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Commentationes Mathematicae Universitatis Carolinae, Vol. 47 (2006), No. 1, 155--158

Persistent URL: <http://dml.cz/dmlcz/119582>

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Another proof of Derriennic’s reverse maximal inequality for the supremum of ergodic ratios

RYOTARO SATO

Abstract. Using the ratio ergodic theorem for a measure preserving transformation in a σ -finite measure space we give a straightforward proof of Derriennic’s reverse maximal inequality for the supremum of ergodic ratios.

Keywords: σ -finite measure space, measure preserving transformation, conservative, ergodic, supremum of ergodic ratios, maximal and reverse maximal inequalities

Classification: Primary 28D05, 47A35

1. Let (X, \mathcal{F}, μ) be a σ -finite measure space and T be a measure preserving transformation in (X, \mathcal{F}, μ) . Given two measurable functions f and g on X such that $0 \leq f, g \leq \infty$ on X and $0 < \int_X g \, d\mu \leq \infty$, let

$$s(f, g)(x) = \sup_{n \geq 0} \frac{\sum_{i=0}^n f(T^i x)}{\sum_{i=0}^n g(T^i x)}.$$

(Throughout this note we define $a/\infty = 0$ and $a/0 = \infty$ for any a , with $0 \leq a \leq \infty$.) In this note we use the ratio ergodic theorem to give a straightforward proof of the following reverse maximal inequality due to Derriennic [1] (cf. also Ornstein [5]). It is interesting to note that the author was inspired by reading Ephremidze’s paper [3].

Theorem. *Suppose that T is conservative and ergodic, and that $\int_X f \, d\mu < \infty$. If $\alpha > \int_X f \, d\mu / \int_X g \, d\mu$, then, letting $E(\alpha) = \{x \mid s(f, g)(x) > \alpha\}$, we have*

$$\int_{E(\alpha)} f \, d\mu \leq \alpha \int_{E(\alpha) \cup T^{-1}E(\alpha)} g \, d\mu.$$

PROOF: We may assume that $\mu(E(\alpha)) > 0$. For $x \in X$, let $K(x) = \{n \geq 0 \mid T^n x \in E(\alpha)\}$ and $L(x) = \{0, 1, \dots\} \setminus K(x)$. Since T is conservative and ergodic, $K(x)$ is infinite for a.a. $x \in X$. To see that $L(x)$ is also infinite for a.a. $x \in X$, suppose there exists $k \geq 0$ such that $i \in K(x)$ for all $i \geq k$. Then clearly we have

$$(1) \quad \limsup_{l \rightarrow \infty} \frac{\sum_{i=k}^l f(T^i x)}{\sum_{i=k}^l g(T^i x)} \geq \alpha.$$

But this is a contradiction, since

$$(2) \quad \lim_{l \rightarrow \infty} \frac{\sum_{i=k}^l f(T^i x)}{\sum_{i=k}^l g(T^i x)} = \frac{\int_X f d\mu}{\int_X g d\mu} < \alpha$$

for a.a. $x \in X$ by the ratio ergodic theorem (cf. Theorem 3.3.4 in [4]).

Since $K(x)$ and $L(x)$ are infinite for a.a. $x \in X$, we can write $K(x) = \bigcup_{n=1}^\infty I_n$ (disjoint union), where $I_n = [k_n, l_n]$ ($= \{i \mid k_n \leq i \leq l_n\}$) and $0 \leq k_n \leq l_n < l_n + 2 \leq k_{n+1}$ for each $n \geq 1$. Hence the set $J(x) = \{n \geq 0 \mid T^n x \in E(\alpha) \cup T^{-1}E(\alpha)\}$ has the form

$$J(x) = \begin{cases} [0, l_1] \cup \bigcup_{n=2}^\infty [k_n - 1, l_n] & \text{if } k_1 = 0, \\ \bigcup_{n=1}^\infty [k_n - 1, l_n] & \text{if } k_1 \geq 1. \end{cases}$$

Since $T^{k_n-1}x \notin E(\alpha)$ for $n \geq 2$, we have

$$(3) \quad \frac{\sum_{i=k_n-1}^{l_n} f(T^i x)}{\sum_{i=k_n-1}^{l_n} g(T^i x)} \leq \alpha \quad (n \geq 2).$$

On the other hand, if h is a function in $L_1(\mu)$ such that $\int_X h d\mu = 1$ and $0 < h < \infty$ on X , then, by the ratio ergodic theorem,

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\sum_{i=0}^n (\chi_{E(\alpha) \cup T^{-1}E(\alpha)} f)(T^i x)}{\sum_{i=0}^n h(T^i x)} = \int_{E(\alpha) \cup T^{-1}E(\alpha)} f d\mu$$

and

$$(5) \quad \lim_{n \rightarrow \infty} \frac{\sum_{i=0}^n (\chi_{E(\alpha) \cup T^{-1}E(\alpha)} g)(T^i x)}{\sum_{i=0}^n h(T^i x)} = \int_{E(\alpha) \cup T^{-1}E(\alpha)} g d\mu$$

for a.a. $x \in X$. Since $\sum_{i=0}^\infty h(T^i x) = \infty$ for a.a. $x \in X$, combining (3), (4) and (5) yields

$$(6) \quad \int_{E(\alpha) \cup T^{-1}E(\alpha)} f d\mu \leq \alpha \int_{E(\alpha) \cup T^{-1}E(\alpha)} g d\mu,$$

and this completes the proof, since $f \geq 0$ on X . □

2. Here we consider the case $g = 1$ on X . Then it follows that $s(f, 1) = f^*$, where $f^*(x) = \sup_{n \geq 1} n^{-1} \sum_{i=0}^{n-1} f(T^i x)$. In this case we have the following reverse maximal inequality.

Proposition. *If $\mu(X) = \infty$, T is ergodic (but not necessarily conservative), and f satisfies $\int_{\{f>t\}} f d\mu < \infty$ for all $t > 0$, then we have $\int_{\{f^* > \alpha\}} f d\mu \leq 2\alpha\mu(\{f^* > \alpha\}) < \infty$ for all $\alpha > 0$.*

PROOF: We first prove that $\mu(\{f^* > \alpha\}) < \infty$. To do this, let $f_1 = f\chi_{\{f \leq \alpha/2\}}$ and $f_2 = f - f_1$. Then we have $f = f_1 + f_2$, $\|f_1\|_\infty \leq \alpha/2$, and $\int_X f_2 d\mu < \infty$. Since $f^* \leq f_1^* + f_2^*$ and $\|f_1^*\|_\infty \leq \alpha/2$, it follows that $\{f^* > \alpha\} \subset \{f_2^* > \alpha/2\}$, and by Hopf's maximal ergodic theorem (cf. Theorem 1.2.1 in [4])

$$\mu(\{f_2^* > \alpha/2\}) \leq (2/\alpha) \int_{\{f_2^* > \alpha/2\}} f_2 d\mu < \infty,$$

so that $\mu(\{f^* > \alpha\}) < \infty$. Putting $F = f - \alpha$, we then have $F^+ = (f - \alpha)^+ \in L_1(\mu)$ and $\{F^* > 0\} = \{f^* > \alpha\}$; furthermore $\int_X F d\mu = \int_X (f - \alpha)^+ d\mu - \int_X (f - \alpha)^- d\mu = -\infty$ because $\mu(X) = \infty$. Hence by Theorem 1.4 in Ephremidze [2] we see that

$$\int_{\{f^* > \alpha\} \cup T^{-1}\{f^* > \alpha\}} (f - \alpha) d\mu \leq 0.$$

Since $f \geq 0$ and $\mu(\{f^* > \alpha\}) < \infty$, we then have

$$\int_{\{f^* > \alpha\}} f d\mu \leq \int_{\{f^* > \alpha\} \cup T^{-1}\{f^* > \alpha\}} f d\mu \leq 2\alpha\mu(\{f^* > \alpha\}) < \infty,$$

completing the proof. □

Corollary. *If $\mu(X) = \infty$, and T is ergodic, then for any $\beta \geq 0$ we have*

$$\int_{\{f^* > t\}} f^* \left(\log \frac{f^*}{t}\right)^\beta d\mu < \infty \quad \text{for all } t > 0$$

if and only if

$$\int_{\{f > t\}} f \left(\log \frac{f}{t}\right)^{\beta+1} d\mu < \infty \quad \text{for all } t > 0.$$

PROOF: See the proof of Theorem 2 in [6]. □

(Of course, as is known, this holds when $\mu(X) < \infty$, by the Theorem.)

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(Received May 2, 2005, revised November 15, 2005)