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A note on paratopological groups


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A note on paratopological groups

CHUAN LIU

Abstract. In this paper, it is proved that a first-countable paratopological group has a regular $G_\delta$-diagonal, which gives an affirmative answer to Arhangel’skii and Burke’s question [Spaces with a regular $G_\delta$-diagonal, Topology Appl. 153 (2006), 1917–1929]. If $G$ is a symmetrizable paratopological group, then $G$ is a developable space. We also discuss copies of $S_\omega$ and of $S_2$ in paratopological groups and generalize some Nyikos [Metrizability and the Fréchet-Urysohn property in topological groups, Proc. Amer. Math. Soc. 83 (1981), no. 4, 793–801] and Svetlichnyi [Intersection of topologies and metrizability in topological groups, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 4 (1989), 79–81] results.

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Classification: Primary 54H13, 54H99

1. Introduction

Recently, paratopological groups have been studied by many topologists ([3], [4], [19]). It is natural to ask what results on topological groups are valid on paratopological groups. In this paper, by discussing copies of $S_\omega$ and of $S_2$ on paratopological groups, we generalize some results from [14], [15] and [18]. We also discuss first-countable paratopological groups and prove that a first-countable paratopological group has a regular $G_\delta$-diagonal, and give an affirmative answer to a question from [3].

Recall that a paratopological group is a group with a topology such that the multiplication is jointly continuous.

All spaces are regular $T_1$ unless stated otherwise. $\mathbb{N}$ denotes natural numbers and $e$ denotes the neutral element of a group. We refer to [6] for notations and terminology not given explicitly.

2. Main results

A space $X$ is said to have a regular $G_\delta$-diagonal if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of $\Delta$ in $X \times X$. According to Zenor [21], a space $X$ has a regular $G_\delta$-diagonal if and only if there exists a sequence $\{G_n : n \in \omega\}$ of open covers of $X$ with the following property:
Since that this is a contradiction.

In [3], Arhangel’skii and Burke proved that every Hausdorff first countable Abelian paratopological group $G$ has a regular $G_\delta$-diagonal. We sharpen the result by showing the following

**Theorem 2.1.** Let $G$ be a Hausdorff first-countable paratopological group. Then $G$ has a regular $G_\delta$-diagonal.

**Proof:** Fix a countable base $\{V_n : n \in \mathbb{N}\}$ at the neutral element $e$ in $G$ with $V_{n+1}^2 \subset V_n$. Let $x \in G$; then $xV_n$, $V_nx$ are open for $n \in \mathbb{N}$ since $G$ is a paratopological group. For $x \in G$, $n \in \mathbb{N}$, let $W_n(x) = xV_n \cap V_n x$. Then $W_n(x)$ is a neighborhood of $x$. Let $G_n = \{W_n(x) : x \in G\}$ for $n \in \mathbb{N}$. Then $\{G_n : n \in \mathbb{N}\}$ is a sequence of open coverings of $G$.

By Zenor’s characterization of regular $G_\delta$-diagonal, we only prove the following

**Claim:** For $y, z \in G$, $y \neq z$, there is $k \in \mathbb{N}$ such that no element of $G_k$ intersects both $yV_k$ and $zV_k$.

Suppose not; for any $n \in \mathbb{N}$, there is an element $W_n(x_n) \in G_n$ such that $yV_n \cap W_n(x_n) \neq \emptyset$ and $W_n(x_n) \cap zV_n \neq \emptyset$. Then there are $a_n, b_n, c_n, d_n$ and $f_n$ in $V_n$ such that $ya_n = x_nb_n$, $x_nc_n = d_nx_n = zf_n$, $ya_n = d_n^{-1}d_nx_nb_n = d_n^{-1}zf_nb_n$. Since $a_n \to e$, we have $ya_n \to y$, hence $d_n^{-1}zf_nb_n \to y$. $d_n \to e$ since $d_n \in V_n$, $G$ is a paratopological group, then $d_n^{-1}zf_nb_n \to ey = y$, hence $zf_nb_n \to y$. Notice that $f_nb_n \in V_n$, thus $f_nb_n \to e$, hence $zd_nb_n \to z$. $G$ is Hausdorff, then $y = z$, this is a contradiction.

Therefore, $G$ has a regular $G_\delta$-diagonal.

A subset $A$ of a space $X$ is said to be bounded [3] in $X$ if every infinite family $\xi$ of open subsets of $X$ such that $V \cap A \neq \emptyset$ for every $V \in \xi$, has an accumulation point $X$. If $X$ is bounded in itself, then we say that $X$ is pseudocompact.

Notice that a pseudocompact or bounded subset of a regular space $X$ is metrizable if $X$ has a regular $G_\delta$-diagonal [3]. We have the following

**Corollary 2.1.** Let $G$ be a regular first-countable paratopological group. Then every pseudocompact subspace of $G$ is a metrizable compactum.

**Corollary 2.2.** Let $G$ be a regular first-countable paratopological group. Then every bounded subspace of $G$ is metrizable.

The above theorem and corollaries give an affirmative answer to Arhangel’skii and Burke’s question [3, Problem 25].

A space $X$ is an $w\Delta$-space [8] if there exists a sequence $(G_n)$ of open covers of $X$ such that if $x_n \in st(x, G_n)$ for each $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point in $X$. 
Since a space $X$ with a regular $G_\delta$-diagonal has a $G_\delta^*$-diagonal, by [8, Theorem 3.3], we have the following

**Corollary 2.3.** Let $G$ be a first-countable paratopological group. Then $G$ is a Moore space if $G$ is an $w\Delta$-space.

A space $X$ is *quasi-developable* [8] if there exists a sequence $(\mathcal{G}_n)$ of families of subsets of $X$ such that for each $x \in X$, $\{\text{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at $x$. Recall that a topological space is said to be *symmetrizable* if its topology is generated by a *symmetric*, that is, by a distance function satisfying all the usual restrictions on a metric, except for the triangle inequality [1].

**Theorem 2.2.** Every symmetrizable paratopological group $G$ is a Moore space.

**Proof:** We fix a symmetric $d$ on the paratopological group $G$ generating the topology on $G$. Since $G$ is weakly first-countable [1], by a result of Nyikos [15], $G$ is first-countable. Put $B(x, 1/n) = \{y \in G : d(x, y) < 1/n\}$, and fix an open base $\{V_n : n \in \mathbb{N}\}$ at $e$ with $V_n \subset \text{int}(B(e, 1/n))$ and $V_n^{2} \subset V_n$. Let $A_{ij} = \{x \in G : V_i x \subset \text{int}(B(x, 1/j))\}$ and $\mathcal{G}_{ij} = \{V_i x : x \in A_{ij}\}$ for $i, j \in \mathbb{N}$. Since $\{V_i x : i \in \mathbb{N}\}$ and $\{\text{int}(B(x, 1/j)) : j \in \mathbb{N}\}$ are bases at $x$, $G = \bigcup \{A_{ij} : i, j \in \mathbb{N}\}$.

We prove that $\{\text{st}(x, \mathcal{G}_{ij}) : i, j \in \mathbb{N}\}$ is a base at $x \in G$. Let $U$ be an open subset of $X$ with $x \in U$. There exists $k \in \mathbb{N}$ such that $x \in \text{int}(B(x, 1/k)) \subset U$ and pick $m, n \in \mathbb{N}$ such that $m < n$, $V_n x \subset V_m x \subset \text{int}(B(x, 1/k))$. We choose $k'$ such that $B(x, 1/k') \subset V_n x$ since $\{B(x, 1/i) : i \in \mathbb{N}\}$ is a weak base at $x$. For $x \in V_n y \in \mathcal{G}_{nk'}$, since $V_n y \subset B(y, 1/k')$, $d(x, y) = d(y, x) < 1/k'$, hence $y \in B(x, 1/k') \subset V_n x$. $V_n y \subset V_n V_n x \subset V_m x \subset \text{int}(B(x, 1/k)) \subset U$, hence $x \in \text{st}(x, \mathcal{G}_{nk'}) \subset U$. Therefore $G$ is quasi-developable.

$G$ is symmetrizable and first-countable, hence $G$ is semi-stratifiable [8, Theorem 9.8], thus every closed subset of $G$ is a $G_\delta$-set. Therefore $G$ is a developable space [8, Theorem 8.6].

We cannot replace “symmetrizable” with “first-countable” in Theorem 2.2, Sorgenfrey line is a first-countable paratopological group but not a Moore space.

Let $S_\kappa$ be the quotient space obtained by identifying all limit points of the topological sum of $\kappa$ many convergent sequences. $S_\omega$ is called sequential fan. The Arens’ space $S_2 = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of $x_n$ is $\{x_n\} \cup \{x_n(m) : m > k, \text{for some } k \in \mathbb{N}\}$; a basic neighborhood of $\infty$ is $\{\infty\} \cup (\bigcup \{V_n : n > k \text{ for some } k \in \mathbb{N}\})$, where $V_n$ is a neighborhood of $x_n$.

In [14], it was proved that a topological group contains a (closed) copy of $S_\omega$ if and only if it contains a (closed) copy of $S_2$. We do not know if the result is still true for paratopological groups, but we have the following theorem by modifying Lemma 2.1 in [14].
Theorem 2.3. Let $G$ be a paratopological group. Then $G$ contains a (closed) copy of $S_\omega$ if $G$ has a (closed) copy of $S_2$.

Proof: Let $A = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ be a closed copy of $S_2$, where $e$ is the neutral element of $G$. For $n, m \in \mathbb{N}$, let $y_n(m) = x_n^{-1}x_n(m)$. Then $y_n(m) \to e$ as $m \to \infty$ for $n \in \mathbb{N}$. For each $n$, let $S_n = \{y_n(m) : m \in \mathbb{N}\}$. Then $F = \{n : S_m \cap S_n$ is infinite \} is finite (otherwise, pick distinct $x_{n_i}^{-1}x_{n_i}(m_i) \in S_m \cap S_{n_i}$ for $n_i \in F$ with $n_i < n_{i+1}$, $x_{n_i}^{-1}x_{n_i}(m_i) \to e$, $x_{n_i} \to e$, hence $x_{n_i}(m_i) \to e$, a contradiction). Without loss of generality, we assume $S_i \cap S_j = \emptyset$ if $i \neq j$. Let $B = \{e\} \cup \{y_n(m) : n, m \in \mathbb{N}\}$.

Claim: $B$ is a closed copy of $S_\omega$.

Suppose $B$ is not closed. Then there is $x \in X \setminus B$ with $x \in \overline{B}$. Since $A$ is closed, there exists an open neighborhood $V$ of the neutral element $e$ such that $Vx$ meets $\{x_n(m) : m \in \mathbb{N}\}$ for at most one $n$. Let $U$ be open neighborhood of $e$ with $U^2 \subset V$; $Ux$ contains an infinite subset $\{y_{n_i}(m_i) : i \in \mathbb{N}\}$ of $B$. Since $x_n \to e$, without loss of generality, $\{x_{n_i} : i \in \mathbb{N}\} \subset U \subset \{x_{n_i}(m_i) : i \in \mathbb{N}\} \subset Ux \subset Vx$, it means $\{x_{n_i}(m_i) : i \in \mathbb{N}\}$ contains an infinite subset $\{y_{n_i}(m_i) : i \in \mathbb{N}\} \subset C$, hence $x_{n_i}(m_i) = x_{n_i}y_{n_i}(m_i) \in Ux \subset Vx$ for each $i \in \mathbb{N}$, which is a contradiction. Hence $B$ is a copy of $S_\omega$. \hfill \Box

Nogura, Shakhmatov and Tanaka proved the following corollary as $G$ is a topological group [14]. By Theorem 2.3, we can see the following corollary is still true for a paratopological group $G$.

Note that a sequential space is an $A$-space\(^1\) if and only if it contains no closed copy of $S_\omega$ [20]. By Theorem 2.3, a paratopological group contains no closed copy of $S_2$ if it is an $A$-space. A sequential space that each point is a $G_\delta$-set or is hereditarily normal is strongly Fréchet if it contains no closed copy of $S_\omega$ and $S_2$ [20, Theorem 3.1]. A strongly Fréchet space is an $\alpha_4$-space\(^2\) [2, Theorem 5.26].

Corollary 2.4. Suppose that $G$ is a sequential paratopological group such that either (a) $e \in G$ is a $G_\delta$-set, or (b) $G$ is hereditarily normal. Then the following

---

\(^1\)A space $X$ is an $A$-space if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of $X$, and $x \in X$ is a point with $x \in \bigcap\{A_n \setminus \{x\} : n \in \mathbb{N}\}$, then for every $n \in \mathbb{N}$ one can find a (possibly empty) set $B_n \subset A_n$ such that $\bigcup\{B_n : n \in \mathbb{N}\}$ is not closed in $X$.

\(^2\)A countable collection $\{S_n : n \in \mathbb{N}\}$ of convergent sequences in a space $X$ is called a sheaf (with a vertex $x$) if each sequence $S_n$ converges to the same point $x \in X$. A space is called $\alpha_4$-space, if for every point $x \in X$ and each sheaf $\{S_n : n \in \mathbb{N}\}$ with the vertex $x$, there exists a sequence converging to $x$ which meets infinitely many sequences $S_n$.\
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are equivalent:

(1) \( G \) is an \( \alpha_4 \)-space;

(2) \( G \) is an \( A \)-space, and

(3) \( G \) is strongly Fréchet.

A paratopological group \( G \) is said to have the property (**), if there exists a sequence \( \{x_n : n \in \mathbb{N}\} \subset G \) such that \( x_n \to e \) and \( x_n^{-1} \to e \). Obviously, every topological group has the property (**). Not every paratopological group has the property (**), for instance, Sorgenfrey line \( S \) does not have the property (**).

A paratopological group having the property (**) need not be a topological group: for instance, if \((\mathbb{R}, +)\) is the real line with the usual topology, then \( S \times \mathbb{R} \) is a paratopological group having the property (**) but not a topological group.

**Theorem 2.4.** Let \( G \) be a paratopological group having the property (**). Then \( G \) has a (closed) copy of \( S_2 \) if it has a (closed) copy of \( S_\omega \).

**Proof:** Let \( A = \{e\} \cup \{y_n(m) : m, n \in \mathbb{N}\} \) be a closed copy of \( S_\omega \), for each \( n, y_n(m) \to e \) as \( m \to \infty \). Since \( G \) has the property (**), there is a sequence \( \{x_n : n \in \mathbb{N}\} \) such that \( x_n \to e \) and \( x_n^{-1} \to e \). Let \( U_n \) be an open neighborhood of \( x_n \) for each \( n \) with \( \overline{U_i} \cap \overline{U_j} = \emptyset \) if \( i \neq j \). Let \( x_n(m) = x_ny_n(m) \) for \( n, m \in \mathbb{N} \). For any \( n \in \mathbb{N} \), we have \( x_n(m) \to x_n \) as \( m \to \infty \). Without loss of generality, we assume \( \{x_n(m) : m \in \mathbb{N}\} \subset U_n \). Let \( B = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : n, m \in \mathbb{N}\} \).

**Claim:** \( B \) is a closed copy of \( S_2 \).

Suppose \( B \) is not closed. Then there exists \( x \notin B, e \neq x \in \overline{B \setminus \{x\}} \). Since \( A \) is closed, there is a neighborhood of \( e \) such that \( V_x \cap (A \setminus \{x\}) = \emptyset \). Let \( U \) be a neighborhood of \( e \) with \( U^2 \subset V \) and \( Ux \) contains at most one \( x_n \). \( Ux \) contains infinitely many elements of \( B \), since \( U \) contains infinitely many \( x_n \)’s, \( UUx \) contains infinitely many \( y_n(m) \). Hence \( Vx \) contains infinitely many elements of \( A \), this is a contradiction.

If \( f : \omega \to \omega \), similarly as in the proof of Theorem 2.3, \( \{x_n(m) : n \geq k \text{ for some } k, m \leq f(n)\} \) is closed. Hence \( B \) is a closed copy of \( S_2 \).

Note that a Fréchet-Urysohn space contains no closed copy of \( S_2 \), then a Fréchet-Urysohn paratopological group having the property (**) contains no closed copy of \( S_\omega \) by Theorem 2.4, hence it is a strongly Fréchet space [20] (or countably bisequential space [13]), therefore it is an \( \alpha_4 \)-space [2, Theorem 5.23].

**Corollary 2.5.** Let \( G \) be a paratopological group with the property (**). If \( G \) is a Fréchet-Urysohn space, then \( G \) is a \( \alpha_4 \)-space.

Corollary 2.5 gives a partial answer to Nyikos’ question [15, Problem 3]: “Is a Fréchet-Urysohn paratopological group an \( \alpha_4 \)-space?”.

**Question 2.1.** Can we omit the property (**) in Theorem 2.4 or in Corollary 2.5?
A space $X$ is called weakly quasi-first countable or $\aleph_0$-weakly first-countable ([17], [18]) if for each $i \in \mathbb{N}$, there exists a mapping $B^i : \mathbb{N} \times X \rightarrow \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of $X$, such that the following (1) and (2) hold:

(1) for $i \in \mathbb{N}$, for each $n \in \mathbb{N}$ and $x \in X$, $B^i(n + 1, x) \subset B^i(n, x)$, and 
   
   \( \{x\} = \bigcap \{B^i(n, x) : n \in \mathbb{N}\} \); and

(2) a subset $V$ of $X$ is open if and only if for each $y \in V$ and for each $i \in \mathbb{N}$ there exists $n(i)$ with $B^i(n(i), y) \subset V$.

If $B^i = B$ for $i \in \mathbb{N}$, then $X$ is called weakly first countable or $g$-first countable. Obviously, a weakly first countable space is weakly quasi-first countable.

**Corollary 2.6.** Let $G$ be a Fréchet-Urysohn paratopological group with the property (**). If $G$ is $\aleph_0$-weakly first-countable, then $G$ is first-countable.

**Proof:** By Corollary 2.5, $G$ is an $\alpha_4$-space, hence $G$ is weakly first-countable [10], thus $G$ is first-countable [15, Theorem 2]. □

By Corollary 2.5, we have the following:

**Corollary 2.7** ([18]). A Fréchet-Urysohn, $\aleph_0$-weakly first-countable topological group is metrizable.

Next, we discuss when we cannot embed a copy of $S_{\omega_1}$ to some paratopological group.

A family $\{B_\alpha : \alpha \in I\}$ of subsets of a space $X$ is hereditarily closure-preserving (weakly hereditarily closure-preserving [5]) (simply, HCP (wHCP)) if

\[
\bigcup \{C_\alpha : \alpha \in J\} = \left( \bigcup \{C_\alpha : \alpha \in J\} \right) \left( \{x_\alpha : \alpha \in J\} \text{ is closed discrete} \right),
\]

whenever $J \subset I$ and $C_\alpha \subset B_\alpha (x_\alpha \in B_\alpha)$ for each $\alpha \in J$. Obviously, a HCP family is wHCP. Spaces with a $\sigma$-wHCP weak base (base) were discussed in [11], [12]. Let $\mathcal{P}$ be a cover of a space $X$. Then $\mathcal{P}$ is a $k$-network for $X$ if whenever $K \subset U$ with $K$ compact and $U$ open in $X$, $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. A $k$-network is a network. A space with a $\sigma$-locally finite $k$-network is an $\aleph$-space [16]. $S_{\omega_1}$ is a closed image of a metric space, hence it has a $\sigma$-HCP closed $k$-network [7] but it is not an $\aleph$-space [9].

**Theorem 2.5.** Let $G$ be a paratopological topological group with the property (**). If $G$ has a $\sigma$-wHCP closed $k$-network, then $G$ contains no closed copy of $S_{\omega_1}$.

**Proof:** Suppose $G$ contains a closed copy of $S_{\omega_1} = \{e\} \cup \{x_n(\alpha) : \alpha < \omega_1, n \in \mathbb{N}\}$, where $e$ is the neutral element of $G$ and $x_n(\alpha) \rightarrow e$ as $n \rightarrow \infty$. Since $G$ has the property (**), there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset G$ such that $x_n \rightarrow e$, $x_n^{-1} \rightarrow e$. $G$ is regular, we take open subsets $U_n$ of $G$ such that $x_n \in U_n$, 

\[
U_n \cap \overline{U_m} = \emptyset \ (n \neq m)
\]

and 

\[
U_n \cap \{x_n : n \in \mathbb{N}\} = \{x_n\}.
\]

For each $m \in \mathbb{N}$, $x_m x_n(\alpha) \rightarrow x_m(\alpha \rightarrow \infty)$, $\{x_m x_n(\alpha) : n \in \mathbb{N}\}$ is eventually in $U_m$ for $\alpha < \omega_1$. Without loss of generality, we assume $\{x_m x_n(\alpha) : n \in \mathbb{N}\} \subset U_m$. 


Claim: $B = \{x_{n(\alpha)}x_{m(\alpha)}(\alpha) : \alpha < \omega_1\}$ is a discrete subset of $G$ for $n(\alpha), m(\alpha) \in \mathbb{N}$.

Case 1: $\{n(\alpha) : \alpha < \omega_1\}$ is finite.

We rewrite $\{n(\alpha) : \alpha < \omega_1\} = \{r_1, \ldots, r_k\}$. Since $\{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}$ is discrete for every $g : \omega_1 \to \mathbb{N}$, then $\{x_{r_i}x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}$ is discrete for each $i \leq k$, hence $B$ is discrete.

Case 2: $\{n(\alpha) : \alpha < \omega_1\}$ is infinite.

Suppose $B$ is not discrete and let $x$ be the cluster point of $B$. For every $g : \omega_1 \to \mathbb{N}$, there exists an open neighborhood $V$ of $e$ such that $|V \cap \{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}| \leq 1$. Let $U$ be an open neighborhood of $e$ with $U^2 \subset V$. Then $C = Ux \cap \{x_{n(\alpha)}x_{m(\alpha)}(\alpha) : \alpha < \omega_1\} \neq \emptyset$ for infinitely many $n(\alpha)$. Since $x_{n^{-1}} \to e$, $\{x_n : n \in \mathbb{N}\}$ is eventually in $U$, $\{x_{n^{-1}} : n \geq k\}C \subset UUx \subset Vx$. Then $|V \cap \{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}| \geq \omega$, a contradiction.

For $\alpha < \omega_1$, let $C_\alpha = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_nx_i(\alpha) : n \in \mathbb{N}, i \geq f(\alpha)\}$. Note that $x_nx_{j_n}(\alpha) \to e(n \to \infty)$, where $j_n \geq f(\alpha)$. Since every infinite subset of $C_\alpha$ has a cluster point in it, $C_\alpha$ is a countably compact. Since every countably compact space with a $\sigma$-wHCP network has a countable network [12, Proposition 6], $C_\alpha$ is compact [11].

Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-wHCP k-network consisting of closed subsets. Then there is a finite $\mathcal{P}' \subset \mathcal{P}$ such that $C_0 \subset \bigcup \mathcal{P}'$. Pick $P_0 \in \mathcal{P}'$ so that $P_0$ contains $k_0 = x_{n(0)}x_{m(0)}(0)$ and infinitely many $x_n$’s. We assume that for each $\alpha < \beta$, there exists $P_\alpha \in \mathcal{P}$ such that $P_\alpha$ contains infinitely many $x_n$’s and a point $k_\alpha = x_{n(\alpha)}x_{m(\alpha)}(\alpha)$. We have $C_\beta \subset G \setminus \{k_\alpha : \alpha < \beta\}$, which is open in $G$ by the Claim. There is a finite $\mathcal{P}'' \subset \mathcal{P}$ such that $C_\beta \subset \bigcup \mathcal{P}'' \subset G \setminus \{k_\alpha : \alpha < \beta\}$, pick $P_\beta \in \mathcal{P}''$ so that $P_\beta$ contains infinitely many $x_n$ and $k_\beta = x_{n(\beta)}x_{m(\beta)}(\beta)$. By induction, we obtain $\{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$ such that $P_\alpha \neq P_\beta$ if $\alpha \neq \beta$ and each $P_\alpha$ contains infinitely many $x_n$’s, hence there are uncountably many $P_\alpha \in \mathcal{P}_n$ for some $n \in \mathbb{N}$. Note that $\mathcal{P}_n$ is wHCP and there is a subsequence $L$ of $\{x_n : n \in \mathbb{N}\}$ such that $L$ is discrete, which is a contradiction. □

References


Department of Mathematics, Ohio University-Zanesville Campus, Zanesville, OH 43701, USA

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