $i = 1, 2, \dots, k$  we have  $(u_1^i, u_2^i)_N^+ \pm sa_N(v_i) \in \mathcal{V}^+$ . Now

$$\left( (\alpha_i^*, \beta_i)_N^+ \right)^* \left( (u_1^i, u_2^i)_N^+ \pm sa_N(v_i) \right) \left( (\alpha_i^*, \beta_i)_N^+ \right) \in \mathcal{V}^+ \text{ for all } i = 1, 2, \dots, k.$$

This means  $(\alpha_i u_1^i \alpha_i^*, \beta_i^* u_2^i \beta_i)_N^+ \pm sa_N(\alpha_i v_i \beta_i) \in \mathcal{V}^+$  for each  $i = 1, 2, \dots, k$ .

Adding  $\left( \sum_{i=1}^k \alpha_i u_1^i \alpha_i^*, \sum_{i=1}^k \beta_i^* u_2^i \beta_i \right)_N^+ \pm sa_N \left( \sum_{i=1}^k \alpha_i v_i \beta_i \right) \in \mathcal{V}^+$ . Since  $\mathcal{A}$  is absolutely convex and  $\sum_{i=1}^k \|\alpha_i\|^2 \leq 1$  and  $\sum_{i=1}^k \|\beta_i\|^2 \leq 1$  we have

$\sum_{i=1}^k \alpha_i u_1^i \alpha_i^* \in \mathcal{A}$  and  $\sum_{i=1}^k \beta_i^* u_2^i \beta_i \in \mathcal{A}$ . Therefore  $\sum_{i=1}^k \alpha_i v_i \beta_i \in \mathcal{S}(\mathcal{A})$ . Therefore  $\mathcal{S}(\mathcal{A})$  is absolutely  $\mathcal{F}$ -convex.  $\square$

**Lemma 2.11.** Let  $\mathcal{V}^+$  be generating. Then  $\mathcal{S}(\mathcal{A})$  is  $\mathcal{F}$ -absorbing if  $\mathcal{A} \subset \mathcal{V}^+$  is positively  $\mathcal{F}$ -absorbing.

PROOF: Let  $\mathcal{A} \subset \mathcal{V}^+$  be positively  $\mathcal{F}$ -absorbing. Let  $v \in \mathcal{V}$ . Since  $\mathcal{V}^+$  is generating, by Proposition 2.4, there exist  $u_1, u_2 \in \mathcal{V}^+$  and a suitable  $N \in \mathbb{N}$  such that  $(u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$ . Since  $\mathcal{A}$  is positively  $\mathcal{F}$ -absorbing and  $u_1, u_2 \in \mathcal{V}^+$  there exist  $\alpha, \beta \in \mathcal{F}$  such that  $\alpha^* u_1 \alpha \in \mathcal{A}$ ,  $\beta^* u_2 \beta \in \mathcal{A}$ . Find  $M \in \mathbb{N}$  such that  $1_M u_1 1_M = u_1$ ,  $1_M u_2 1_M = u_2$ ,  $1_M v 1_M = v$ ,  $1_M \alpha 1_M = \alpha$ ,



$1_M \beta 1_M = \beta$ . Then  $\left( (\alpha, \beta)_M^+ \right)^* \left( (u_1, u_2)_M^+ \pm sa_M(v) \right) (\alpha, \beta)_M^+ \in \mathcal{V}^+$ . This gives  $(\alpha^* u_1 \alpha, \beta^* u_2 \beta)_M^+ \pm sa_M(\alpha^* v \beta) \in \mathcal{V}^+$ . Since  $\alpha^* u_1 \alpha \in \mathcal{A}$  and  $\beta^* u_2 \beta \in \mathcal{A}$ , we get  $\alpha^* v \beta \in \mathcal{S}(\mathcal{A})$ . Hence  $\mathcal{S}(\mathcal{A})$  is  $\mathcal{F}$ -absorbing.  $\square$

Some more concepts will be needed in the sequel.

**Definition 2.12.** Let  $\mathcal{A} \subset \mathcal{V}^+$ .  $\mathcal{A}$  is called *positively bounded* if for any  $v \in \mathcal{V}_{sa}$ ,  $v + k_n a_n \in \mathcal{V}^+$  for all  $n \in \mathbb{N}$  implies  $v \in \mathcal{V}^+$ , where  $\{a_n\}$  is a sequence in  $\mathcal{A}$  and  $\{k_n\}$  is a sequence in  $(0, \infty)$  with  $\inf k_n = 0$ .

**Definition 2.13.** Let  $\mathcal{A} \subset \mathcal{V}^+$ .  $\mathcal{A}$  is called *almost positively bounded* if  $(k_n u_1^n, k_n u_2^n)_{N_n}^+ \pm sa_{N_n}(v) \in \mathcal{V}^+$  for all  $n \in \mathbb{N}$  implies  $v = 0$  where  $\{u_1^n\}$ ,  $\{u_2^n\}$  are sequences in  $\mathcal{A}$  and  $\{k_n\}$  is a sequence in  $(0, \infty)$  with  $\inf k_n = 0$ ,  $\{N_n\}$  is a sequence in  $\mathbb{N}$ .

**Lemma 2.14.** Let  $\mathcal{V}^+$  be proper. Let  $\mathcal{A} \subset \mathcal{V}^+$  be order absolutely  $\mathcal{F}$ -convex and positively bounded. Then  $\mathcal{A}$  is almost positively bounded.

PROOF: Let  $v \in \mathcal{V}$ , sequences  $\{u_1^n\}$ ,  $\{u_2^n\}$  be in  $\mathcal{A}$ ,  $\{k_n\}$  be a sequence in  $(0, \infty)$  with  $\inf k_n = 0$  and  $\{N_n\}$  be a sequence in  $\mathbb{N}$  such that

$$Z_{N_n} = (k_n u_1^n, k_n u_2^n)_{N_n}^+ \pm sa_{N_n}(v) \in \mathcal{V}^+$$

for all  $n \in \mathbb{N}$ . Then

$$(1) \quad (I_{N_n} + J_{N_n}) Z_{N_n} (I_{N_n} + J_{N_n})^* = k_n u_1^n + k_n u_2^n \pm (v + v^*)$$

and

$$(2) \quad (I_{N_n} + iJ_{N_n}) Z_{N_n} (I_{N_n} + iJ_{N_n})^* = k_n u_1^n + k_n u_2^n \pm i(v - v^*).$$

Put  $u_1^n + u_2^n = 2u_n$  for all  $n \in \mathbb{N}$ . From (1) and (2) we get

$$(3) \quad k_n u_n \pm \operatorname{Re}(v), k_n u_n \pm \operatorname{Im}(v) \in \mathcal{V}^+.$$

Since  $\mathcal{A}$  is convex as it is order absolutely  $\mathcal{F}$ -convex,  $u_n \in \mathcal{A}$  for all  $n \in \mathbb{N}$ . As  $\mathcal{A}$  is positively bounded, from (3) we get  $\pm \operatorname{Re} v, \pm \operatorname{Im} v \in \mathcal{V}^+$ . Finally as  $\mathcal{V}^+$  is proper, we have  $\operatorname{Re} v = 0$ ,  $\operatorname{Im} v = 0$ . That is  $v = 0$ . Hence  $\mathcal{A}$  is almost positively bounded.  $\square$

**Remark.** It may be noted that the notion of (almost-)positively bounded sets is introduced to generalize the notion of (almost-)Archimedean property of the cone ([5]).

Now we are in a position to characterize  $\mathcal{F}$ -Riesz norms.

**Theorem 2.15.** *Let  $(\mathcal{V}, \mathcal{V}^+)$  be a non-degenerate positively generated ordered  $\mathcal{F}$ -bimodule. Let  $\mathcal{A} \subset \mathcal{V}^+$  be order absolutely  $\mathcal{F}$ -convex, almost positively bounded and positively  $\mathcal{F}$ -absorbing. Also assume that  $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$ . Let  $p(\cdot)$  be the gauge of  $\mathcal{S}(\mathcal{A})$ . Then  $p(\cdot)$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$ .*

*Conversely, let  $\|\cdot\|$  be an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$  where  $(\mathcal{V}, \mathcal{V}^+)$  is a positively generated ordered  $\mathcal{F}$ -bimodule. Also let  $\mathcal{U}^+ = \{v \in \mathcal{V}^+ \mid \|v\| < 1\} = \mathcal{U} \cap \mathcal{V}^+$ , where  $\mathcal{U}$  is the open unit ball of  $(\mathcal{V}, \|\cdot\|)$ . Then  $\mathcal{U}^+$  is order absolutely  $\mathcal{F}$ -convex, almost positively bounded and positively  $\mathcal{F}$ -absorbing.*

PROOF: First assume that  $(\mathcal{V}, \mathcal{V}^+)$  is a non-degenerate positively generated ordered  $\mathcal{F}$ -bimodule. Let  $\mathcal{A} \subset \mathcal{V}^+$  be order absolutely  $\mathcal{F}$ -convex, almost positively bounded and positively  $\mathcal{F}$ -absorbing. Also assume that  $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$ . Let  $p(\cdot)$  be the gauge of  $\mathcal{S}(\mathcal{A})$ . We show that  $p(\cdot)$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$ . In the light of Theorem 2.3, Lemmas 2.10 and 2.11 we note that  $p(\cdot)$  is a  $\mathcal{F}$ -semi-norm on  $\mathcal{V}$ . Let  $v \in \mathcal{V}$ . We show that

$$p(v) = \inf \{ \max(p(u_1), p(u_2)) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ \text{for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N} \}.$$

Since  $\mathcal{S}(\mathcal{A})$  is  $\mathcal{F}$ -absorbing there exists some  $\lambda > 0$  such that  $\lambda v \in \mathcal{S}(\mathcal{A})$ . This gives some  $u_1, u_2 \in \mathcal{A}$  and a  $N \in \mathbb{N}$  such that  $(u_1, u_2)_N^+ \pm sa_N(\lambda v) \in \mathcal{V}^+$ . That is  $(\lambda^{-1}u_1, \lambda^{-1}u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$ . Also  $p(\lambda^{-1}u_1) = \lambda^{-1}p(u_1)$ . Since  $p(\cdot)$  is the gauge of  $\mathcal{S}(\mathcal{A})$  and  $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$ , we have  $p(u_1) \leq 1$  and  $p(u_2) \leq 1$ . Therefore  $p(\lambda^{-1}u_1) \leq \lambda^{-1}$ ,  $p(\lambda^{-1}u_2) \leq \lambda^{-1}$ . That is  $\max\{p(\lambda^{-1}u_1), p(\lambda^{-1}u_2)\} \leq \lambda^{-1}$ . Let  $\epsilon > 0$ . Then  $(p(v) + \epsilon)^{-1}v \in \mathcal{S}(\mathcal{A})$ . Replacing  $\lambda$  by  $(p(v) + \epsilon)$  in the above discussion, there exist  $u_1, u_2 \in \mathcal{V}^+$  and some  $N \in \mathbb{N}$  such that  $(u_1, u_2)_N^+ \pm sa_N(\lambda v) \in \mathcal{V}^+$  and  $\max\{p(u_1), p(u_2)\} \leq (p(v) + \epsilon)$ . That is,

$$p(v) \geq \inf \{ \max(p(u_1), p(u_2)) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ \text{for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N} \}.$$

Let  $u_1, u_2 \in \mathcal{V}^+$  and  $(u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$  for some  $N \in \mathbb{N}$ . Find a  $\lambda > 0$  such that  $\lambda u_1, \lambda u_2 \in \mathcal{S}(\mathcal{A})$ . This gives  $(\lambda u_1, \lambda u_2)_N^+ \pm sa_N(\lambda v) \in \mathcal{V}^+$ . Since  $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$ , we get  $\lambda u_1, \lambda u_2 \in \mathcal{A}$ . That is  $\lambda v \in \mathcal{S}(\mathcal{A})$ . Therefore  $p(v) \leq \lambda^{-1}$ . Let  $\epsilon > 0$ . Put  $\lambda = (\max\{p(u_1), p(u_2)\} + \epsilon)^{-1}$ . Then  $\lambda u_1, \lambda u_2 \in \mathcal{S}(\mathcal{A})$  so that  $p(v) \leq \max\{p(u_1), p(u_2)\} + \epsilon$ . This gives

$$p(v) \leq \inf \{ \max(p(u_1), p(u_2)) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ \text{for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N} \}.$$

Therefore  $p(\cdot)$  is  $\mathcal{F}$ -Riesz semi-norm on  $\mathcal{V}$ . Now let  $v \in \mathcal{V}$  be such that  $p(v) = 0$ . Then there is a sequence  $\{k_n\}$  in  $(0, \infty)$  with  $\inf k_n = 0$  such that  $k_n^{-1}v \in \mathcal{S}(\mathcal{A})$ .

Thus for every  $n \in \mathbb{N}$ , there exist  $u_1^n, u_2^n \in \mathcal{A}$  such that  $(u_1^n, u_2^n)_{N_n}^+ \pm sa_{N_n}(k_n^{-1}v) \in \mathcal{V}^+$  for suitable  $N_n \in \mathbb{N}$ . This means that  $(k_n u_1^n, k_n u_2^n)_{N_n}^+ \pm sa_{N_n}(v) \in \mathcal{V}^+$ . Since  $\mathcal{A}$  is almost positively bounded, we get  $v = 0$ . Hence  $p(\cdot)$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$ .

Conversely, let  $\|\cdot\|$  be an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$  where  $(\mathcal{V}, \mathcal{V}^+)$  is a positively generated ordered  $\mathcal{F}$ -bimodule. Also let  $\mathcal{U}^+ = \{v \in \mathcal{V}^+ \mid \|v\| < 1\} = \mathcal{U} \cap \mathcal{V}^+$ , where  $\mathcal{U}$  is the open unit ball of  $(\mathcal{V}, \|\cdot\|)$ . We show that  $\mathcal{U}^+$  is order absolutely  $\mathcal{F}$ -convex, almost positively bounded and positively  $\mathcal{F}$ -absorbing.

Let  $u \in \mathcal{U}$ . Find an  $\epsilon > 0$  such that  $\|u\| + \epsilon < 1$ . Since  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm there exist  $u_1, u_2 \in \mathcal{V}^+$ , a suitable  $N \in \mathbb{N}$  such that  $(u_1, u_2)_N^+ \pm sa_N(u) \in \mathcal{V}^+$  and  $\max\{\|u_1\|, \|u_2\|\} < \|u\| + \epsilon < 1$ . That is  $\|u_1\| < 1, \|u_2\| < 1$ . This means  $u_1, u_2 \in \mathcal{U}^+$ . That is  $u \in \mathcal{S}(\mathcal{A})$ . Thus  $\mathcal{U} \subset \mathcal{S}(\mathcal{U}^+)$ . Let  $v \in \mathcal{S}(\mathcal{U}^+)$ . Then there exist  $u_1, u_2 \in \mathcal{U}^+$  and a suitable  $N \in \mathbb{N}$  such that  $(u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$ . Since  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm, we have  $\|v\| \leq \max\{\|u_1\|, \|u_2\|\} < 1$ . Therefore  $v \in \mathcal{U}$  or  $\mathcal{S}(\mathcal{U}^+) \subset \mathcal{U}$ . Therefore  $\mathcal{S}(\mathcal{U}^+) = \mathcal{U}$ . Next, let  $u_1, u_2, \dots, u_k \in \mathcal{U}^+$  and  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathcal{F}$  with  $\sum_{i=1}^k \|\alpha_i^* \alpha_i\| \leq 1$ . Put  $u = \sum_{i=1}^k \alpha_i^* u_i \alpha_i$ . Then  $u \in \mathcal{V}$  and

$$\|u\| \leq \sum_{i=1}^k \|\alpha_i\|^2 \|u_i\| < \sum_{i=1}^k \|\alpha_i\|^2 \leq 1.$$

It follows  $\mathcal{U}^+$  is order absolutely  $\mathcal{F}$ -convex. We now prove that  $\mathcal{U}^+$  is almost positively bounded. Let  $v \in \mathcal{V}$  and sequences  $\{u_1^n\}, \{u_2^n\}$  be in  $\mathcal{U}^+$  and  $\{k_n\}$  in  $(0, \infty)$  with  $\inf k_n = 0$  and  $\{N_n\}$  a sequence in  $\mathbb{N}$  such that  $(k_n u_1^n, k_n u_2^n)_{N_n}^+ \pm sa_{N_n}(v) \in \mathcal{V}^+$  for all  $n \in \mathbb{N}$ . We show that  $\|v\| = 0$ . Let  $\epsilon > 0$ . Since  $\inf k_n = 0$  there exists a  $n_0 \in \mathbb{N}$  such that  $k_{n_0} < \epsilon$ . As  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm and  $\|u_1^{n_0}\| < 1, \|u_2^{n_0}\| < 1$ , we have  $\|v\| \leq \max\{\|k_{n_0} u_1^{n_0}\|, \|k_{n_0} u_2^{n_0}\|\} < k_{n_0} < \epsilon$ . Since  $\epsilon > 0$  is arbitrary,  $\|v\| = 0$ . Since  $\|\cdot\|$  is a norm,  $v = 0$ . Hence  $\mathcal{U}^+$  is almost-positively bounded. Finally, let  $v \in \mathcal{V}^+$  and  $\epsilon > 0$ . Put  $\alpha = (\|v\| + \epsilon)^{-\frac{1}{2}} 1_n$  where  $1_n v 1_n = v$ . Then  $\alpha^* v \alpha = \frac{1}{(\|v\| + \epsilon)} 1_n v 1_n = \frac{v}{(\|v\| + \epsilon)} \in \mathcal{U}^+$ . Therefore  $\mathcal{U}^+$  is positively  $\mathcal{F}$ -absorbing.  $\square$

**Theorem 2.16.** *Let  $(\mathcal{V}, \mathcal{V}^+)$  be a non-degenerate ordered  $\mathcal{F}$ -bimodule. Let  $\mathcal{V}^+$  be proper and generating. Let  $\mathcal{A} \subset \mathcal{V}^+$  be order absolutely  $\mathcal{F}$ -convex, positively bounded and  $\mathcal{F}$ -absorbing. Assume that  $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$ . Let  $p(\cdot)$  be the gauge of  $\mathcal{S}(\mathcal{A})$ . Then  $p(\cdot)$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$  such that  $\mathcal{V}^+$  is  $p$ -closed.*

*Conversely, let  $(\mathcal{V}, \mathcal{V}^+)$  be an ordered  $\mathcal{F}$ -bimodule and  $\mathcal{V}^+$  be generating. Let  $\|\cdot\|$  be an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$  such that  $\mathcal{V}^+$  is closed. Let  $\mathcal{U}^+ = \{v \in \mathcal{V}^+ \mid \|v\| < 1\}$ . Then  $\mathcal{U}^+$  is order absolutely  $\mathcal{F}$ -convex, positively bounded and positively  $\mathcal{F}$ -absorbing such that  $\mathcal{S}(\mathcal{U}^+) \cap \mathcal{V}^+ = \mathcal{U}^+$ . Moreover  $\mathcal{V}^+$  is proper.*

PROOF: First assume that  $\mathcal{V}^+$  is proper and generating. Let  $\mathcal{A} \subset \mathcal{V}^+$  be order absolutely  $\mathcal{F}$ -convex, positively bounded and  $\mathcal{F}$ -absorbing. Assume that  $\mathcal{S}(\mathcal{A}) \cap$

$\mathcal{V}^+ = \mathcal{A}$ . Let  $p(\cdot)$  be the gauge of  $\mathcal{S}(\mathcal{A})$ . We show that  $p(\cdot)$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}^+$  such that  $\mathcal{V}^+$  is  $p$ -closed. In the light of Lemma 2.14 and Theorem 2.15 it suffices to prove that  $\mathcal{V}^+$  is  $p$ -closed. We shall show that  $\mathcal{V}_{sa} \setminus \mathcal{V}^+$  is  $p$ -open. Define for  $v \in \mathcal{V}_{sa}$ ,

$$r(v) = \inf\{\alpha \in \mathcal{R} \mid v + \alpha a \in \mathcal{V}^+ \text{ for some } a \in \mathcal{A}\}.$$

We first show that  $r(v) \leq 0$  if and only if  $v \in \mathcal{V}^+$ . Let  $v \in \mathcal{V}^+$ . Then  $v + 0a \in \mathcal{V}^+$  for all  $a \in \mathcal{A}$ . That is  $r(v) \leq 0$ . To show the other way let  $r(v) \leq 0$ . Then for every  $n \in \mathbb{N}$  there exists an  $a_n \in \mathcal{A}$  such that  $v + (r(v) + \frac{1}{n})a_n \in \mathcal{V}^+$ . Also  $v + (r(v) + \frac{1}{n})a_n \leq v + (\frac{1}{n})a_n$  as  $r(v) \leq 0$ . That is  $v + (\frac{1}{n})a_n \in \mathcal{V}^+$  for every  $n \in \mathbb{N}$ . As  $\mathcal{A}$  is positively bounded,  $v \in \mathcal{V}^+$ . We now show that  $p(v) - r(v) \geq 0$  for all  $v \in \mathcal{V}_{sa}$ . Suppose  $p(v) - r(v) < 0$  for some  $v \in \mathcal{V}_{sa}$ . Put  $\epsilon = \frac{1}{2}(r(v) - p(v)) > 0$ . Since  $p(\cdot)$  is  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$ , there exists an  $a \in \mathcal{A}$  such that  $(p(v) + \epsilon)a \pm v \in \mathcal{V}^+$ . Then  $(r(v) - \epsilon)a \pm v \in \mathcal{V}^+$ . In particular  $(r(v) - \epsilon)a + v \in \mathcal{V}^+$ . This contradicts the definition of  $r(v)$ . Thus  $p(v) \geq r(v)$  for all  $v \in \mathcal{V}_{sa}$ . Finally we show that  $\mathcal{V}_{sa} \setminus \mathcal{V}^+$  is  $p$ -open. Let  $v \in \mathcal{V}_{sa}$ ,  $v \notin \mathcal{V}^+$ . Since  $v \notin \mathcal{V}^+$ ,  $r(v) > 0$ . Let  $\delta = \frac{1}{2}r(v)$ . Let  $\mathcal{D} = \{w \in \mathcal{V}_{sa} \mid p(v - w) < \delta\}$ . Let  $w \in \mathcal{D}$ . Then  $\delta > p(v - w) \geq r(v - w)$ . So there exists an  $a \in \mathcal{A}$  such that  $\delta a + (v - w) \in \mathcal{V}^+$ . If  $w \in \mathcal{V}^+$ , then  $\delta a + v \in \mathcal{V}^+$ . Thus  $r(v) \leq \delta = \frac{r(v)}{2}$ , which is a contradiction. Therefore  $w \notin \mathcal{V}^+$ . That is  $\mathcal{V}_{sa} \setminus \mathcal{V}^+$  is  $p$ -open.

For the converse it suffices to prove that  $\mathcal{U}^+$  is positively bounded and that  $\mathcal{V}^+$  is proper in light of Theorem 2.15. We show that  $\mathcal{U}^+$  is positively bounded. Let  $v \in \mathcal{V}^+$  and  $w_n = v + k_n u_n \in \mathcal{V}^+$  for all  $n \in \mathbb{N}$ , where  $\{u_n\}$  is a sequence in  $\mathcal{U}^+$  and  $\{k_n\}$  is a sequence in  $(0, \infty)$  with  $\inf k_n = 0$ . Without loss of generality we can take  $\{k_n\}$  to be decreasing. Now  $\{w_n\}$  is a convergent sequence because  $\|v - w_n\| = \|k_n u_n\| < k_n \rightarrow 0$ . Therefore  $w_n \rightarrow v$ . Since  $\mathcal{V}^+$  is closed,  $v \in \mathcal{V}^+$ . Therefore  $\mathcal{U}^+$  is positively bounded.

Finally we show that  $\mathcal{V}^+$  is proper. Let  $\pm v \in \mathcal{V}^+$ . Then as  $v$  is self-adjoint,  $\|v\| = \inf\{\|u\| \mid u \in \mathcal{V}^+, u \pm v \in \mathcal{V}^+\}$ . Also  $0 \in \mathcal{V}^+$  and  $0 \pm v \in \mathcal{V}^+$ . That is  $\|v\| \leq \|0\| = 0$ . That is  $v = 0$ . Therefore  $\mathcal{V}^+$  is proper.  $\square$

Now we move to the final result of the paper.

**Definition 2.17** ( $\mathcal{F}$ -Riesz normed bimodule). Let  $(\mathcal{V}, \mathcal{V}^+)$  be a non-degenerate ordered  $\mathcal{F}$ -bimodule such that  $\mathcal{V}^+$  is proper and generating. Assume that  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$  such that  $\mathcal{V}^+$  is norm closed. Then the triple  $(\mathcal{V}, \mathcal{V}^+, \|\cdot\|)$  is called an  $\mathcal{F}$ -Riesz normed bimodule.

**Definition 2.18** (Matricially Riesz normed space). Let  $(V, \{M_n(V)^+\})$  be a positively generated matrix ordered space and suppose that  $\{\|\cdot\|_n\}$  is a matrix norm on  $V$ . Then the triplet  $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$  is called a *matricially normed space* if for each  $n \in \mathbb{N}$ ,  $\|\cdot\|_n$  is a Riesz norm on  $M_n(V)$  and  $M_n(V)^+$  is closed.

**Theorem 2.19.** *Let  $(V, \{M_n(V)^+\}, \{\|\cdot\|_n\})$  be a matricially Riesz normed space. Let  $(\mathcal{V}, \mathcal{V}^+)$  be the matricial inductive limit of the matrix ordered space  $(V, \{M_n(V)^+\})$  and let  $(\mathcal{V}, \|\cdot\|)$  be the matricial inductive limit of matrix normed space  $(V, \{\|\cdot\|_n\})$ . Then  $(\mathcal{V}, \mathcal{V}^+, \|\cdot\|)$  is a non-degenerate  $\mathcal{F}$ -Riesz normed bimodule. Conversely, let  $(\mathcal{W}, \mathcal{W}^+, \|\cdot\|)$  be a non-degenerate  $\mathcal{F}$ -Riesz normed bimodule. Let  $W = 1_1\mathcal{W}1_1$  and  $M_n(W)^+ = 1_n\mathcal{W}^+1_n$  and  $\|\cdot\|_n = \|\cdot\| |_{M_n(W)}$  for all  $n \in \mathbb{N}$ . Then  $(W, \{M_n(W)^+\}, \{\|\cdot\|_n\})$  is a matricially Riesz normed space whose inductive limit is  $(\mathcal{W}, \mathcal{W}^+, \|\cdot\|)$ .*

PROOF: Let  $(V, \{M_n(V)^+\}, \{\|\cdot\|_n\})$  be a matricially Riesz normed space. We show that  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$ . Let  $v \in \mathcal{V}$ . Then there exists a smallest  $n \in \mathbb{N}$  such that  $1_nv1_n = v$ . Then

$$\|v\| = \|v\|_n = \inf \{ \max(\|u_1\|_n, \|u_2\|_n) \mid (u_1, u_2)_n^+ \pm sa_n(v) \in M_{2n}(V)^+ \text{ for some } u_1, u_2 \in M_n(V)^+ \}.$$

Let

$$p(v) = \inf \{ \max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \text{ for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N} \}.$$

Then  $p(v) \leq \|v\|$ . Let  $\epsilon > 0$ . Then there exist  $u_1, u_2 \in \mathcal{V}^+$ ,  $N \in \mathbb{N}$  such that  $(u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$  and  $\max(\|u_1\|, \|u_2\|) < p(v) + \epsilon$ . In this case  $N \geq n$ . Put  $u'_1 = 1_n u_1 1_n$ ,  $u'_2 = 1_n u_2 1_n$ . Then  $u'_1, u'_2 \in M_n(V)^+$ . Also

$$((1_n, 1_n)_n^+)^* \left[ (u_1, u_2)_N^+ \pm sa_N(v) \right] ((1_n, 1_n)_n^+) = (u'_1, u'_2)_n^+ \pm sa_n(v) \in M_{2n}(V)^+$$

as  $1_nv1_n = v$ . Next  $\|u'_1\|_n \leq \|u_1\|$ ,  $\|u'_2\|_n \leq \|u_2\|$  so that

$$\|v\| = \|v\|_n \leq \max(\|u'_1\|_n, \|u'_2\|_n) \leq \max(\|u_1\|, \|u_2\|) < p(v) + \epsilon.$$

Since  $\epsilon > 0$  is arbitrary,  $\|v\| \leq p(v)$ . Therefore  $p(v) = \|v\|$ . Hence  $\|\cdot\|$  is an  $\mathcal{F}$ -Riesz norm on  $\mathcal{V}$ . We show that  $\mathcal{V}^+$  is  $\|\cdot\|$  closed. Let  $v \in \mathcal{V}^+$ . Then there exists a sequence  $\{v_k\} \subset \mathcal{V}^+$  such that  $v_k \rightarrow v$  in  $\|\cdot\|$ . Hence  $v \in \mathcal{V}_{sa}$ . Find an  $n \in \mathbb{N}$  such that  $1_nv1_n = v$ . Then  $v'_k = 1_n v_k 1_n \rightarrow 1_nv1_n = v$  in  $\|\cdot\|_n$ . Since  $M_n(V)^+$  is closed, we have  $v \in M_n(V)^+ \subset \mathcal{V}^+$ . Therefore  $\mathcal{V}^+$  is closed.

For the converse it is enough to show that  $\|\cdot\|_n$  is a Riesz norm on  $M_n(W)$  for all  $n \in \mathbb{N}$ . Fix an  $n \in \mathbb{N}$  and  $w \in M_n(W)$ . Let

$$r(w) = \inf \{ \max(\|u_1\|_n, \|u_2\|_n) \mid (u_1, u_2)_n^+ \pm sa_n(w) \in M_{2n}(W)^+ \text{ for some } u_1, u_2 \in M_n(W)^+ \}.$$

Recall that

$$\|w\|_n = \|w\| = \inf \{ \max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm sa_N(w) \in \mathcal{W}^+ \}$$

for some  $u_1, u_2 \in \mathcal{W}^+$  and a suitable  $N \in \mathbb{N}$ .

Then  $\|w\|_n \leq r(w)$ . Let  $\epsilon > 0$ . Then as above using  $(1_n, 1_n)_n^+$ , we may conclude that  $r(w) \leq \|w\|_n + \epsilon$ . Therefore  $r(w) = \|w\|_n$ . That is  $\|\cdot\|_n$  is a Riesz norm on  $M_n(W)$ . Also  $M_n(W)^+$  is  $\|\cdot\|_n$  closed.  $\square$

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#### REFERENCES

- [1] Choi M.D., Effros E.G., *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209.
- [2] Effros E.G., Ruan Z.J., *On matricially normed spaces*, Pacific J. Math. **132** (1988), no. 2, 243–264.
- [3] Karn A.K., *Approximate matrix order unit spaces*, Ph.D. Thesis, University of Delhi, Delhi, 1997.
- [4] Karn A.K., Vasudevan R., *Approximate matrix order unit spaces*, Yokohama Math. J. **44** (1997), 73–91.
- [5] Karn A.K., Vasudevan R., *Characterization of matricially Riesz normed spaces*, Yokohama Math. J. **47** (2000), 143–153.
- [6] Ramani J.V., Karn A.K., Yadav S., *Direct limit of matrix ordered spaces*, Glasnik Matematički **40** (2005), no. 2, 303–312.
- [7] Ruan Z.J., *Subspaces of  $C^*$ -algebras*, J. Funct. Anal. **76** (1988), 217–230.

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