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Allami Benyaiche, Salma Ghiate

Abstract. In this paper, we study the Martin boundary associated with a harmonic structure given by a coupled partial differential equations system. We give an integral representation for non negative harmonic functions of this structure. In particular, we obtain such results for biharmonic functions (i.e. $\triangle^2\varphi = 0$) and for non negative solutions of the equation $\triangle^2\varphi = \varphi$.

Keywords: Martin boundary, biharmonic functions, coupled partial differential equations

Classification: Primary 31C35; Secondary 31B30, 31B10, 60J50

1. Introduction

Let D be a domain in $\mathbb{R}^d$, $d \geq 1$, and let L_i, $i = 1, 2$, be two second order elliptic differential operators on D leading to harmonic spaces (D, H_{L_i}) with Green functions G_i (see [18]). Moreover, we assume that every ball $B \subset \bar{B} \subset D$ is an L_i-regular set. Throughout this paper we consider two positive Radon measures μ_1 and μ_2 such that $K_D^{\mu_i} = \int_D G_i(\cdot, y)\mu_i(dy)$ is a bounded continuous real function on D, $i = 1, 2$, and

$$\left\|K_D^{\mu_1}\right\|_\infty \left\|K_D^{\mu_2}\right\|_\infty < 1.$$

We consider the system:

$$(S) \begin{cases} L_1 u = -v\mu_1, \\ L_2 v = -u\mu_2. \end{cases}$$

Note that if U is a relatively compact open subset of D, $\mu_1 = \lambda^d$, where λ^d is the Lebesgue measure, $\mu_2 = 0$ and $L_1 = L_2 = \triangle$, then we obtain the classical biharmonic case on U. In the case when $\mu_1 = \mu_2 = \lambda^d$ and $\lambda^d(D) < \infty$, we obtain equations of type $\triangle^2\varphi = \varphi$. In this work, we shall study the Martin boundary associated with the balayage space given by the system (S) (see [7], [14] and [19]), and we shall characterize minimal points of this boundary in order to give an integral representation for non negative solutions of the system (S).

Let us note that the notion of a balayage space defined by J. Bliedtner and W. Hansen in [7] is more general than that of a P-harmonic space. It covers harmonic structures given by elliptic or parabolic partial differential equations, Riesz potentials, and biharmonic equations (which are a particular case of this work). In the biharmonic case, a similar study can be done using couples of functions as presented in [3], [5], [8], [9], [21] and [22].

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2. Notations and preliminaries

For $j=1,2$, let $X_j = D\times\{j\}$, and let $X = X_1 \bigcup X_2$. Moreover, let i_j and π_j be the mappings defined by

$$i_j : \begin{cases} D \longrightarrow X_j \\ x \longmapsto (x,j) \end{cases} \quad \text{and} \quad \pi_j : \begin{cases} X_j \longrightarrow D \\ (x,j) \longmapsto x. \end{cases}.$$

Let $\mathcal{U}_0$ be the set of all balls B such that $B\subset\bar{B}\subset D$, $\mathcal{U}_j$ be the image of $\mathcal{U}_0$ by i_j, $j=1,2$, and $\mathcal{U}=\mathcal{U}_1\cup\mathcal{U}_2$.

Definition 2.1. Let v be a measurable function on X. For $U\in\mathcal{U}_1$, we define the kernel S_U by

$$S_U v = (H^1_{\pi_1(U)}(v\circ i_1))\circ\pi_1 + (K^{\mu_1}_{\pi_1(U)}(v\circ i_2))\circ\pi_1.$$

For $U\in\mathcal{U}_2$, we define the kernel S_U by

$$S_U v = (H^2_{\pi_2(U)}(v\circ i_2))\circ\pi_2 + (K^{\mu_2}_{\pi_2(U)}(v\circ i_1))\circ\pi_2,$$

where $H^j_{\pi_j(U)}$, $j=1,2$, denote the harmonic kernels associated with (D,H_{L_j}) and

$$K^{\mu_i}_{\pi_i(U)}(w) = \int G_i^{\pi_i(U)}(\cdot,y)w(y)\mu_i\,(dy) \quad i=1,2,$$

where w is a measurable function on D and $G_i^{\pi_i(U)}$ is the Green function associated with the operator L_i on $\pi_i(U)$. Let G_j, $j=1,2$, be the Green kernel associated with L_j on D. The family of kernels $(S_U)_{U\in\mathcal{U}}$ yields a balayage space on X as defined in [7] and [14].

Let ${}^*\mathcal{H}(X)$ denote the set of all hyperharmonic functions on X, i.e.

$${}^*\mathcal{H}(X) := \{v\in\mathcal{B}(X) : v \text{ is l.s.c. and } S_U v\le v \ \ \forall U\in\mathcal{U}\},$$

where $\mathcal{B}(X)$ denotes the set of all Borel functions on X. Let $\mathcal{S}(X)$ be the set of all superharmonic functions on X, i.e.

$$\mathcal{S}(X) := \{v\in {}^*\mathcal{H}(X) : (S_U v)\mid_U \in C(U) \ \ \forall U\in\mathcal{U}\},$$

and let $\mathcal{H}(X)$ be the set of all harmonic functions on X:

$$\mathcal{H}(X) := \{h\in\mathcal{S}(X) : S_U h = h \ \ \forall U\in\mathcal{U}\}.$$

Denoting $\mathcal{W} := {}^*\mathcal{H}^+(X)$, the space $(X, \mathcal{W})$ is a balayage space (see [7] and [14]).

For every positive numerical function φ on X and for every $U \in \mathcal{U}$, the reduit R_φ^U is defined by

$$R_\varphi^U := \inf\{v \in {}^*\mathcal{H}(X) : v \geq \varphi \ \text{ on } \ U\}.$$

Let $\widehat{R}_\varphi^U$ be the lower semi-continuous regularization of R_φ^U, i.e.

$$\widehat{R}_\varphi^U(x) := \liminf_{y \to x} R_\varphi^U(y), \quad x \in X.$$

Theorem 2.1. *Let s be a function on X such that*

$$K_D^{\mu_j}(s \circ i_k) < \infty, \quad j \neq k, \quad j,k = 1,2.$$

The following statements are equivalent.

1. *s is a superharmonic function on X.*
2. *$s_j := s \circ i_j - K_D^{\mu_j}(s \circ i_k)$, $j \neq k$, $j,k \in \{1,2\}$, are L_j-superharmonic on D.*

Proof: Let s be a superharmonic function on X and let $U \in \mathcal{U}_0$. We have

$$i_1(U) \in \mathcal{U}_1 \ \text{ and } \ \pi_1(i_1(U)) = U.$$

Since $S_{i_1(U)} s \leq s$, we have

$$H_U^1(s \circ i_1) + K_U^{\mu_1}(s \circ i_2) \leq s \circ i_1.$$

Knowing that

$$K_U^{\mu_1}(s \circ i_2) = K_D^{\mu_1}(s \circ i_2) - H_U^1(K_D^{\mu_1}(s \circ i_2)),$$

we obtain

$$H_U^1(s \circ i_1) + K_D^{\mu_1}(s \circ i_2) - H_U^1(K_D^{\mu_1}(s \circ i_2)) \leq s \circ i_1.$$

Therefore

$$H_U^1(s \circ i_1 - K_D^{\mu_1}(s \circ i_2)) \leq s \circ i_1 - K_D^{\mu_1}(s \circ i_2).$$

So, $s_1 := s \circ i_1 - K_D^{\mu_1}(s \circ i_2)$ is an L_1-superharmonic function on D. Similarly, we prove that $s_2 := s \circ i_2 - K_D^{\mu_2}(s \circ i_1)$ is L_2-superharmonic on D. Conversely, we assume that s_i, $i = 1,2$, are L_i-superharmonic functions. Let $U \in \mathcal{U}_j$, $j = 1,2$ and $k \neq j$. Since s_j is an L_j-superharmonic function,

$$H_{\pi_j(U)}^j s_j \leq s_j.$$

Hence

$$H_{\pi_j(U)}^j(s \circ i_j - K_D^{\mu_j}(s \circ i_k)) \leq s \circ i_j - K_D^{\mu_j}(s \circ i_k).$$

Therefore

$$H_{\pi_j(U)}^j(s \circ i_j) + K_{\pi_j(U)}^{\mu_j}(s \circ i_k) \leq s \circ i_j.$$

So,

$$S_U s \leq s, \quad \forall\, U \in \mathcal{U}.$$

Thus s is superharmonic on X. □

Corollary 2.1. *Let v be a function on X such that $K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j,k \in \{1,2\}$, is a finite function. Then the following properties are equivalent.*

1. *v is harmonic on X.*
2. *$v \circ i_1 - K_D^{\mu_1}(v \circ i_2)$ and $v \circ i_2 - K_D^{\mu_2}(v \circ i_1)$ are L_1-harmonic and L_2-harmonic function on D, respectively.*

Remarks 2.1. (1) Note that if v is a positive harmonic function on X, then $K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j,k \in \{1,2\}$, is a finite function.
(2) If $v \in \mathcal{H}(X)$, then the couple $(v \circ i_1, v \circ i_2)$ is a solution of (S).

Corollary 2.2. *Let v be a positive function defined on X. Then the following properties are equivalent.*

1. *v is hyperharmonic on X.*
2. *The function*
$$v_j := \begin{cases} v \circ i_j - K_D^{\mu_j}(v \circ i_k) & \text{if } K_D^{\mu_j}(v \circ i_k) < \infty, \\ +\infty & \text{otherwise} \end{cases}$$
is a positive L_j-hyperharmonic function on D, $j \neq k$, $j,k \in \{1,2\}$.

If we identify a function s on X with the couple $(s \circ i_1, s \circ i_2)$ defined on D, then we get the following N. Bouleau's decomposition [9]:

Theorem 2.2. *Any superharmonic function s on X can be written as $s = t + Vs$, where*
$$V = \begin{pmatrix} 0 & K_D^{\mu_1} \\ K_D^{\mu_2} & 0 \end{pmatrix}$$
and t is a function on X defined by
$$t := \begin{cases} s_1 \circ \pi_1 & \text{on } X_1, \\ s_2 \circ \pi_2 & \text{on } X_2, \end{cases}$$
where $s_j := s \circ i_j - K_D^{\mu_j}(s \circ i_k)$, $j \neq k$, $j,k \in \{1,2\}$.

Proof: It follows from Theorem 2.1 that s_j, $j = 1,2$, is L_j-superharmonic on D. Then, if we identify the function s with the couple $(s \circ i_1, s \circ i_2)$ defined on D and the function t with the couple $(t \circ i_1, t \circ i_2) = (s_1, s_2)$ defined on D, we have
$$\begin{pmatrix} 0 & K_D^{\mu_1} \\ K_D^{\mu_2} & 0 \end{pmatrix} \begin{pmatrix} s \circ i_1 \\ s \circ i_2 \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} s \circ i_1 \\ s \circ i_2 \end{pmatrix}.$$
□

Remark 2.1. In the classical biharmonic case, we obtain the N. Bouleau's decomposition [9]. Indeed, if we identify a function s on X with the couple $(s \circ i_1, s \circ i_2)$ on D, then
$$s \circ i_1 = s_1 + K_D^{\mu_1}(s \circ i_2),$$
with s_1 L_1-superharmonic on D and the N. Bouleau's kernel V is given by $V = K_D^{\mu_1}$.

3. Martin boundary associated with (S)

Let us fix $x_0 \in D$ and set for all $x, y \in D$

$$g^1(x,y) := \begin{cases} \frac{G_1(x,y)}{G_1(x_0,y)} & \text{if } x \neq x_0 \text{ or } y \neq x_0, \\ 1 & \text{if } x = y = x_0, \end{cases}$$

and

$$g^2(x,y) := \begin{cases} \frac{G_2(x,y)}{G_2(x_0,y)} & \text{if } x \neq x_0 \text{ or } y \neq x_0, \\ 1 & \text{if } x = y = x_0. \end{cases}$$

Let $\mathcal{A}_1 = \{g^1(x,\cdot),\ x \in D\}$, $\mathcal{A}_2 = \{g^2(x,\cdot),\ x \in D\}$ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

As in [10] and [12], we consider the Martin compactification $\widehat{D}$ of D associated with $\mathcal{A}$. The boundary $\triangle = \widehat{D} \setminus D$ of D is called the Martin boundary of D associated with the system (S).

The function $g^k(x,\cdot)$, $k = 1, 2$, $x \in D$ can be extended, on $\widehat{D}$, to a continuous function denoted $g^k(x,\cdot)$, $k = 1, 2$, $x \in D$ as well.

In the following, we denote $Q := \sum_{n=0}^{+\infty}(K_D^{\mu_1}K_D^{\mu_2})^n$ (resp. $T := \sum_{n=0}^{+\infty}(K_D^{\mu_2}K_D^{\mu_1})^n$) which coincides with $(I - K_D^{\mu_1}K_D^{\mu_2})^{-1}$ (resp. $(I - K_D^{\mu_2}K_D^{\mu_1})^{-1}$) on $\mathcal{B}_b(D)$, where $(I - K_D^{\mu_1}K_D^{\mu_2})^{-1}$ (resp. $(I - K_D^{\mu_2}K_D^{\mu_1})^{-1}$) is the inverse of the operator $(I - K_D^{\mu_1}K_D^{\mu_2})$ (resp. $(I - K_D^{\mu_2}K_D^{\mu_1})$) on $\mathcal{B}_b(D)$, and $\mathcal{B}_b(D)$ denotes the set of all bounded Borel measurable functions on D. We recall the following equalities

$$(K_D^{\mu_1}K_D^{\mu_2})Q = Q(K_D^{\mu_1}K_D^{\mu_2}),$$
$$(K_D^{\mu_1}K_D^{\mu_2})Q + I = Q.$$

Similarly we have

$$(K_D^{\mu_2}K_D^{\mu_1})T = T(K_D^{\mu_2}K_D^{\mu_1}),$$
$$(K_D^{\mu_2}K_D^{\mu_1})T + I = T,$$
$$K_D^{\mu_2}Q = TK_D^{\mu_2}$$

and

$$K_D^{\mu_1}T = QK_D^{\mu_1}.$$

Remark 3.1. Note that if φ is a finite positive Borel measurable function on D such that $K_D^{\mu_1}K_D^{\mu_2}\varphi$ is bounded, then $Q\varphi < +\infty$.

Theorem 3.1. *Let t_i, $i = 1, 2$, be two L_i-harmonic functions on D such that $K_D^{\mu_j} t_k$ is finite and $K_D^{\mu_k} K_D^{\mu_j} t_k$ is bounded, $j \neq k$, $j, k \in \{1, 2\}$, on D. Then the functions v and w defined on X by*

$$v := \begin{cases} (Qt_1) \circ \pi_1 & \textit{on} \ \ X_1, \\ (K_D^{\mu_2} Qt_1) \circ \pi_2 & \textit{on} \ \ X_2, \end{cases}$$

and

$$w := \begin{cases} (QK_D^{\mu_1} t_2) \circ \pi_1 & \textit{on} \ \ X_1, \\ (Tt_2) \circ \pi_2 & \textit{on} \ \ X_2 \end{cases}$$

are harmonic on X.

Remark 3.2. In the biharmonic case, if we assume that $K_D^{\lambda^d} t_2 < \infty$, then $(t_1, 0)$ and $(K_D^{\lambda^d} t_2, t_2)$ are biharmonic.

Proof: Let us prove first that v and w are finite.
(i) We have

$$(Qt_1) \circ \pi_1 = (QK_D^{\mu_1} K_D^{\mu_2} t_1) \circ \pi_1 + t_1 \circ \pi_1.$$

Since $K_D^{\mu_1} K_D^{\mu_2} t_1$ is bounded and t_1 is finite,

$$(Qt_1) \circ \pi_1 < \infty.$$

(ii) We have also

$$(K_D^{\mu_2} Qt_1) \circ \pi_2 = (TK_D^{\mu_2} t_1) \circ \pi_2,$$

hence

$$(K_D^{\mu_2} Qt_1) \circ \pi_2 = (TK_D^{\mu_2} K_D^{\mu_1} K_D^{\mu_2} t_1) \circ \pi_2 + (K_D^{\mu_2} t_1) \circ \pi_2.$$

Since $K_D^{\mu_1} K_D^{\mu_2} t_1$ is bounded and $K_D^{\mu_2} t_1$ is finite,

$$(K_D^{\mu_2} Qt_1) \circ \pi_2 < \infty.$$

(iii) We have

$$(QK_D^{\mu_1} t_2) \circ \pi_1 = (QK_D^{\mu_1} K_D^{\mu_2} K_D^{\mu_1} t_2) \circ \pi_1 + (K_D^{\mu_1} t_2) \circ \pi_1.$$

Knowing that $K_D^{\mu_2} K_D^{\mu_1} t_2$ is bounded and $K_D^{\mu_1} t_2$ is finite, we have

$$(QK_D^{\mu_1} t_2) \circ \pi_1 < \infty.$$

(iv) We have

$$(Tt_2) \circ \pi_2 = (TK_D^{\mu_2} K_D^{\mu_1} t_2) \circ \pi_2 + t_2 \circ \pi_2.$$

Since $K_D^{\mu_2}K_D^{\mu_1}t_2$ is bounded and t_2 is finite,

$$(Tt_2)\circ\pi_2<\infty.$$

Let us show now that v and w are harmonic. From Corollary 2.1, it suffices to show that $v\circ i_j - K_D^{\mu_j}(v\circ i_k)$ and $w\circ i_j - K_D^{\mu_j}(w\circ i_k)$, $j\neq k$, $j,k\in\{1,2\}$, are L_j-harmonic functions on D.

(v) On the one hand,

$$v\circ i_1 - K_D^{\mu_1}(v\circ i_2) = Qt_1 - (K_D^{\mu_1}K_D^{\mu_2})Qt_1.$$

As

$$Qt_1 = (K_D^{\mu_1}K_D^{\mu_2})Qt_1 + t_1,$$

we get

$$v\circ i_1 - K_D^{\mu_1}(v\circ i_2) = t_1.$$

Since t_1 is an L_1-harmonic function on D, $v\circ i_1 - K_D^{\mu_1}(v\circ i_2)$ is L_1-harmonic on D.

On the other hand,

$$v\circ i_2 - K_D^{\mu_2}(v\circ i_1) = K_D^{\mu_2}Qt_1 - K_D^{\mu_2}Qt_1 = 0,$$

i.e. $v\circ i_2 - K_D^{\mu_2}(v\circ i_1)$ is L_2-harmonic on D. Then we conclude that v is harmonic on X.

(vi) Since

$$T = K_D^{\mu_2}QK_D^{\mu_1} + I, \tag{$*$}$$

we have

$$w\circ i_1 - K_D^{\mu_1}(w\circ i_2) = (QK_D^{\mu_1} - K_D^{\mu_1}K_D^{\mu_2}QK_D^{\mu_1} - K_D^{\mu_1})t_2.$$

As

$$Q = (K_D^{\mu_1}K_D^{\mu_2})Q + I,$$

we obtain

$$w\circ i_1 - K_D^{\mu_1}(w\circ i_2) = 0.$$

Using $(*)$, we have

$$w\circ i_2 - K_D^{\mu_2}(w\circ i_1) = (K_D^{\mu_2}QK_D^{\mu_1} + I - K_D^{\mu_2}QK_D^{\mu_1})t_2 = t_2.$$

Then $w\circ i_j - K_D^{\mu_j}(w\circ i_k)$ is L_j-harmonic on D and therefore, w is a harmonic function on X. □

Corollary 3.1. *Let t_i, $i = 1, 2$, be two positive L_i-hyperharmonic functions on D. Then the functions v and w defined on D by*

$$v := \begin{cases} (Qt_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2} Qt_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$w := \begin{cases} (QK_D^{\mu_1} t_2) \circ \pi_1 & \text{on } X_1, \\ (Tt_2) \circ \pi_2 & \text{on } X_2 \end{cases}$$

are hyperharmonic on X.

Theorem 3.2. *Let ν_1 and ν_2 be two positive Radon measures on $\triangle$ such that*

$$\int_\triangle K_D^{\mu_j} g^k(\cdot, y)\, d\nu_k(y) < \infty$$

and

$$\int_\triangle K_D^{\mu_j} K_D^{\mu_k} g^j(\cdot, y)\, d\nu_j(y)$$

is bounded on D, $j \neq k$, $j, k \in \{1, 2\}$. Then the function v defined on X_1 by

$$v := \int_\triangle (Qg^1(\cdot, y)) \circ \pi_1\, d\nu_1(y) + \int_\triangle (QK_D^{\mu_1} g^2(\cdot, y)) \circ \pi_1\, d\nu_2(y)$$

and on X_2 by

$$v := \int_\triangle (K_D^{\mu_2} Qg^1(\cdot, y)) \circ \pi_2\, d\nu_1(y) + \int_\triangle (Tg^2(\cdot, y)) \circ \pi_2\, d\nu_2(y)$$

is harmonic on X.

Proof: It suffices to replace the functions t_j from Theorem 3.1 with the L_j-harmonic functions $\int_\triangle g^j(\cdot, y)\, d\nu_j(y)$.

□

Corollary 3.2. *Let ν_1 and ν_2 be two positive Radon measures on $\triangle$ such that $\int_\triangle K_D^{\mu_1} g^2(\cdot, y)\, d\nu_2(y) < \infty$. Then*

$$(v, w) = \Big(\int_\triangle g^1(\cdot, y)\, d\nu_1(y) + \int_\triangle K_D^{\mu_1} g^2(\cdot, y)\, d\nu_2(y), \int_\triangle g^2(\cdot, y)\, d\nu_2(y) \Big)$$

is a biharmonic couple in the classical sense.

Theorem 3.3. *Let v be a positive harmonic function on X such that $K_D^{\mu_j} K_D^{\mu_k}(v \circ i_j)$ is bounded on D, $j,k \in \{1,2\}$, $j \neq k$. Then there exist two positive Radon measures ν_1 and ν_2 supported by $\triangle$ such that v can be represented on X_1 by*

$$v = \int_\triangle (Qg^1(\cdot,y)) \circ \pi_1 \, d\nu_1(y) + \int_\triangle (QK_D^{\mu_1} g^2(\cdot,y)) \circ \pi_1 \, d\nu_2(y)$$

and on X_2 by

$$v = \int_\triangle (K_D^{\mu_2} Qg^1(\cdot,y)) \circ \pi_2 \, d\nu_1(y) + \int_\triangle (Tg^2(\cdot,y)) \circ \pi_2 \, d\nu_2(y).$$

Proof: Let $(D_n)_n$ be an increasing sequence of relatively compact open subsets of D such that $D = \bigcup D_n$, and let v be a positive harmonic function on X. From Corollary 2.1, the positive functions $v \circ i_1 - K_D^{\mu_1}(v \circ i_2)$ and $v \circ i_2 - K_D^{\mu_2}(v \circ i_1)$ are L_1-harmonic and L_2-harmonic on D, respectively. Then for all $n \in \mathbb{N}$, both $\widehat{R}^{D_n}_{v \circ i_1 - K_D^{\mu_1}(v \circ i_2)}$ and $\widehat{R}^{D_n}_{v \circ i_2 - K_D^{\mu_2}(v \circ i_1)}$ are L_1-potential and L_2-potential on D, respectively. Therefore, there exist two positive Radon measures μ_n^1 and μ_n^2 on D such that

$$\widehat{R}^{D_n}_{v \circ i_1 - K_D^{\mu_1}(v \circ i_2)} = \int_D G_1(\cdot,y)\, d\mu_n^1(y)$$

and

$$\widehat{R}^{D_n}_{v \circ i_2 - K_D^{\mu_2}(v \circ i_1)} = \int_D G_2(\cdot,y)\, d\mu_n^2(y).$$

Then we have

$$\widehat{R}^{D_n}_{v \circ i_1 - K_D^{\mu_1}(v \circ i_2)} = \int_D g^1(\cdot,y)\, d\nu_n^1(y)$$

and

$$\widehat{R}^{D_n}_{v \circ i_2 - K_D^{\mu_2}(v \circ i_1)} = \int_D g^2(\cdot,y)\, d\nu_n^2(y)$$

with

$$d\nu_1(y) = G_1(x_0,\cdot) d\mu_n^1(y)$$

and

$$d\nu_2(y) = G_2(x_0,\cdot) d\mu_n^2(y).$$

Since $\widehat{R}^{D_n}_{v \circ i_j - K_D^{\mu_j}(v \circ i_k)}$ is L_j-harmonic on $D \setminus D_n$, $j \neq k$, $j,k \in \{1,2\}$, ν_n^1 and ν_n^2 are necessarily supported by $D \setminus D_n$.

Because of $\|\nu_n^j\| \leq (v \circ i_j)(x_0) - K_D^{\mu_j}(v \circ i_k)(x_0)$, $j = 1,2$, we may extract two subsequences $(\nu^1_{p(n)})$ and $(\nu^2_{p(n)})$ converging vaguely to two positive Radon measures ν^1 and ν^2 on $\bar{D} = \widehat{D}$. So, ν^1 and ν^2 are supported by $\triangle$. Therefore

$$\begin{cases} v \circ i_1 - K_D^{\mu_1}(v \circ i_2) = \int_\triangle g^1(\cdot,y)\, d\nu^1(y), \\ v \circ i_2 - K_D^{\mu_2}(v \circ i_1) = \int_\triangle g^2(\cdot,y)\, d\nu^2(y). \end{cases}$$

Hence

$$\begin{cases} v \circ i_1 = \int_\triangle g^1(\cdot,y)\,d\nu^1(y) + K_D^{\mu_1}\big(\int_\triangle g^2(\cdot,y)\,d\nu^2(y) + K_D^{\mu_2}(v\circ i_1)\big), \\ v \circ i_2 = \int_\triangle g^2(\cdot,y)\,d\nu^2(y) + K_D^{\mu_2}(v\circ i_1), \end{cases}$$

and

$$\begin{cases} v \circ i_1 = \int_\triangle g^1(\cdot,y)\,d\nu^1(y) + \int_\triangle K_D^{\mu_1} g^2(\cdot,y)\,d\nu^2(y) + K_D^{\mu_1}K_D^{\mu_2}(v\circ i_1), \\ v \circ i_2 = \int_\triangle g^2(\cdot,y)\,d\nu^2(y) + K_D^{\mu_2}(v\circ i_1). \end{cases}$$

Thus,

$$\begin{cases} Q(v \circ i_1) = \int_\triangle Qg^1(\cdot,y)\,d\nu^1(y) + \int_\triangle QK_D^{\mu_1} g^2(\cdot,y)\,d\nu^2(y) \\ \qquad\qquad +QK_D^{\mu_1}K_D^{\mu_2}(v\circ i_1), \\ v \circ i_2 = \int_\triangle g^2(\cdot,y)\,d\nu^2(y) + K_D^{\mu_2}(v\circ i_1). \end{cases}$$

Since

$$QK_D^{\mu_1}K_D^{\mu_2} + I = Q,$$

we obtain

$$\begin{cases} K_D^{\mu_1}K_D^{\mu_2}Q(v \circ i_1) + v\circ i_1 = \int_\triangle Qg^1(\cdot,y)\,d\nu^1(y) + \int_\triangle QK_D^{\mu_1} g^2(\cdot,y)\,d\nu^2(y) \\ \qquad\qquad +QK_D^{\mu_1}K_D^{\mu_2}(v\circ i_1), \\ v \circ i_2 = \int_\triangle g^2(\cdot,y)\,d\nu^2(y) + K_D^{\mu_2}(v\circ i_1). \end{cases}$$

Since $K_D^{\mu_1}K_D^{\mu_2}(v\circ i_1)$ is bounded,

$$\begin{cases} v \circ i_1 = \int_\triangle Qg^1(\cdot,y)\,d\nu_1(y) + \int_\triangle QK_D^{\mu_1} g^2(\cdot,y)\,d\nu_2(y), \\ v \circ i_2 = \int_\triangle K_D^{\mu_2}Qg^1(\cdot,y)\,d\nu_1(y) + \int_\triangle Tg^2(\cdot,y)\,d\nu_2(y). \end{cases}$$

So the function v can be written on X_1 as

$$v = \int_\triangle (Qg^1(\cdot,y))\circ\pi_1\,d\nu_1(y) + \int_\triangle (QK_D^{\mu_1}g^2(\cdot,y))\circ\pi_1\,d\nu_2(y)$$

and on X_2 as

$$v = \int_\triangle (K_D^{\mu_2}Qg^1(\cdot,y))\circ\pi_2 d\nu_1(y) + \int_\triangle (Tg^2(\cdot,y))\circ\pi_2\,d\nu_2(y).$$

□

Corollary 3.3 ([5])**.** *Let (v,w) be a positive biharmonic couple in the classical sense. Then there exist two positive Radon measures μ and ν supported by $\triangle$ such that*

$$\begin{cases} v = \int_\triangle g^1(\cdot,y)\,d\mu(y) + \int_\triangle K_D^{\mu_1}g^2(\cdot,y)\,d\nu(y), \\ w = \int_\triangle g^2(\cdot,y)\,d\nu(y). \end{cases}$$

4. Minimal points and uniqueness of the integral representation

Definition 4.1. (1) A positive L_1-harmonic (resp. L_2-harmonic) function h on D is called L_1-*minimal* (resp. L_2-*minimal*) if for any positive L_1-harmonic (resp. L_2-harmonic) function u on D, $u \le h$ implies $u = \alpha h$ with a factor $\alpha > 0$.

(2) A positive harmonic function h on X is called *minimal* if for any positive harmonic function u on X, $u \le h$ implies $u = \alpha h$ with a factor $\alpha > 0$.

Denote

$$\triangle_1 = \{y \in \triangle : g^1(\cdot, y) \text{ is } L_1\text{-minimal}\},$$
$$\triangle_2 = \{y \in \triangle : g^2(\cdot, y) \text{ is } L_2\text{-minimal}\}.$$

Note that for all $y \in \triangle$, the function $g^1(\cdot, y)$ (resp. $g^2(\cdot, y)$) is L_1-harmonic (resp. L_2-harmonic) on D.

Proposition 4.1. *Any positive harmonic function v on X such that $K_D^{\mu_k} K_D^{\mu_j}(v \circ i_k)$ is bounded for all $j \neq k$, $j, k \in \{1, 2\}$, can be written as $v = w + s$, where w and s are defined by*

$$w := \begin{cases} (Qv_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2} Q v_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$s := \begin{cases} (QK_D^{\mu_1} v_2) \circ \pi_1 & \text{on } X_1, \\ (Tv_2) \circ \pi_2 & \text{on } X_2, \end{cases}$$

with $v_j := v \circ i_j - K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$.

Remark 4.1. (1) Note that if $v = w' + s'$ is another decomposition of v with

$$w' := \begin{cases} (Qt_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2} Q t_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$s' := \begin{cases} (QK_D^{\mu_1} t_2) \circ \pi_1 & \text{on } X_1, \\ (Tt_2) \circ \pi_2 & \text{on } X_2, \end{cases}$$

where t_j, $j = 1, 2$, are L_j-harmonic on D, then $t_1 = v_1$ and $t_2 = v_2$.

(2) In the classical case, for any biharmonic couple (h_1, h_2) the following holds:

$$(h_1, h_2) = (t, 0) + (K_D^{\mu_1} h_2, h_2),$$

where t is a harmonic function on D. Note that $(K_D^{\mu_1} h_2, h_2)$ is a pure biharmonic couple (see [3] and [21], [22]).

Corollary 4.1. *Let v be a positive minimal harmonic function on X such that $K_D^{\mu_k} K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is bounded. Then $v = \alpha w$ or $v = \beta s$, where α and β are positive constants; w and s are defined as in Proposition* 4.1.

Proposition 4.2. *Let v be a positive function on X such that $K_D^{\mu_j}(v \circ i_k)$ is finite and $K_D^{\mu_k} K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is bounded. The following statements are equivalent.*

(1) *v is a minimal harmonic function on X.*
(2) *v_1 is a positive minimal L_1-harmonic function on D, or v_2 is a positive minimal L_2-harmonic function on D, where $v_j := v \circ i_j - K_D^{\mu_j}(v \circ i_k)$.*

Proof: Let v be a positive minimal harmonic function on X. Then we have $v = \alpha w$ or $v = \beta s$ by Corollary 4.1.

We shall show that if $v = \alpha w$, then v_1 is L_1-minimal and if $v = \beta s$, then v_2 is L_2-minimal.

(i) Case $v = \alpha w$:

Suppose that v_1 is not L_1-minimal. Then there exist two L_1-harmonic functions u_1 and u_2 such that $v_1 = u_1 + u_2$. So $v = \alpha f_1 + \alpha f_2$, with

$$f_1 = \begin{cases} (Qu_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2} Qu_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$f_2 = \begin{cases} (Qu_2) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2} Qu_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

It follows from Theorem 3.1 that f_1 and f_2 are harmonic on X. This contradicts that v is minimal.

(ii) Case $v = \beta s$:

Suppose that v_2 is not L_2-minimal. Then there exist two L_2-harmonic functions u_1 and u_2 such that $v_2 = u_1 + u_2$. Therefore $v = \beta s_1 + \beta s_2$, with

$$s_1 = \begin{cases} (QK_D^{\mu_1} u_1) \circ \pi_1 & \text{on } X_1, \\ (Tu_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$s_2 = \begin{cases} (QK_D^{\mu_1} u_2) \circ \pi_1 & \text{on } X_1, \\ (Tu_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

It follows from Theorem 3.1 that s_1 and s_2 are harmonic on X. This contradicts that v is minimal.

Conversely, suppose that v_1 is L_1-minimal and let us show that v is minimal. Assume the contrary and put $v = g_1 + g_2$, where g_1 and g_2 are harmonic functions on X. Then, from Proposition 4.1, there exist two L_1-harmonic functions s_1 and s_2, and two L_2-harmonic functions w_1 and w_2 such that

$$g_1 = \begin{cases} (Qs_1)\circ\pi_1 + (QK_D^{\mu_1}w_1)\circ\pi_1 & \text{on } X_1, \\ (K_D^{\mu_2}Qs_1)\circ\pi_2 + (Tw_1)\circ\pi_2 & \text{on } X_2, \end{cases}$$

and

$$g_2 = \begin{cases} (Qs_2)\circ\pi_1 + (QK_D^{\mu_1}w_2)\circ\pi_1 & \text{on } X_1, \\ (K_D^{\mu_2}Qs_2)\circ\pi_2 + (Tw_2)\circ\pi_2 & \text{on } X_2. \end{cases}$$

Therefore the function $g_1 + g_2$ is defined on X_1 by

$$g_1 + g_2 := (Q(s_1+s_2))\circ\pi_1 + (QK_D^{\mu_1}(w_1+w_2))\circ\pi_1$$

and on X_2 by

$$g_1 + g_2 := (K_D^{\mu_2}Q(s_1+s_2))\circ\pi_2 + (T(w_1+w_2))\circ\pi_2.$$

We deduce, from Proposition 4.1 and Remark 4.1.1, that $v_1 = s_1 + s_2$, which leads to a contradiction because v_1 is L_1-minimal.

In the same way, we suppose that v_2 is an L_2-minimal function and we show that v is a minimal function. □

By using the fact that any positive minimal L_j-harmonic function on D is proportional to $g^j(\cdot, y)$, $y \in \triangle_j$ (see [10]), w and s from Corollary 4.1 can be given more precisely.

Corollary 4.2. *Let v be a positive minimal harmonic function defined on X such that the function $K_D^{\mu_k}K_D^{\mu_j}(v\circ i_k)$, $j \neq k$, $j,k \in \{1,2\}$, is bounded. Then*

$$v = \alpha w \quad \textit{or} \quad v = \beta s,$$

with

$$w := \begin{cases} (Qg^1(\cdot,y))\circ\pi_1 & \textit{on } X_1,\ y\in\triangle_1, \\ (K_D^{\mu_2}Qg^1(\cdot,y))\circ\pi_2 & \textit{on } X_2,\ y\in\triangle_1, \end{cases}$$

and

$$s := \begin{cases} (QK_D^{\mu_1}g^2(\cdot,y))\circ\pi_1 & \textit{on } X_1,\ y\in\triangle_2, \\ (Tg^2(\cdot,y))\circ\pi_2, & \textit{on } X_2,\ y\in\triangle_2. \end{cases}$$

Proof: This result follows immediately from Proposition 4.2 and Corollary 4.1. □

Remark 4.2. Note that $K_D^{\mu_j}(v \circ i_k) < \infty$, $j \neq k$, $j,k \in \{1,2\}$, because v is a positive harmonic function on X.

Consider the family of mappings on the real vector space $\mathcal{H}(X)$ defined by

$$\varphi_K : \begin{cases} \mathcal{H}(X) & \longrightarrow & \mathbb{R}^+, \\ h & \longmapsto & \varphi_K(h), \end{cases}$$

where

$$\varphi_K(h) = \sup_{x \in K} (|h \circ i_1(x)| + |h \circ i_2(x)|),$$

and K is a compact subset of D. (φ_K) is a family of semi-norms on $\mathcal{H}(X)$ and these semi-norms define a topology that makes $\mathcal{H}(X)$ a metrizable topological space. It follows that this space is locally convex.

The cone $\mathcal{H}^+(X) = \{h \in \mathcal{H}(X) : h \geq 0\}$ defines on $\mathcal{H}(X)$ an order relation called specific order:

$$h_1 \prec h_2 \quad \Longleftrightarrow \quad h_2 = h_1 + g, \quad g \in \mathcal{H}^+(X).$$

Equipped with this order, $\mathcal{H}^+(X)$ is a lattice. The minimal harmonic functions are the points of the extreme generatrices of $\mathcal{H}^+(X)$. We recall that a base of $\mathcal{H}^+(X)$ is the intersection of $\mathcal{H}^+(X)$ with a closed hyperplane.

Let us consider the set

$$B := \{h \in \mathcal{H}^+(X) : (h \circ i_1)(x_o) + (h \circ i_2)(x_o) = 1\}, \quad x_o \in D.$$

B is a compact base of the cone $\mathcal{H}^+(X)$. Indeed, the mapping

$$\phi_{x_o} : \begin{cases} \mathcal{H}^+(X) & \longrightarrow & \mathbb{R}, \\ h & \longmapsto & (h \circ i_1)(x_o) + (h \circ i_2)(x_o) = 1 \end{cases}$$

is a continuous linear form. Then it defines a closed hyperplane B such that the origin $0 \notin B$. Then, B is equicontinuous at any point $x \in X$. So, we conclude, by Ascoli's theorem, that B is compact. Note that $\mathcal{H}^+(X) = \mathbb{R}^+ B$. Let $\mathcal{E}(B)$ denote the set of all extreme points of $\mathcal{H}^+(X)$ belonging to B (see [11]). Moreover, using Corollary 4.2, we have

$$\mathcal{E}(B) = \mathcal{E}_1(B) \cup \mathcal{E}_2(B),$$

where

$$\mathcal{E}_1(B) = \left\{ h \in \mathcal{E}(B) : \exists\, \alpha \in \mathbb{R}^+,\ \exists\, y \in \triangle_1 : h = \begin{cases} (\alpha Q g^1(\cdot, y)) \circ \pi_1 & \text{on } X_1 \\ (\alpha K_D^{\mu_2} Q g^1(\cdot, y)) \circ \pi_2 & \text{on } X_2 \end{cases} \right\}$$

and

$$\mathcal{E}_2(B) = \left\{ h \in \mathcal{E}(B) : \exists\, \beta \in \mathbb{R}^+,\ \exists\, y \in \triangle_2 : h = \begin{cases} (\beta Q K_D^{\mu_1} g^2(\cdot, y)) \circ \pi_1 & \text{on } X_1 \\ (\beta T g^2(\cdot, y)) \circ \pi_2 & \text{on } X_2 \end{cases} \right\}.$$

We recall the following results which are useful for showing the uniqueness of an integral representation (see [16]).

Definition 4.2 ([16]). Let Γ a closed convex cone. A mapping $\ell : \lambda \longmapsto e_\lambda$ of a separated topological space Ω in $\mathcal{E}(\Gamma)$ is called a *parametrization* of $\mathcal{E}(\Gamma)$, if any element $\gamma \in \mathcal{E}(\Gamma)$ is proportional to a unique element e_λ. It is called *admissible* if it is continuous and the inverse mapping $\mathcal{E}(\Gamma) \longrightarrow \Omega$ is universally measurable.

Theorem A ([16]). *Let a closed cone convex Γ and an admissible parametrization ℓ of $\mathcal{E}(\Gamma)$ be given. For any $\gamma \in \Gamma$, there exist a positive Radon measure μ on Ω such that*

$$\gamma = \int_\Omega e_\lambda d\mu(\lambda).$$

Theorem B ([16]). *The measure μ given by Theorem A is unique for any $\gamma \in \Gamma$, if and only if the cone Γ is a lattice.*

Theorem 4.1. *If $g^1(x,\cdot)$, $x \in D$, separates $\triangle_1$ and $g^2(x,\cdot)$, $x \in D$, separates $\triangle_2$, then for any positive harmonic function v on X such that the function $K_D^{\mu_k} K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j,k \in \{1,2\}$, is bounded, there exist two unique measures ν_1 and ν_2 supported respectively by $\triangle_1$ and $\triangle_2$ such that v can be represented on X_1 by*

$$v = \int_{\triangle_1} (Qg^1(\cdot,y)) \circ \pi_1 \, d\nu_1(y) + \int_{\triangle_2} (QK_D^{\mu_1} g^2(\cdot,y)) \circ \pi_1 \, d\nu_2(y)$$

and on X_2 by

$$v = \int_{\triangle_1} (K_D^{\mu_2} Qg^1(\cdot,y)) \circ \pi_2 \, d\nu_1(y) + \int_{\triangle_2} (Tg^2(\cdot,y)) \circ \pi_2 \, d\nu_2(y).$$

Proof: If $v = 0$, we have $\nu_1 = \nu_2 = 0$.

If $v \neq 0$, we may assume without loss of generality that $v \in B$. Consider the mapping

$$\Psi : \begin{cases} \triangle_1 \cup \triangle_2 \longrightarrow \mathcal{E}(B) \\ y \longmapsto \Psi(y) \end{cases}$$

where $\Psi(y)$ is defined by

$$\Psi(y) := \begin{cases} (Qg^1(\cdot,y)) \circ \pi_1 & \text{on } X_1 \\ (K_D^{\mu_2} Qg^1(\cdot,y)) \circ \pi_2 & \text{on } X_2 \end{cases}, \quad y \in \triangle_1,$$

$$\Psi(y) := \begin{cases} (QK_D^{\mu_1} g^2(\cdot,y)) \circ \pi_1 & \text{on } X_1 \\ (Tg^2(\cdot,y)) \circ \pi_2 & \text{on } X_2 \end{cases}, \quad y \in \triangle_2.$$

The mapping Ψ is bijective because $g^1(x,\cdot)$ and $g^2(x,\cdot)$ separate $\triangle_1$ and $\triangle_2$, respectively. Ψ and its inverse Ψ^{-1} are continuous because g^1 and g^2 are continuous on $\triangle \times D$. Then there exists, by Theorem B, a unique measure ν supported by $\triangle_1 \cup \triangle_2$ such that

$$v = \int_{\triangle_1 \cup \triangle_2} \Psi(y) \, d\nu(y).$$

Let ν_j, $j = 1, 2$, be the restriction of the measure ν to $\triangle_j$. Then v may be written on X_1 as

$$v = \int_{\triangle_1} (Qg^1(\cdot, y)) \circ \pi_1 \, d\nu_1(y) + \int_{\triangle_2} (QK_D^{\mu_1} g^2(\cdot, y)) \circ \pi_1 \, d\nu_2(y)$$

and on X_2 as

$$v = \int_{\triangle_1} (K_D^{\mu_2} Qg^1(\cdot, y)) \circ \pi_2 \, d\nu_1(y) + \int_{\triangle_2} (Tg^2(\cdot, y)) \circ \pi_2 \, d\nu_2(y).$$

□

Let t_i, $i = 1, 2$, be two positive L_i-harmonic functions on D such that the function $K_D^{\mu_j} t_k$ is finite and the function $K_D^{\mu_k} K_D^{\mu_j} t_k$, $j \neq k$, $j, k \in \{1, 2\}$, is bounded on D. By [10] and [12], there exists a unique measure ν_{t_j}, supported by $\triangle_j$, such that $t_j = \int_{\triangle_j} g^j(\cdot, y) \, d\nu_{t_j}(y)$, $j = 1, 2$. We consider the harmonic function w from Theorem 3.1 defined on X by

$$w := \begin{cases} (Qt_1 + QK_D^{\mu_1} t_2) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2} Qt_1 + Tt_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

Corollary 4.3. *If the functions $g^j(x, \cdot)$, $x \in D$, separate $\triangle_j$, $j = 1, 2$, then w is written on X_1 by*

$$w = \int_{\triangle_1} (Qg^1(\cdot, y)) \circ \pi_1 \, d\nu_{t_1}(y) + \int_{\triangle_2} (QK_D^{\mu_1} g^2(\cdot, y)) \circ \pi_1 \, d\nu_{t_2}(y),$$

and on X_2 by

$$w = \int_{\triangle_1} (K_D^{\mu_2} Qg^1(\cdot, y)) \circ \pi_2 \, d\nu_{t_1}(y) + \int_{\triangle_2} (Tg^2(\cdot, y) \circ \pi_2 \, d\nu_{t_2}(y).$$

PROOF: It suffices to replace t_j, $j = 1, 2$, with their Martin representations in the expression of w, and the result follows from the uniqueness of the measures ν_j in Theorem 4.1. □

Remark 4.3. By Corollary 4.3 , we have $\nu_{t_j}(\triangle \backslash \triangle_j) = 0$, thus $\nu_{t_j}(\triangle \backslash (\triangle_1 \cup \triangle_2)) = 0$, $j = 1, 2$.

5. Dirichlet problem on the Martin boundary associated with (S)

Given a couple of functions (u_1, u_2) defined on $\triangle$, the Dirichlet problem on $\triangle$ consists to find a couple of functions (h_1, h_2) solving the system (S) such that

$$\lim_{x \longrightarrow y} h_i(x) = u_i(y) \quad \forall\, y \in \triangle.$$

The couple (u_1, u_2) can be identified with a function f on $\bar{\triangle} := \bigcup_{j=1}^{2} \triangle \times \{j\}$ such that $f \circ i_j = u_j$, where i_j, $j = 1, 2$, denote always the mappings of $\triangle$ in $\triangle \times \{j\}$ defined by $i_j(z) := (z, j)$, $z \in \triangle$. The Dirichlet problem may be stated as follows: for a given function f defined on $\bar{\triangle}$, determine, if possible, a harmonic function H_f on X such that $H_f(x) \longrightarrow f(y)$ as $x \longrightarrow y$ for each $y \in \bar{\triangle}$. As in harmonic and biharmonic cases, there are some examples where there is no solution of this problem. In this section, we will discuss the Perron-Wiener-Brelot (PWB) approach to the Dirichlet problem. To this end, we give the following definition.

Definition 5.1. Let h_1 (resp. h_2) be a strictly positive L_1-harmonic (resp. L_2-harmonic) function on D, and let h be the function defined on X by

$$h := \begin{cases} h_1 \circ \pi_1 & \text{on } X_1, \\ h_2 \circ \pi_2 & \text{on } X_2. \end{cases}$$

A function v on X is called *h-harmonic* (resp. *h-hyperharmonic*, *h-superharmonic*) *on* X if and only if the function u defined on X by

$$u := \begin{cases} (h_1(v \circ i_1)) \circ \pi_1 & \text{on } X_1, \\ (h_2(v \circ i_2)) \circ \pi_2 & \text{on } X_2 \end{cases}$$

is harmonic (resp. hyperharmonic, superharmonic) on X.

We also define the upper and lower class associated with a function defined on $\bar{\triangle}$. Let f be a function defined on $\bar{\triangle}$ and let h be a function defined on X as in Definition 5.1. We define:

$$\bar{U}_f := \{v : v \text{ is } h\text{-hyperharmonic and bounded from below on } X \text{ and } \liminf_{x \to y} v(x) \geq f(y),\ \forall\, y \in \bar{\triangle}\}$$

and

$$\underline{U}_f := \{s : s \text{ is } h\text{-hypoharmonic and bounded from above on } X \text{ and } \limsup_{x \to y} v(x) \leq f(y),\ \forall\, y \in \bar{\triangle}\}.$$

We note that $\bar{U}_f$ and $\underline{U}_f$ are never empty since they contain the constant functions $+\infty$ and $-\infty$ respectively, and that $\bar{U}_f = -\underline{U}_{-f}$. Put

$$\bar{H}_f := \inf \bar{U}_f \quad \text{and} \quad \underline{H}_f := \sup \underline{U}_f.$$

f is called *h-resolutive* if $\bar{H}_f$ and $\underline{H}_f$ are equal and h-harmonic on X. If f is h-resolutive, then we define $H_f^h := \bar{H}_f = \underline{H}_f$ and call H_f^h the PWB-*solution of the Dirichlet problem on* X with boundary function f. If $f \circ i_j$ is h_j-resolutive on $\triangle$, we call $H_{f\circ i_j}^{h_j}$ the PWB-solution of Dirichlet problem on D *associated with* $f \circ i_j$, $j = 1, 2$.

Further properties of PWB solutions.

Let f and g be two functions defined on $\bar{\triangle}$. Then we have

(i) $\underline{H}_f^h = -\bar{H}_{-f}^h$.

(ii) $\underline{H}_f^h \leq \bar{H}_f^h$.

(iii) $\underline{H}_f^h \leq \underline{H}_g^h$ and $\bar{H}_f^h \leq \bar{H}_g^h$ if $f \leq g$.

(iv) Let f, g be two h-resolutive functions and $\alpha \in \mathbb{R}$. Then $f + g$ and αf are h-resolutive and

$$H_{f+g}^h = H_f^h + H_g^h, \quad H_{\alpha f}^h = \alpha H_f^h.$$

(v) If $\underline{U}_f \cap (-S(X)) \neq \emptyset$ (resp. $\bar{U}_f \cap S(X) \neq \emptyset$), then the function $\bar{H}_f^h$ (resp. $\underline{H}_f^h$) is identically ∞, or h-harmonic on X.

Let f be a positive function on $\bar{\triangle}$ such that $f \circ i_2 = 0$ and w the function defined on X by

$$w := \begin{cases} (\frac{1}{h_1} Q(h_1.\bar{H}_{f\circ i_1}^{h_1})) \circ \pi_1 & \text{on} \quad X_1, \\ (\frac{1}{h_2} K_D^{\mu_2} Q(h_1.\bar{H}_{f\circ i_1}^{h_1})) \circ \pi_2 & \text{on} \quad X_2. \end{cases}$$

We have $\bar{H}_f^h \leq w$. Indeed, it follows from Corollary 3.1 that w is a positive h-hyperharmonic function on X and moreover, we have

$$\liminf_{x \longrightarrow y} (w \circ i_1)(x) \geq (f \circ i_1)(y), \quad \text{for all} \quad y \in \triangle$$

and

$$\liminf_{x \longrightarrow y} (w \circ i_2)(x) \geq 0, \quad \text{for all} \quad y \in \triangle.$$

Hence, $w \in \bar{U}_f$. Thus $\bar{H}_f^h \leq w$ and therefore if $\bar{H}_f^h = +\infty$ then $w = +\infty$. If $\bar{H}_f^h < \infty$, we have

Lemma 5.1. *Let f be a positive function on $\bar{\triangle}$ such that $f \circ i_2 = 0$ and $K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1))$ is bounded on D. Then we have*

$$\bar{H}_f^h = \begin{cases} (\frac{1}{h_1} Q(h_1 \bar{H}^{h_1}_{f \circ i_1})) \circ \pi_1 & \text{on } X_1, \\ (\frac{1}{h_2} K_D^{\mu_2} Q(h_1 \bar{H}^{h_1}_{f \circ i_1})) \circ \pi_2 & \text{on } X_2. \end{cases}$$

Proof: It suffices to show that $w \leq \bar{H}_f^h$.

(a) Let us show that $w \circ i_1 \leq \bar{H}_f^h \circ i_1$.

It follows from property (v) of PWB solutions that the function $\bar{H}_f^h$ is h-harmonic on X. Then the function

$$\bar{u} := \begin{cases} (h_1(\bar{H}_f^h \circ i_1)) \circ \pi_1 & \text{on } X_1, \\ (h_2(\bar{H}_f^h \circ i_2)) \circ \pi_2 & \text{on } X_2 \end{cases}$$

is a positive harmonic function on X, and by Corollary 2.1, the functions $\bar{u}_j = h_j(\bar{H}_f^h \circ i_j) - K_D^{\mu_j}(h_k(\bar{H}_f^h \circ i_k))$, $j, k \in \{1, 2\}$, $j \neq k$ are positive and L_j-harmonic on D. Put $v_j := \frac{1}{h_j} \bar{u}_j$. On the one hand, we have

$$K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \leq (h_2(\bar{H}_f^h \circ i_2)),$$

hence

$$K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \leq K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)),$$

i.e.

$$K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \leq (h_1(\bar{H}_f^h \circ i_1) - h_1.v_1).$$

So,

$$Q(h_1.v_1) + Q K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \leq Q(h_1(\bar{H}_f^h \circ i_1)).$$

Since

$$Q K_D^{\mu_1} K_D^{\mu_2} + I = Q,$$

we get

$$Q(h_1 v_1) + Q K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \leq Q K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) + h_1(\bar{H}_f^h \circ i_1).$$

Therefore,

$$Q(h_1 v_1) \leq h_1(\bar{H}_f^h \circ i_1). \tag{5.1.1}$$

On the other hand,

$$\liminf_{x\longrightarrow y} v_1(x) = \liminf_{x\longrightarrow y}(\bar{H}^h_f \circ i_1 - \frac{1}{h_1}K^{\mu_1}_D(h_2(\bar{H}^h_f \circ i_2)))(x)$$
$$\geq (f\circ i_1)(y) - \limsup_{x\longrightarrow y}(\frac{1}{h_1}K^{\mu_1}_D(h_2(\bar{H}^h_f \circ i_2)))(x)$$

for all $y \in \triangle$. Since

$$\limsup_{x\longrightarrow y}(\frac{1}{h_1}K^{\mu_1}_D(h_2(\bar{H}^h_f \circ i_2)))(x)$$
$$\leq \int_D \limsup_{x\longrightarrow y} \frac{1}{h_1(x)} G_1(x,z)h_2(z)(\bar{H}^h_f \circ i_2)(z)\, d\mu_1(z),$$

and $\limsup_{x\longrightarrow y} \frac{1}{h_1(x)}G_1(x,z) = 0$ ν_{h_1}-a.e. on $\triangle_1$, where ν_{h_1} is the measure associated with h_1 in the Martin representation ([13, p. 218]), we have, by Remark 4.3, $\nu_{h_1}(\triangle\backslash\triangle_1) = 0$. Hence $\limsup_{x\longrightarrow y} \frac{1}{h_1(x)}G_1(x,z) = 0$ ν_{h_1}-a.e. on $\triangle$. Thus $\liminf_{x\longrightarrow y} v_1(x) \geq (f\circ i_1)(y)$ ν_{h_1}-a.e. on $\triangle$. Hence v_1 is a positive h_1-L_1-hyperharmonic function on D and $\liminf_{x\longrightarrow y} v_1(x) \geq (f\circ i_1)(y)$ ν_{h_1}-a.e. on $\triangle$. So

$$v_1 \geq \bar{H}^{h_1}_{f\circ i_1}. \tag{5.1.2}$$

Thus, by (5.1.1), we have

$$Q(h_1\bar{H}^{h_1}_{f\circ i_1}) \leq (h_1(\bar{H}^h_f \circ i_1)).$$

(b) Let us show that $w\circ i_2 \leq (\bar{H}^h_f \circ i_2)$.

It follows from (a) that

$$Q(h_1\bar{H}^{h_1}_{f\circ i_1}) \leq (h_1(\bar{H}^h_f \circ i_1)).$$

Then,

$$K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f\circ i_1}) \leq K^{\mu_2}_D(h_1(\bar{H}^h_f \circ i_1)) \leq (h_2(\bar{H}^h_f \circ i_2)).$$

This finishes the proof. □

Remark 5.1. The result of Lemma 5.1 is still valid if instead of the assumption $K^{\mu_1}_D K^{\mu_2}_D(h_1(\bar{H}^h_f \circ i_1))$ is bounded, we suppose only that $Q(h_1(\bar{H}^h_f \circ i_1))$ is finite.

Let f be a positive function on $\bar{\triangle}$ such that $f\circ i_1 = 0$ and $\tilde{w}$ the function defined on X by

$$\tilde{w} := \begin{cases} (\frac{1}{h_1}QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f\circ i_2}))\circ\pi_1 & \text{on } X_1, \\ (\frac{1}{h_2}T(h_2\bar{H}^{h_2}_{f\circ i_2}))\circ\pi_2 & \text{on } X_2. \end{cases}$$

We have $\bar{H}^h_f \leq \tilde{w}$. Therefore if $\bar{H}^h_f = +\infty$, then $\tilde{w} = +\infty$. If $\bar{H}^h_f < \infty$, we have:

Lemma 5.2. *Let f be a positive function on $\bar{\triangle}$ such that $f \circ i_1 = 0$ and $K_D^{\mu_2} K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2))$ is bounded on D. Then*

$$\bar{H}_f^h = \begin{cases} (\frac{1}{h_1} Q K_D^{\mu_1}(h_2 \bar{H}_{f\circ i_2}^{h_2})) \circ \pi_1 & \text{on } X_1, \\ (\frac{1}{h_2} T(h_2 \bar{H}_{f\circ i_2}^{h_2})) \circ \pi_2 & \text{on } X_2. \end{cases}$$

PROOF: It suffices to show that $\tilde{w} \le \bar{H}_f^h$.
(a) Let us show that $\tilde{w} \circ i_1 \le \bar{H}_f^h \circ i_1$.

By the property (v) of PWB solutions, the function $\bar{H}_f^h$ is h-harmonic on X. Then the function

$$\bar{u} := \begin{cases} (h_1(\bar{H}_f^h \circ i_1)) \circ \pi_1 & \text{on } X_1, \\ (h_2(\bar{H}_f^h \circ i_2)) \circ \pi_2 & \text{on } X_2 \end{cases}$$

is a positive harmonic function on X and by Corollary 2.1, $\bar{u}_j = h_j(\bar{H}_f^h \circ i_j) - K_D^{\mu_j}(h_k(\bar{H}_f^h \circ i_k))$, $j, k \in \{1, 2\}$, $j \ne k$, are positive and L_j-harmonic functions on D. Put $v_j := \frac{1}{h_j}\bar{u}_j$. On the one hand, we have

$$K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) \le (h_1(\bar{H}_f^h \circ i_1)),$$

hence

$$K_D^{\mu_1}(h_2 v_2 + K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1))) \le h_1(\bar{H}_f^h \circ i_1)$$

and

$$Q K_D^{\mu_1}(h_2 v_2) + Q K_D^{\mu_1} K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \le Q(h_1(\bar{H}_f^h \circ i_1)).$$

Since

$$Q K_D^{\mu_1} K_D^{\mu_2} + I = Q,$$

we get

$$Q K_D^{\mu_1}(h_2 . v_2) \le h_1(\bar{H}_f^h \circ i_1).$$

As in the proof of Lemma 5.1, we show that $\liminf_{x \longrightarrow y} v_2(x) \ge (f \circ i_2)(y)$ ν_{h_2}-a.e. on $\triangle$. Since v_2 is a positive $h_2 - L_2$-hyperharmonic function and $\liminf_{x \longrightarrow y} v_2(x) \ge (f \circ i_2)(y)$, ν_{h_2}-a.e. on $\triangle$, we obtain

$$v_2 \ge \bar{H}_{f\circ i_2}^{h_2}, \tag{5.1.2}$$

hence

$$Q K_D^{\mu_1}(h_2 \bar{H}_{f\circ i_2}^{h_2}) \le (h_1(\bar{H}_f^h \circ i_1)).$$

(b) Let us show that $\tilde{w} \circ i_2 \leq (\bar{H}_f^h \circ i_2))$. We have

$$K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) \leq h_1(\bar{H}_f^h \circ i_1).$$

So

$$K_D^{\mu_2} K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) \leq K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) = h_2(\bar{H}_f^h \circ i_2) - h_2 v_2.$$

Hence

$$T(h_2.v_2) + TK_D^{\mu_2} K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) \leq T(h_2(\bar{H}_f^h \circ i_2)).$$

Since

$$TK_D^{\mu_2} K_D^{\mu_1} + I = T,$$

we get

$$T(h_2 \bar{H}_{f\circ i_2}^{h_2}) \leq (h_2(\bar{H}_f^h \circ i_2)).$$

□

Remark 5.2. The result of Lemma 5.2 is still valid if instead of the assumption $K_D^{\mu_2} K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2))$ is bounded, we suppose only that $T(h_2(\bar{H}_f^h \circ i_2))$ is finite.

Let f be a positive function on $\bar{\triangle}$ and let w' be the function defined on X by

$$w' := \begin{cases} \frac{1}{h_1}(Q(h_1 \bar{H}_{f\circ i_1}^{h_1}) + QK_D^{\mu_1}(h_2 \bar{H}_{f\circ i_2}^{h_2})) \circ \pi_1 & \text{on } X_1, \\ \frac{1}{h_2}(K_D^{\mu_2} Q(h_1 \bar{H}_{f\circ i_1}^{h_1}) + T(h_2 \bar{H}_{f\circ i_2}^{h_2})) \circ \pi_2 & \text{on } X_2. \end{cases}$$

We have $\bar{H}_f^h \leq w'$. Therefore, if $\bar{H}_f^h = +\infty$ then $w' = +\infty$. If $\bar{H}_f^h < \infty$, we have

Proposition 5.1. *Let f be a positive function on $\bar{\triangle}$ such that $K_D^{\mu_j} K_D^{\mu_k}(h_j(\bar{H}_f^h \circ i_j))$ is bounded on D, $j, k \in \{1,2\}$, $j \neq k$. Then we have*

$$\bar{H}_f^h = \begin{cases} \frac{1}{h_1}(Q(h_1 \bar{H}_{f\circ i_1}^{h_1}) + QK_D^{\mu_1}(h_2 \bar{H}_{f\circ i_2}^{h_2})) \circ \pi_1 & \textit{on } X_1, \\ \frac{1}{h_2}(K_D^{\mu_2} Q(h_1 \bar{H}_{f\circ i_1}^{h_1}) + T(h_2 \bar{H}_{f\circ i_2}^{h_2})) \circ \pi_2 & \textit{on } X_2. \end{cases}$$

Proof: It suffices to show that $w' \leq \bar{H}_f^h$.
(a) Let us show that $w' \circ i_1 \leq \bar{H}_f^h \circ i_1$.

By the property (v) of PWB solutions, the function $\bar{H}_f^h$ is h-harmonic on X. Then the function

$$\bar{u} := \begin{cases} (h_1(\bar{H}_f^h \circ i_1)) \circ \pi_1 & \text{on } X_1, \\ (h_2(\bar{H}_f^h \circ i_2)) \circ \pi_2 & \text{on } X_2 \end{cases}$$

is a positive harmonic on X and by Corollary 2.1, $\bar{u}_j = h_j(\bar{H}_f^h \circ i_j) - K_D^{\mu_j}(h_k(\bar{H}_f^h \circ i_k))$, $j,k \in \{1,2\}$, $j \neq k$, are positive L_j-harmonic on D. Put $v_j = \frac{1}{h_j}\bar{u}_j$. On the one hand,

$$h_1.v_1 + K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) = h_1(\bar{H}_f^h \circ i_1)$$

and

$$h_2 v_2 + K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) = h_2(\bar{H}_f^h \circ i_2).$$

Hence

$$Q(h_1 v_1) + QK_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) = Q(h_1(\bar{H}_f^h \circ i_1))$$

and

$$QK_D^{\mu_1}(h_2.v_2) + QK_D^{\mu_1}K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) = QK_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)).$$

Since

$$QK_D^{\mu_1}K_D^{\mu_2} + I = Q,$$

we have

$$Q(h_1.v_1) + QK_D^{\mu_1}(h_2.v_2) = h_1(\bar{H}_f^h \circ i_1).$$

It follows from (5.1.2) and (5.2.1) that

$$Q(h_1 \bar{H}_{f\circ i_1}^{h_1}) + QK_D^{\mu_1}(h_2 \bar{H}_{f\circ i_2}^{h_2}) \leq h_1(\bar{H}_f^h \circ i_1).$$

Similarly, we show that

$$\frac{1}{h_2}(K_D^{\mu_2} Q(h_1 \bar{H}_{f\circ i_1}^{h_1}) + T(h_2 \bar{H}_{f\circ i_2}^{h_2})) \leq h_2(\bar{H}_f^h \circ i_2).$$

□

Remark 5.3. The result of Proposition 5.1 is still valid if instead of the assumption $K_D^{\mu_j}K_D^{\mu_k}(h_j(\bar{H}_f^h \circ i_j))$ is bounded on D, $j,k \in \{1,2\}$, $j \neq k$, we suppose that $Q(h_1(\bar{H}_f^h \circ i_1)) < \infty$ and $T(h_2(\bar{H}_f^h \circ i_2)) < \infty$.

h-negligible sets.

Definition 5.2. Let e be a subset of $\bar{\triangle}$. e is called *h-negligible* if $\bar{H}_{1_e}^h = 0$, where 1_e is the indicator of the set e.

Let $\tilde{e}$ be a subset of $\triangle$. $\tilde{e}$ is called *h_j-negligible* if and only if $\bar{H}_{1_{\tilde{e}}}^{h_j} = 0$, $j = 1,2$.

Proposition 5.2. *Let $e \subset \bar{\triangle} = (\triangle \times \{1\}) \cup (\triangle \times \{2\})$ be such that $e = (e_1 \times \{1\}) \cup (e_2 \times \{2\})$, where $e_j \subset \triangle$, $j = 1,2$. The following are equivalent:*

(1) *e is h-negligible;*
(2) *e_j is h_j-negligible, $j = 1,2$.*

Proof: Suppose that e is h-negligible; then $\bar{H}^h_{1_e} = 0$. By Proposition 5.1, we have

$$\bar{H}^h_{1_e} = \begin{cases} \frac{1}{h_1}(Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{1_e\circ i_2}))\circ\pi_1 & \text{on } X_1, \\ \frac{1}{h_2}(K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) + T(h_2\bar{H}^{h_2}_{1_e\circ i_2}))\circ\pi_2 & \text{on } X_2, \end{cases}$$

hence

$$Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) = -QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{1_e\circ i_2}),\ K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) = -T(h_2\bar{H}^{h_2}_{1_e\circ i_2}).$$

Since the functions $h_j\bar{H}^{h_j}_{1_e\circ i_j}$, $j=1,2$, are positive, $\bar{H}^{h_j}_{1_e\circ i_j} = 0$, $j=1,2$. Since $1_e\circ i_j = 1_{e_j}$, $\bar{H}^{h_j}_{1_{e_j}} = 0$, i.e., the set e_j is h_j-negligible. The converse is obvious. □

Proposition 5.3. *Let f and $\tilde{f}$ be two positive functions defined on $\bar{\triangle}$ such that $e=\{f\neq\tilde{f}\}$ is a h-negligible set. Then $\bar{H}^h_f = \bar{H}^h_{\tilde{f}}$.*

Proof: We have $e=\{f\neq\tilde{f}\}=(e_1\times\{1\})\cup(e_2\times\{2\})$, where $e_j=\{f\circ i_j\neq\tilde{f}\circ i_j\}$, $j=1,2$, and e is h-negligible. Then, by Proposition 5.2, e_j is h_j-negligible. Thus $\bar{H}^{h_j}_{f\circ i_j} = \bar{H}^{h_j}_{\tilde{f}\circ i_j}$, $j=1,2$. Therefore, by Proposition 5.1, $\bar{H}^h_f = \bar{H}^h_{\tilde{f}}$. □

Lemma 5.3. *Let f be a positive function on $\bar{\triangle}$ such that $K^{\mu_j}_D K^{\mu_k}_D(h_j(\bar{H}^h_f\circ i_j))$ is bounded on D, $j,k\in\{1,2\}$, $j\neq k$. Then we have*

$$h_j\bar{H}^{h_j}_{f\circ i_j} = h_j(\bar{H}^h_f\circ i_j) - K^{\mu_j}_D(h_k(\bar{H}^h_f\circ i_k)).$$

Proof: By Proposition 5.1, we have

$$\begin{cases} \bar{H}^h_f\circ i_1 = \frac{1}{h_1}(Q(h_1\bar{H}^{h_1}_{f\circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f\circ i_2})), \\ \bar{H}^h_f\circ i_2 = \frac{1}{h_2}(K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f\circ i_1}) + T(h_2\bar{H}^{h_2}_{f\circ i_2})). \end{cases}$$

Then

$$\begin{cases} h_1\bar{H}^h_f\circ i_1 = (Q(h_1\bar{H}^{h_1}_{f\circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f\circ i_2})), \\ h_2\bar{H}^h_f\circ i_2 = (K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f\circ i_1}) + T(h_2\bar{H}^{h_2}_{f\circ i_2})). \end{cases}$$

Hence

$$\begin{cases} K^{\mu_2}_D(h_1.\bar{H}^h_f\circ i_1) = K^{\mu_2}_D(Q(h_1\bar{H}^{h_1}_{f\circ i_1})) + K^{\mu_2}_D(QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f\circ i_2})), \\ h_2\bar{H}^h_f\circ i_2 = (K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f\circ i_1}) + T(h_2\bar{H}^{h_2}_{f\circ i_2})). \end{cases}$$

Since $\bar{H}_f^h$ is h-harmonic on X, $K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) < \infty$. Thus,

$$h_2(\bar{H}_f^h \circ i_2) - K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) = T(h_2\bar{H}_{f\circ i_2}^{h_2}) - K_D^{\mu_2}QK_D^{\mu_1}(h_2\bar{H}_{f\circ i_2}^{h_2}).$$

Since

$$T = K_D^{\mu_2}QK_D^{\mu_1} + I,$$

we get

$$h_2(\bar{H}_f^h \circ i_2) - K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) = h_2\bar{H}_{f\circ i_2}^{h_2}.$$

Similarly, we show that

$$h_1(\bar{H}_f^h \circ i_1) - K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) = h_1\bar{H}_{f\circ i_1}^{h_1}.$$

□

Theorem 5.1. *Let f be a positive function defined on $\bar{\triangle}$ such that $K_D^{\mu_j}K_D^{\mu_k}(h_j(\bar{H}_f^h \circ i_j))$ is bounded, $j \neq k$, $j,k \in \{1,2\}$. The following are equivalent:*

(a) *f is h-resolutive;*
(b) (1) *$f \circ i_j$ is h_j-resolutive on $\triangle$, $j = 1,2$, and*
 (2) *$K_D^{\mu_k}(h_jH_{f\circ i_j}^{h_j})$ is finite, $j \neq k$, $j,k \in \{1,2\}$.*

PROOF: Suppose that (b) holds. Then the function $h_jH_{f\circ i_j}^{h_j}$ is L_j-harmonic, $j = 1,2$. Moreover, we have

$$h_jH_{f\circ i_j}^{h_j} \leq h_j(\bar{H}_f^h \circ i_j).$$

Since $K_D^{\mu_j}K_D^{\mu_k}(h_j(\bar{H}_f^h \circ i_j))$ is bounded, $j \neq k$, $j,k \in \{1,2\}$, $K_D^{\mu_j}K_D^{\mu_k}(h_jH_{f\circ i_j}^{h_j})$ is bounded, $j \neq k$, $j,k \in \{1,2\}$. Hence, by Theorem 3.1, the function

$$\bar{H}_f^h = \begin{cases} \frac{1}{h_1}(Q(h_1H_{f\circ i_1}^{h_1}) + QK_D^{\mu_1}(h_2H_{f\circ i_2}^{h_2})) \circ \pi_1 & \text{on } X_1, \\ \frac{1}{h_2}(K_D^{\mu_2}Q(h_1H_{f\circ i_1}^{h_1}) + T(h_2H_{f\circ i_2}^{h_2})) \circ \pi_2 & \text{on } X_2 \end{cases}$$

is h-harmonic on X, moreover $\bar{H}_f^h = \underline{H}_f^h = H_f^h$, therefore f is h-resolutive.

Conversely, suppose that f is h-resolutive. Then $\bar{H}_f^h = \underline{H}_f^h = H_f^h$ and H_f^h is h-harmonic. On the one hand, it follows from Lemma 5.3 that

$$h_j\bar{H}_{f\circ i_j}^{h_j} = h_j(H_f^h \circ i_j) - K_D^{\mu_j}(h_k(H_f^h \circ i_k)),$$

and by Corollary 2.1, the function $H^{h_j}_{f\circ i_j}$ is $h_j - L_j$-harmonic on D, i.e. $f \circ i_j$ is h_j-resolutive on $\triangle$. On the other hand,

$$K^{\mu_k}_D(h_j H^{h_j}_{f\circ i_j}) \le K^{\mu_k}_D(h_j(H^h_f \circ i_j)) \le h_k H^h_f \circ i_k,$$

thus

$$K^{\mu_k}_D(h_j H^{h_j}_{f\circ i_j}) < \infty.$$

□

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