Dimitrios A. Kandilakis; Manolis Magiropoulos
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A $p$-Laplacian system with resonance and nonlinear boundary conditions on an unbounded domain

D.A. Kandilakis, M. Magiropoulos

Abstract. We study a nonlinear elliptic system with resonance part and nonlinear boundary conditions on an unbounded domain. Our approach is variational and is based on the well known Landesman-Laser type conditions.

Keywords: quasilinear problem, $p$-Laplacian system, Landesman-Laser condition, resonance

Classification: 35D05, 35J45, 35J50

1. Introduction and statement of results

Let $\Omega$ be an unbounded domain in $\mathbb{R}^N$, $N \geq 3$, with a noncompact and smooth boundary $\partial \Omega$. In this paper we consider the following quasilinear elliptic system

$$
\begin{align*}
-\Delta_p u &= \lambda_1 a(x)|u|^{p-2}u + \lambda_1 b(x)|u|^\alpha|v|^\beta v + g_1(x,u) - h_1(x), \quad x \in \Omega \\
-\Delta_p v &= \lambda_1 d(x)|v|^{p-2}v + \lambda_1 b(x)|u|^\alpha|v|^\beta u + g_2(x,u) - h_2(x), \quad x \in \Omega
\end{align*}
$$

subject to the nonlinear boundary conditions

$$
\begin{align*}
|\nabla u|^{p-2}\nabla u \cdot \eta + c_1(x)|u|^{p-2}u &= 0, \quad x \in \partial \Omega \\
|\nabla v|^{p-2}\nabla v \cdot \eta + c_2(x)|v|^{p-2}v &= 0, \quad x \in \partial \Omega
\end{align*}
$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ and $\eta$ is the unit outward normal vector on $\partial \Omega$. On a single equation level with $\Omega$ bounded and Dirichlet boundary conditions, the problem has been studied by Arcoya and Orsina [1] taking into consideration the well known Landesman-Laser type conditions for the resonance part. The extension to the case of a system, again with $\Omega$ bounded and Dirichlet boundary conditions, was first considered by Zographopoulos in [7].

In order to confront with our problem we need a suitable space setting which we describe next.

For $\xi \in \mathbb{R}$, we set $w_\xi(x) := \frac{1}{(1+|x|)^\xi}$, and assume that the space $L^r(w_\xi, \Omega) := \{u : \int_\Omega w_\xi(x)|u|^r < +\infty\}$, $r \geq 1$, is supplied with the norm

$$
\|u\|_{w_\xi, r} = \left(\int_\Omega w_\xi(x)|u|^r\right)^{1/r}.
$$
Let $C_0^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted on $\Omega$. For $p \in (1, +\infty)$, the weighted Sobolev space $E_p$ is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p + \int_\Omega w_p(x)|u|^p\right)^{1/p}.$$  

By Lemma 2 in [5], we see that if $c(\cdot)$ is a positive continuous function defined on $\mathbb{R}^N$ such that

$$kw_p(x) \leq c(x) \leq Kw_p(x),$$

for some positive constants $k$ and $K$, then the norm

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p + \int_{\partial\Omega} c(x)|u|^m\right)^{1/p}$$

is equivalent to $\|\cdot\|_p$.

We will consider our system on the space $E = E_p \times E_p$, supplied with the norm

$$\|(u, v)\| = \|u\|_{1,p} + \|v\|_{1,p}.$$  

The following lemma is useful for our compactness arguments.

**Lemma 1.** (i) If

$$p \leq r \leq \frac{pN}{N - p} \quad \text{and} \quad N > \alpha \geq N - r \frac{N-p}{p},$$

then the embedding $E \subseteq L^r(w_\alpha, \Omega)$ is continuous. If the upper bound for $r$ in the first inequality and the upper bound for $\alpha$ in the second are strict, then the embedding is compact.

(ii) If

$$p \leq m \leq \frac{p(N-1)}{N-p} \quad \text{and} \quad N > \beta \geq N - 1 - m \frac{N-p}{p},$$

then the trace operator $T : E \rightarrow L^m(w_\beta, \partial\Omega)$ is continuous. If the upper bound for $m$ in the first inequality and the lower bound for $\beta$ are strict, then the trace operator is compact.

(iii) If

$$1 \leq q < p \quad \text{and} \quad \frac{\alpha_1 - N}{\alpha_2 - N} \geq \frac{p}{q},$$

then the embedding $L^p(w_{\alpha_1}, \Omega) \subseteq L^q(w_{\alpha_2}, \Omega)$ is continuous.

**Proof:** The first and second part of the lemma is Theorem 1 in [5], while the third is a consequence of the following inequality

$$\int_\Omega \frac{1}{(1 + |x|)^{\alpha_2}} |u|^q \, dx \leq \left(\int_\Omega \frac{1}{(1 + |x|)^d} \, dx\right)^{\frac{p-q}{p}} \left(\int_\Omega \frac{1}{(1 + |x|)^{\alpha_1}} |u|^p \, dx\right)^{\frac{q}{p}},$$
where \( d = \frac{\alpha_2 p - \alpha_1 q}{p-q} \). Note that the integral \( \int_{\Omega} \frac{1}{(1+|x|)^d} \, dx \) converges since \( d > N \).

\[ \square \]

We study (1)–(2) in connection with the eigenvalue problem

\[
\begin{aligned}
-\Delta_p u &= \lambda_1 a(x)|u|^{p-2}u + \lambda_1 b(x)|u|^\alpha |v|^{\beta}v, \\
-\Delta_p v &= \lambda_1 d(x)|v|^{p-2}v + \lambda_1 b(x)|u|^\alpha |v|^{\beta}u,
\end{aligned}
\]

subject to the boundary conditions (2), which was considered in \([4]\) under the following set of assumptions, also needed for the present problem:

(H1) \( 2 < p < N, \alpha, \beta \geq 0 \) with \( \alpha + \beta = p - 2 \) and \( \alpha + 1, \beta + 1 \leq \frac{p^*}{N}, \) where \( p^* = \frac{Np}{N-p} \).

(H2) (i) There exist positive constants \( \alpha_1, A \) with \( \alpha_1 \in \left( p + \frac{(\beta+1)(N-p)}{p^*}, N \right) \) such that \( 0 < a(x) \leq A w_{\alpha_1}(x) \) a.e. in \( \Omega \).

(ii) There exist positive constants \( \alpha_2, D \) with \( \alpha_2 \in \left( p + \frac{(\alpha+1)(N-p)}{p^*}, N \right) \) such that

\[ 0 < d(x) \leq D w_{\alpha_2}(x) \] a.e. in \( \Omega \).

(iii) \( m\{x \in \Omega : b(x) > 0\} > 0 \) and

\[ 0 \leq b(x) \leq B w_s(x) \] a.e. in \( \Omega \),

where \( B > 0 \) and \( s \in (p, N) \).

(H3) \( c_1(\cdot) \) and \( c_2(\cdot) \) are positive continuous functions defined on \( \mathbb{R}^N \) with

\[ kw_{p-1}(x) \leq c_1(x), c_2(x) \leq K w_{p-1}(x), \]

for some positive constants \( k \) and \( K \).

Let

\[
I(u, v) = \frac{\alpha+1}{p} \int_\Omega |\nabla u|^p + \frac{\alpha+1}{p} \int_{\partial\Omega} c_1(x)|u|^p + \frac{\beta+1}{p} \int_\Omega |\nabla v|^p + \frac{\beta+1}{p} \int_{\partial\Omega} c_2(x)|v|^p
\]

and

\[
J(u, v) = \frac{\alpha+1}{p} \int_\Omega a(x)|u|^p + \frac{\beta+1}{p} \int_\Omega d(x)|v|^p + \int_\Omega b(x)|u|^\alpha |v|^{\beta}uv.
\]
**Theorem 2** ([4]). Let $\Omega$ be an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, with a non-compact and smooth boundary $\partial \Omega$. Assume that hypotheses (H1), (H2) and (H3) hold. Then

(a) the system (3) admits a positive principal eigenvalue $\lambda_1$ given by
\[
\lambda_1 = \inf \{ I(u,v) : J(u,v) = 1 \}.
\]

Each component of the associated normalized eigenfunction $(u_1,v_1)$ is positive on $\bar{\Omega}$ and of class $C^{1,\delta}_{loc}(\Omega)$ for some $\delta \in (0,1)$.

(b) the set of eigenfunctions corresponding to $\lambda_1$ forms a one dimensional manifold $X \subseteq E$ defined by
\[
X = \{ c(u_1,v_1) ; c \in \mathbb{R} \setminus \{0\} \}.
\]

(c) $\lambda_1$ is isolated, in the sense that there exists $\eta > 0$ such that the interval $(0, \lambda_1 + \eta)$ does not contain any other eigenvalue than $\lambda_1$.

We make the following assumptions concerning the resonance part:

**H4** (i) $g_1(\cdot,\cdot), g_2(\cdot,\cdot)$ are Caratheodory functions such that
\[
|g_1(x,s)| \leq \frac{C_1}{(1+|x|)^{\alpha_3}} \quad \text{and} \quad |g_2(x,s)| \leq \frac{C_2}{(1+|x|)^{\alpha_4}},
\]
where $\alpha_3 > N - \frac{N}{p} - \alpha_1$, $\alpha_4 > N - \frac{N}{p} - \alpha_2$, $C_1$, $C_2$ are positive constants, and the limits
\[
\lim_{s \to \pm \infty} g_i(x,s) = g_i^\pm(x), \quad i = 1, 2,
\]
exist for almost every $x \in \Omega$.

(ii) $|h_1(x)| \leq \frac{H_1}{(1+|x|)^{\alpha_3}}$ and $|h_2(x)| \leq \frac{H_2}{(1+|x|)^{\alpha_4}}$ for some positive constants $H_1, H_2$.

Furthermore, we will need the following inequalities

\[
L^+ < (\alpha + 1) \int_{\Omega} h_1(x)u_1 + (\beta + 1) \int_{\Omega} h_2(x)v_1 < L^-,
\]

(4)

\[
L^- < (\alpha + 1) \int_{\Omega} h_1(x)u_1 + (\beta + 1) \int_{\Omega} h_2(x)v_1 < L^+,
\]

(5)

where $(u_1,v_1)$ is the normalized eigenfunction of (3)–(2) with positive components and

\[
L^+ = (\alpha + 1) \int_{\Omega} g_1^+(x)u_1 + (\beta + 1) \int_{\Omega} g_2^+(x)v_1,
\]

\[
L^- = (\alpha + 1) \int_{\Omega} g_1^-(x)u_1 + (\beta + 1) \int_{\Omega} g_2^-(x)v_1.
\]
Inequalities (4) and (5) are the adaptation to the case of systems of the Landesman-Laser type conditions for scalar equations.

The energy functional of the problem (1)–(2) is

\[
\Phi(u, v) = \frac{\alpha+1}{p} \int_\Omega |\nabla u|^p + \frac{\alpha+1}{p} \int_\partial \Omega c_1(x)|u|^p - \lambda_1 \frac{\alpha+1}{p} \int_\Omega a(x)|u|^p \\
- (\alpha+1) \int_\Omega G_1(x, u) + (\alpha+1) \int_\Omega h_1(x)u \\
+ \frac{\beta+1}{p} \int_\Omega |\nabla v|^p + \frac{\beta+1}{p} \int_\partial \Omega c_2(x)|v|^p - \lambda_1 \frac{\beta+1}{p} \int_\Omega d(x)|v|^p \\
- (\beta+1) \int_\Omega G_2(x, v) + (\beta+1) \int_\Omega h_2(x)v - \lambda_1 \int_\Omega b(x)|u|^\alpha|v|^\beta uv,
\]

where

\[
G_i(x, s) = \int_0^s g_i(x, t) \, dt, \quad i = 1, 2.
\]

In view of (H1)–(H3), the functional $\Phi$ is well defined and continuously differentiable on $E$. By a \textit{weak solution} of (1)–(2) we mean an element of $E$ which is a critical point of $\Phi$.

The main result of this work is the following theorem:

**Theorem 3.** (i) Assume that hypotheses (H1)–(H3) and inequality (4) or (5) hold. Then the system (1)–(2) admits a weak solution.

2. The main result

In view of Theorem 2(a), it is clear that $\lambda_1 \leq \min\{\lambda_u, \lambda_v\}$, where $\lambda_u, \lambda_v$ are the first eigenvalues of the problems $-\Delta_p u = \lambda a(x)|u|^{p-2}u$ and $-\Delta_p v = \lambda d(x)|v|^{p-2}v$, with the boundary conditions (2), respectively. The following lemma shows that this inequality is actually strict.

**Lemma 4.** $\lambda_1 < \min\{\lambda_u, \lambda_v\}$.

**Proof:** Let $u_0 > 0$ be an eigenfunction corresponding to $\lambda_u$ and $v_0 > 0$ an eigenfunction corresponding to $\lambda_v$. If $\lambda_u = \lambda_v$, then

\[
\lambda_1 \leq \frac{I(u_0, v_0)}{J(u_0, v_0)} < \lambda_u,
\]

so without loss of generality we may assume that $\lambda_u < \lambda_v$. Let $t > 0$ be such that

\[
\frac{\beta+1}{p} \int_\Omega d(x)|v_0|^p < \frac{\lambda_u}{\lambda_v - \lambda_u} \int_\Omega b(x)|tu_0|^{\alpha}|v_0|^{\beta}tu_0v_0.
\]
Then, in view of (6),
\[ \lambda_1 \leq \frac{I(tu_0, v_0)}{F(tu_0, v_0)} < \lambda_u = \min\{\lambda_u, \lambda_v\}. \]

\(\square\)

Note that due to assumptions H(1)–H(4), the operators \(A, N, B, C : E \to E^*\) given by
\[
\langle A(u, v), (\varphi, \psi) \rangle := \int_\Omega |\nabla u|^{p-2} u \nabla \varphi + \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \psi,
\]
\[
\langle N(u, v), (\varphi, \psi) \rangle := \int_\Omega a(x)|u|^{p-2} u \varphi - \int_{\partial \Omega} c_1(x)|u|^{p-2} u \varphi + \int_\Omega d(x)|v|^{p-2} v \psi - \int_{\partial \Omega} c_2(x)|v|^{p-2} v \psi,
\]
\[
\langle B(u, v), (\varphi, \psi) \rangle := \int_\Omega b(x)|u|^\alpha |v|^\beta \varphi + \int_\Omega b(x)|u|^\alpha |v|^\beta \psi,
\]
\[
\langle C(u, v), (\varphi, \psi) \rangle := \int_\Omega (g_1(x, u) - h_1(x)) \varphi + \int_\Omega (g_2(x, v) - h_2(x)) \psi,
\]
are well defined. Following standard arguments based on the embeddings given in Lemma 1, we have:

**Lemma 5.** The operators \(A, N, B\) and \(C\) are continuous. Moreover, \(N, B\) and \(C\) are compact.

We can now proceed with the proof of the main result:

**Proof of Theorem 3:** We assume first that (4) holds. We claim that \(\Phi\) satisfies the PS-condition. Indeed, let \(\{(u_n, v_n)\}_{n \in \mathbb{N}}\) be a PS-sequence in \(E\). Then

\[ -c \leq \Phi(u_n, v_n) \leq c, \]

for some \(c > 0\), and there exists a sequence \(\{\varepsilon_n\}_{n \in \mathbb{N}}\) converging to \(0^+\), such that

\[ -\varepsilon_n \| (u, v) \| \leq \Phi'(u_n, v_n)(u, v) \leq \varepsilon_n \| (u, v) \| \]

for every \((u, v) \in E\).

We will show that the sequence \(\{(u_n, v_n)\}_{n \in \mathbb{N}}\) is bounded in \(E\). Assume the contrary, that is \(\|(u_n, v_n)\| \to +\infty\). Let

\[ \hat{u}_n := \frac{u_n}{\|(u_n, v_n)\|}, \quad \hat{v}_n := \frac{v_n}{\|(u_n, v_n)\|}. \]
Since \( \|\hat{u}_n\|_{E_p} \leq 1 \) and \( \|\hat{v}_n\|_{E_p} \leq 1 \), by passing to subsequences if necessary, we may assume that \( \hat{u}_n \rightarrow \hat{u} \) and \( \hat{v}_n \rightarrow \hat{v} \) weakly in \( E_p \). Due to our hypotheses on \( h_1 \) and \( g_1 \) we obtain

\[
(10) \quad \lim_{n \to +\infty} \int_{\Omega} \frac{G_1(x, u_n)}{\|(u_n, v_n)\|^p} = \lim_{n \to +\infty} \int_{\Omega} \frac{h_1 u_n}{\|(u_n, v_n)\|^p} = 0
\]

and similarly for \( G_2(\cdot, \cdot) \) and \( h_2(\cdot) \). Dividing (7) by \( \|(u_n, v_n)\|^p \) and using (10), we arrive at

\[
\limsup_{n \to +\infty} \left[ \frac{\alpha+1}{p} \left\{ \int_{\Omega} |\nabla \hat{u}_n|^p + \int_{\partial \Omega} c_1(x) |\hat{u}_n|^p - \lambda_1 \int_{\Omega} a(x) |\hat{u}_n|^p \right\} 
+ \frac{\beta+1}{p} \left\{ \int_{\Omega} |\nabla \hat{v}_n|^p + \int_{\partial \Omega} c_2(x) |\hat{v}_n|^p - \lambda_1 \int_{\Omega} d(x) |\hat{v}_n|^p \right\} 
- \lambda_1 \int_{\Omega} b(x) |\hat{u}_n|^\alpha |\hat{v}_n|^\beta \hat{u}_n \hat{v}_n \right] \leq 0,
\]

and Lemma 1 gives

\[
\limsup_{n \to +\infty} \left[ \frac{\alpha+1}{p} \left\{ \int_{\Omega} |\nabla \hat{u}_n|^p + \int_{\partial \Omega} c_1(x) |\hat{u}_n|^p \right\} 
+ \frac{\beta+1}{p} \left\{ \int_{\Omega} |\nabla \hat{v}_n|^p + \int_{\partial \Omega} c_2(x) |\hat{v}_n|^p \right\} \right] 
\leq \lambda_1 \left( \frac{\alpha+1}{p} \int_{\Omega} a(x) |\hat{u}|^p + \frac{\beta+1}{p} \int_{\Omega} d(x) |\hat{v}|^p + \int_{\Omega} b(x) |\hat{u}|^\alpha |\hat{v}|^\beta \hat{u} \hat{v} \right).
\]

The reverse inequality (with the limsup replaced by liminf) also holds due to the lower semicontinuity of the norms. Thus \((\hat{u}, \hat{v})\) is a nonzero solution of (3) with \( \|(\hat{u}, \hat{v})\| = 1 \). In view of Lemma 4, \( \hat{u} \neq 0 \) and \( \hat{v} \neq 0 \). By Theorem 2, \( \hat{u} \) and \( \hat{v} \) have the same sign. Suppose that both \( \hat{u} \) and \( \hat{v} \) are positive, the other case can be treated similarly. Thus \( \hat{u} = u_1 \) and \( \hat{v} = v_1 \). If we replace \((u, v)\) by \((u_n, v_n)\) in (8), write the relation for \(-\Phi'\), multiply the members of (7) by \( p \), add memberwise the resulting inequalities, and divide by \( \|(u_n, v_n)\| \), we obtain

\[
\left| (\alpha+1)(p-1) \int_{\Omega} h_1(x) \hat{u}_n + (\beta+1)(p-1) \int_{\Omega} h_2(x) \hat{v}_n - (\alpha+1) p \int_{\Omega} \hat{g}_1(x, u_n) \hat{u}_n + (\alpha+1) p \int_{\Omega} g_1(x, u_n) \hat{u}_n - (\beta+1) p \int_{\Omega} \hat{g}_2(x, v_n) \hat{v}_n + (\beta+1) p \int_{\Omega} g_2(x, v_n) \hat{v}_n \right| \leq \frac{c}{\|(u_n, v_n)\|} + \varepsilon_n,
\]
where
\[
\hat{g}_i(x,s) := \begin{cases} \frac{G_i(x,s)}{s} & \text{if } s \neq 0, \\ g_i(x,0) & \text{if } s = 0, \end{cases} \quad i = 1, 2.
\]

By letting \( n \to +\infty \), we get
\[
\lim_{n \to +\infty} \left\{ (\alpha + 1) \int_\Omega [g_1(x,u_n)\hat{u}_n - p\hat{g}_1(x,u_n)\hat{u}_n] \right. \\
+ (\beta + 1) \int_\Omega [g_2(x,v_n)\hat{v}_n - p\hat{g}_2(x,v_n)\hat{v}_n] \right. \\
\left. = (\alpha + 1)(1 - p) \int_\Omega h_1(x)\hat{u} + (\beta + 1)(1 - p) \int_\Omega h_2(x)\hat{v}. \right.
\]

By (9), \( u_n(x) \) and \( v_n(x) \) tend to \(+\infty\), so \( g_1(x,u_n) \to g_1^+(x) \) and \( g_2(x,v_n) \to g_2^+(x) \) a.e. in \( \Omega \).

Therefore
\[
\lim_{n \to +\infty} \int_\Omega [g_1(x,u_n)\hat{u}_n - p\hat{g}_1(x,u_n)\hat{u}_n] = (1 - p) \int_\Omega g_1^+(x)\hat{u},
\]
with a similar relation holding for \( g_2(\cdot, \cdot) \) as well. In view of (11) and (12), we have
\[
(\alpha + 1) \int_\Omega g_1^+(x)u_1 + (\beta + 1) \int_\Omega g_2^+(x)v_1 = (\alpha + 1) \int_\Omega h_1(x)u_1 + (\beta + 1) \int_\Omega h_2(x)v_1,
\]
contradicting (4). Thus \( \{ (u_n,v_n) \}_{n \in \mathbb{N}} \) is bounded. Therefore, up to subsequences, \( u_n \to u_0 \) and \( v_n \to v_0 \) weakly in \( E_p \) and strongly in \( L^p(w_{\alpha_2},\Omega) \), respectively. By taking \( (u,v) = (u_n,v_n) - (u_0,v_0) \) in (8), and using Lemma 1, we derive that
\[
(\alpha + 1) \left\{ \int_\Omega \left( |\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0 \right) (\nabla u_n - \nabla u_0) \right. \\
+ \int_{\partial \Omega} c_1 \left( |u_n|^{p-2}u_n - |u_0|^{p-2}u_0 \right) (u_n - u_0) \right. \\
+ (\beta + 1) \left\{ \int_\Omega \left( |\nabla v_n|^{p-2}\nabla v_n - |\nabla v_0|^{p-2}\nabla v_0 \right) (\nabla v_n - \nabla v_0) \right. \\
+ \int_{\partial \Omega} c_1 \left( |v_n|^{p-2}v_n - |v_0|^{p-2}v_0 \right) (v_n - v_0) \right. \to 0,
\]
which, in view of inequality 2.5 in [2], implies that \((u_n, v_n) \to (u_0, v_0)\) in \(E\). We show next that \(\Phi\) is coercive. Indeed, if this were not the case, there would exist a sequence \(\{(u_n, v_n)\}_{n \in \mathbb{N}}\) with \(\|(u_n, v_n)\| \to +\infty\) and

\[
|\Phi(u_n, v_n)| \leq M, \text{ for some } M > 0.
\]

Working as before, we get that

\[
\lim_{n \to +\infty} \left[ \frac{\alpha+1}{p} \left\{ \int_{\Omega} |\nabla \hat{u}_n|^p + \int_{\partial \Omega} c_1(x)|\hat{u}_n|^p - \lambda_1 \int_{\Omega} a(x)|\hat{u}_n|^p \right\} \right.
\]

\[
+ \frac{\beta+1}{p} \left\{ \int_{\Omega} |\nabla \hat{v}_n|^p + \int_{\partial \Omega} c_2(x)|\hat{v}_n|^p - \lambda_1 \int_{\Omega} d(x)|\hat{v}_n|^p \right\}
\]

\[
- \lambda_1 \int_{\Omega} b(x)|\hat{u}_n|^\alpha |\hat{v}_n|^\beta \hat{u}_n \hat{v}_n
\]

\[
= 0,
\]

where \(\hat{u}_n\) and \(\hat{v}_n\) are defined in (9). Thus \((\hat{u}_n, \hat{v}_n) \to (u_1, v_1)\) or \((\hat{u}_n, \hat{v}_n) \to -(u_1, v_1)\) in \(E\). If \((\hat{u}_n, \hat{v}_n) \to (u_1, v_1)\), by (13) and the variational characterization of \(\lambda_1\), we obtain

\[
(\alpha+1) \int_{\Omega} g_1^+(x)u_1 + (\beta+1) \int_{\Omega} g_2^+(x)v_1 \geq (\alpha+1) \int_{\Omega} h_1(x)u_1 + (\beta+1) \int_{\Omega} h_2(x)v_1,
\]

while if \((\hat{u}_n, \hat{v}_n) \to -(u_1, v_1)\), we get

\[
(\alpha+1) \int_{\Omega} g_1^-(x)u_1 + (\beta+1) \int_{\Omega} g_2^-(x)v_1 \leq (\alpha+1) \int_{\Omega} h_1(x)u_1 + (\beta+1) \int_{\Omega} h_2(x)v_1,
\]

contradicting (4). We can now use Theorem 4.7 in [3] to get a weak solution of (1)–(2).

Assume next that (5) holds. We split \(E\) as the direct sum of the eigenspace \(X\) and \(Y = \{(u, v) \in E : \int_{\Omega} uu_1^{p-1} + \int_{\Omega} vv_1^{p-1} = 0\}\). Then \(\Phi\) has a saddle point geometry, i.e.,

(i) \(\Phi(t(u_1, v_1)) \to -\infty\) if \(|t| \to +\infty\), and

(ii) \(\Phi\) is bounded from below on \(Y\).

Indeed, since

\[
\Phi(t(u_1, v_1)) = (\alpha+1) \left[ \int_{\Omega} h_1(x)tu_1 - \int_{\Omega} G_1(x, tu_1) \right]
\]

\[
+ (\beta+1) \left[ \int_{\Omega} h_2(x)tv_1 - \int_{\Omega} G_2(x, tv_1) \right]
\]

\[
= (\alpha+1)t \left[ \int_{\Omega} h_1(x)u_1 - \int_{\Omega} \frac{G_1(x, tu_1)}{tu_1} u_1 \right]
\]

\[
+ (\beta+1)t \left[ \int_{\Omega} h_2(x)v_1 - \int_{\Omega} \frac{G_2(x, tv_1)}{tv_1} v_1 \right],
\]
by taking the limit as $|t| \to \infty$ and working as in the first part of the proof, we can use (5) to get (i). To prove (ii) we exploit the isolation of $\lambda_1$, see Theorem 2, to derive that there exists $\hat{\lambda} > \lambda_1$ such that

$$\hat{\lambda} < \frac{I(u, v)}{J(u, v)}$$

for every $(u, v) \in Y$. If $(u, v) \in Y$, in view of Lemma 1,

$$\Phi(u, v) = I(u, v) - \lambda_1 J(u, v) + (\alpha + 1) \left[ \int_{\Omega} h_1(x)u - \int_{\Omega} G_1(x, u) \right]$$

$$+ (\beta + 1) \left[ \int_{\Omega} h_2(x)v - \int_{\Omega} G_2(x, v) \right]$$

$$> \left(1 - \frac{\lambda_1}{\hat{\lambda}}\right) I(u, v) - (\alpha + 1)c_1\|u\|_{1,p} - (\beta + 1)c_2\|v\|_{1,p},$$

for some $c_1, c_2 > 0$. Consequently, $\Phi$ is bounded from below on $Y$. An application of the saddle point theorem, see [6], provides a weak solution of (1)–(2). □

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**References**


Department of Sciences, Technical University of Crete, 73100 Chania, Greece

E-mail: dkan@science.tuc.gr

Science Department, Technological and Educational Institute of Crete, 71500 Heraclion, Greece

E-mail: mageir@stef.teiher.gr

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