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Quasi-concave copulas, asymmetry and transformations

Elisabetta Alvoni, Pier Luigi Papini

Abstract. In this paper we consider a class of copulas, called quasi-concave; we compare them with other classes of copulas and we study conditions implying symmetry for them.

Recently, a measure of asymmetry for copulas has been introduced and the maximum degree of asymmetry for them in this sense has been computed: see Nelsen R.B., Extremes of nonexchangeability, Statist. Papers 48 (2007), 329–336; Klement E.P., Mesiar R., How non-symmetric can a copula be?, Comment. Math. Univ. Carolin. 47 (2006), 141–148. Here we compute the maximum degree of asymmetry that quasi-concave copulas can have; we prove that the supremum of \( \{ |C(x, y) - C(y, x)|; x, y \text{ in } [0, 1]; C \text{ is quasi-concave} \} \) is \( \frac{1}{5} \). Also, we show by suitable examples that such supremum is a maximum and we indicate copulas for which the maximum is achieved.

Moreover, we show that the class of quasi-concave copulas is preserved by simple transformations, often considered in the literature.

Keywords: copula, quasi-concave, asymmetry

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1. Copulas and asymmetry

As known, copulas link the joint distribution function of a random vector to the corresponding marginal distribution functions. Moreover, from some years, in Finance, Statistics and Probability there is a growing interest on nonexchangeability of random variables, and this can be studied in terms of non-symmetric copulas.

We recall some definitions.

A (bivariate) copula is a function \( C: [0, 1]^2 \rightarrow [0, 1] \) satisfying:

(1) \( C(1, y) = y, \ C(x, 1) = x, \) for \( 0 \leq x, y \leq 1, \)

(2) \( C(x', y') - C(x, y') \geq C(x', y) - C(x, y) \) for \( 0 \leq x \leq x' \leq 1, \ 0 \leq y \leq y' \leq 1. \)

In particular, condition (2), usually called 2-increasingness, together with (1) implies:

(3) \( C(x, y) \) is increasing in each variable

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and
\[ C(0, y) = 0, \ C(x, 0) = 0, \ \text{for} \ 0 \leq x, \ y \leq 1. \]

Also, we obtain from (2) (set \( y' = 1 \) or \( x' = 1 \)):
\[ C(x, y) \] is 1-Lipschitz in each variable.

A copula is the restriction to the unit square of a distribution function with uniform marginals on \([0,1]\). We refer to [5] for general results on copulas.

A copula \( C(x, y) \) is commutative or symmetric, if
\[ C(x, y) = C(y, x) \text{ for all } x, y \in [0,1]. \]

If a copula is not commutative, it can be interesting to know how large the difference between \( C(x, y) \) and \( C(y, x) \) can be.

According to [6], we set, for a copula \( C \):
\[ \beta_C = \sup \{|C(x, y) - C(y, x)|; \ x, y \in [0,1]\}. \]

As proved in [6, Theorem 2.2] and in [3], we have:
\[ \sup \{\beta_C; \ C \text{ is a copula}\} = \frac{1}{3}; \]
due to this fact, it was suggested to use \( 3\beta_C \) as a normalized measure of asymmetry for copulas.

Moreover, the supremum is achieved: the set of copulas for which such value is attained, was characterized in [6]; their elements were called maximally nonexchangeable copulas. These and other results on asymmetry have been considered also in [3].

To see how and where the interest in asymmetry can arise, we recall that for example in [2] it was explained why it can be suitable to change symmetric copulas into asymmetric ones.

2. Quasi-concave copulas, symmetry and other related classes of copulas

We define a class of copulas, described in [5, Section 3.4.3].

**Definition 1.** We say that a copula is quasi-concave if for all \((x, y), (x', y')\) in \([0,1]^2\) and all \(\lambda \in [0,1]\), we have:
\[ C(\lambda x + (1-\lambda)x', \ \lambda y + (1-\lambda)y') \geq \min\{C(x, y), \ C(x', y')\}. \]

Another, more popular, class of copulas can be described in the following way (see for example [5, Definition 3.4.6]):
Definition 2. We say that a copula is Schur-concave if for all \(x, y, \lambda \) in \([0, 1]\), we have:

\[
C(x, y) \leq C(\lambda x + (1 - \lambda)y, \lambda y + (1 - \lambda)x).
\]

It is clear that a copula satisfying (10) is commutative.

A weakening of condition (10) has been considered, mainly in a context different from that of copulas (see [4, (4.1)]):

\[
C\left(\frac{x + y}{2}, \frac{x + y}{2}\right) \geq C(x, y) \quad \text{for all } x, y \in [0, 1].
\]

For copulas, an “asymmetric” version of (11) was considered in [1].

The meaning of (11) and, respectively, (10), is the following. Consider the values of \(C\) along the line segment \(x + y = 2\alpha\) (\(0 \leq \alpha \leq 1\)); if (11) holds, then \(C(x, y)\) takes the maximum value at the point \((\alpha, \alpha)\); if (10) holds, then \(C(x, y)\) is also increasing in the upper part of the line \(x + y = 2\alpha\), from the border of the unit square to the diagonal, and decreasing along the lower part (from \((\alpha, \alpha)\) to the border).

We recall (see [5, p.104]) that quasi-concave copulas are also Schur-concave if they are symmetric (but not in general).

Now we prove that also quasi-concavity together with (11) implies Schur-concavity. Thus we obtain a description of symmetric quasi-concave copulas.

Theorem 1. If a quasi-concave copula satisfies (11), then it is Schur-concave.

Proof: Let \(C(x, y)\) be quasi-concave and satisfy (11); assume, by contradiction that \(C\) is not Schur-concave, and let be \((x, y), (x', y')\) points along the segment \(x + y = 2\alpha\) (\(0 \leq \alpha \leq 1\)) such that:

\[
0 \leq x < x' \leq \frac{x + y}{2}; \quad C(x, y) > C(x', y').
\]

Since, according to (11):

\[
C\left(\frac{x + y}{2}, \frac{x + y}{2}\right) \geq C(x, y),
\]

the quasi-concavity of \(C\) implies

\[
C(x', y') \geq \min\{C(x, y), C\left(\frac{x + y}{2}, \frac{x + y}{2}\right)\} = C(x, y),
\]

against \((*)\); so we have a contradiction.

Analogously, we obtain a contradiction starting from \(\frac{x+y}{2} \leq x < x' \leq 1\). This concludes the proof of the theorem. \(\square\)

We have immediately the following consequence.
Corollary. For a quasi-concave copula $C$ the following are equivalent:

(i) $C$ is symmetric,
(ii) $C$ satisfies (11),
(iii) $C$ is Schur-concave.

Remark. For an example of a (symmetric) Schur-concave copula which is not quasi-concave see [5, Example 3.28(a)]; so condition (11) does not imply quasi-concavity. Example 2 in [1] describes a symmetric copula satisfying (11), which is not Schur-concave (so neither quasi-concave).

We can ask for some other condition implying the quasi-concavity of a copula. We give below one possible answer.

We consider another class of copulas, satisfying a condition which also has a statistical meaning (see [5, Definition 5.2.9 and Corollary 5.2.11]).

Definition 3. We say that a copula is *stochastically increasing* in $x$ and $y$, (SI) for short, if it is concave in each variable; namely:

\[
C(x, y) \text{ is a concave function of } y \text{ for any fixed } x, \\
\text{and a concave function of } x \text{ for any fixed } y \text{ (} x, y \in [0, 1])
\] (12)

We have the following result.

Theorem 2. (SI) copulas are quasi-concave.

Proof: We recall that, since we are dealing with continuous functions, quasi-concavity for copulas is equivalent to Jensen (midpoint) quasi-concavity, that is to

\[
(9') \quad C\left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) \geq \min\{C(x, y), C(x', y')\}.
\]

We prove now a simple claim.

**Claim.** If a copula $C(x, y)$ is not quasi-concave, then $(9')$ is violated by a pair of points $(x, y), (x', y')$ such that the line joining them has a negative slope.

**Proof of the claim:** Let $C(x, y)$ be a copula. Let the points $(x, y), (x', y')$ be such that the line joining them has a non-negative slope: if $x \leq x'$ and $y \leq y'$, then (3) implies: $\min\{C(x, y), C(x', y')\} = C(x, y) \leq C(x', y')$; moreover $C(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y'), \lambda \in [0, 1]$, is an increasing function of $\lambda$.

Thus in this case $(9')$ is satisfied; this proves the claim.

**Proof of Theorem 2:** We deal with Jensen concavity. Let $C(x, y)$ be an (SI) copula; assume, by contradiction, that $C(x, y)$ is not quasi-concave: according to
the claim, there are two points \((x, y), (x', y')\) such that the line joining them has a negative slope and moreover:

\[
\min\{C(x, y), C(x', y')\} > C\left(\frac{x + x'}{2}, \frac{y + y'}{2}\right).
\]

Assume that, for example, \(x < x'\) and \(y > y'\) (a similar reasoning applies in the case \(x > x'\) and \(y < y'\)).

According to (12), we have

\[
C\left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) \geq \frac{1}{2}(C(x, \frac{y + y'}{2}) + C(x', \frac{y + y'}{2}));
\]

therefore:

\[
\frac{1}{2}(C(x, \frac{y + y'}{2}) + C(x', \frac{y + y'}{2})) < \min\{C(x, y), C(x', y')\}
\]

\[
\leq \frac{1}{2}(C(x, y) + C(x', y')).
\]

Since (by (12))

\[
C(x', y) - C(x', \frac{y + y'}{2}) \leq C(x', \frac{y + y'}{2}) - C(x', y'),
\]

we obtain

\[
C(x', y) - C(x', \frac{y + y'}{2}) < C(x, y) - C(x, \frac{y + y'}{2})
\]

or

\[
C(x', y) + C(x', \frac{y + y'}{2}) < C(x, y) + C(x', \frac{y + y'}{2}).
\]

The last inequality contradicts 2-increasingness. This completes the proof of the theorem.

\(\square\)

**Remark 1.** Following the lines of the now given proof, also the following fact can be proved:

*If a copula is (SI), then it is also concave along lines with a negative slope.*

Recall that (SI) does not imply concavity of a copula (the definition of concavity being the usual one, which implies (SI)): in fact, there exists a unique concave copula (which is the greatest copula: see [5, Example 3.26.(a)]). Also: Schur concavity and concavity are independent notions for functions (see [7, p.258]); but the unique concave copula is Schur-concave.

**Remark 2.** Note that in general (SI) copulas are not symmetric (or equivalently, according to Theorem 2 and the Corollary to Theorem 1, they do not satisfy (10) or (11)). An example of an asymmetric, (SI) copula, is the following:
Example 1. Consider the following copula:

$$C(x, y) = \begin{cases} 
xy^{3/4} & \text{if } x \leq y^{1/2} \\
yx^{1/2} & \text{if } x > y^{1/2}.
\end{cases}$$

Remark 3. It is also possible to see that symmetric, quasi-concave copulas (see the Corollary) are not in general (SI) copulas: consider for example as $C(x, y)$ a copula whose level lines, which are broken lines, join $(x^2, 1)$, $(x, x)$, $(1, x^2)$, $x \in [0, 1]$.

Remark 4. Recall that a copula is associative if for all $x, y, z \in [0, 1]$ we have:

$$C(C(x, y), z) = C(x, C(y, z))$$

for all $x, y, z \in [0, 1]$.

The following copula is (SI), symmetric but not associative:

$$C(x, y) = \begin{cases} 
x\sqrt{y} & \text{if } x \leq y, \\
y\sqrt{x} & \text{if } x \geq y;
\end{cases}$$

to see this, it is enough to consider for example in the above definition $x = y = \frac{1}{2}$; $z = \frac{1}{4}$.

3. Quasi-concave copulas and asymmetry

In this section we want to study the quantity

$$\beta(Q) = \sup \{\beta_C; \ C \text{ is a quasi-concave copula}\}.$$  

We recall the following result (see [6, Lemma 2.1]).

Lemma. For any copula $C$ and any $x, y \in [0, 1]$ we have:

$$|C(x, y) - C(y, x)| \leq \min\{x, y, 1-x, 1-y, |x-y|\}.$$  

Now we prove the main result of this section.

Theorem 3. We have:

$$\beta(Q) = 1/5.$$  

Proof: Let $\beta(C) = \beta > 0$ for some quasi-concave copula $C$ and let $\beta = C(x, y) - C(y, x)$ for a pair $x, y$. It is not a restriction to assume $x < y$ (otherwise, by
symmetry, we may construct a copula $C'$, with same asymmetry, for which this holds true.

Let $C(y, x) = \alpha$ and $C(x, y) = \alpha + \beta$. According to the Lemma, $P \equiv (x, y)$ belongs to the triangle $T : \{(x, y); \ x \geq \beta; \ y \leq 1 - \beta; \ y \geq x + \beta\}$.

The points $(1, \alpha + \beta)$, $(\alpha + \beta, 1)$, $(x, y)$ all belong to the level sets $L = \{(u, v); \ C(u, v) = \alpha + \beta\}$.

Recall that $C(y, x) = \alpha$; let $(y, z')$ be the lower point of abscissa $y$ that belongs to $L$, and $(y, z)$ the point of abscissa $y$ that belongs to the segment of extremes $(x, y)$, $(1, x)$ $(x \geq \alpha + \beta)$. Considering that segment, if we write $y = \frac{1-y}{1-x}x + \frac{y-x}{1-x}1$, we see that $z' = \frac{1-y}{1-x}y + \frac{y-x}{1-x}x$.

Now the quasi-concavity of $C$ implies $C(y, z') \geq \alpha + \beta$, so $z' \geq z > x$, and then (by using (5)) $z' - x \geq z - x \geq C(y, z) - C(y, x) = \beta$; then

$$\frac{1-y}{1-x}y + \frac{y-x}{1-x}x - x \geq \beta; \ \text{equivalently} \ \frac{y-y^2+yx-x}{1-x} \geq \beta.$$  

Now consider the function $f(x, y) = \frac{-y^2+y(1+x)-x}{1-x}$ in the triangle $T$; simple computations show that it attains its maximum at the point $(\beta, \frac{1+\beta}{2})$. So we have

$$\frac{1-\beta}{4} \geq \beta, \ \text{which is equivalent to} \ \beta \leq 1/5.$$  

We have proved that 1/5 is an upper bound for $\beta(Q)$. To conclude the proof we must produce an example of a quasi-concave copula $C$ such that

$$\beta_C = \sup\{|C(x, y) - C(y, x)|; \ x, y \in [0, 1]\} = 1/5.$$  

This is done by the example below.

**Example 2.** The above proof shows that the value 1/5 for asymmetry can only be attained, in the upper triangle $y \geq x$ of the unit square, at the point $(\frac{1}{2}, \frac{3}{4})$.

Note that the copulas we are considering are related to examples in Section 3.2.1 of [5].

We define a copula $C_1(x, y)$, whose asymmetry is 1/5, in the following way:

$$C_1(x, y) = \begin{cases} \max\{y + (x - 1)/2, 0\} & \text{if } 0 \leq y \leq \frac{x+1}{2}, \\ x & \text{if } \frac{x+1}{2} < y \leq 1. \end{cases}$$  

The copula $C_1$ is quasi-concave (the upper boundary of level sets are convex: see [5, Theorem 3.4.5]). The support of $C_1$ consists of the two line segments in $I^2$:

$$\{(x, y) \in I^2; \ y = \frac{1+x}{2}\} \cup \{(x, y) \in I^2; \ y = \frac{1-x}{2}\}.$$
We can also consider a copula $C_2$, with the same asymmetry, whose support is distributed along some line segments: weight $4/5$ spread along the line joining $(1/5, 1)$ and $(1, 1/5)$; weight $1/5$ along the segment joining $(0, 3/5)$ and $(1/5, 2/5)$; weight $2/5$ along the segment joining $(1/5, 2/5)$ and $(1, 0)$; finally, negative weight $2/5$ spread along the segment of extremes $(1/5, 3/5)$ and $(1, 1/5)$.

The copulas $C_1$ and $C_2$ seem to be, respectively, the largest and the smallest quasi-concave copulas among all quasi-concave copulas such that $C(3/5, 1/5) = 0$, $C(1/5, 3/5) = 1/5$.

Analogously we can construct, by symmetry, another pair of copulas $C_3$ and $C_4$ so that, by using these 4 copulas, we can indicate all quasi-concave copulas attaining the largest values for asymmetry. All of this can be done following the scheme of [6].

**Remark.** Our last result also says how far a quasi-concave copula can be from being Schur-concave. For example, given any quasi-concave copula $C(x, y)$, the copula $C'(x, y) = 1/2(C(x, y) + C(y, x))$ is a symmetric copula such that

$$|C(x, y) - C'(x, y)| \leq \frac{1}{10}$$

for all $x, y$.

But we can observe that in this way the copula we obtain is not in general a quasi-concave copula. This can be seen by starting, for example, from the copula in Exercise 3.8 in [5], with $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$.

### 4. Quasi-concave copulas and transformations

The following transformations have often been considered for aggregation operators, in particular for copulas.

Let $\varphi$ be a strictly increasing bijection of $[0,1]$. Set

$$C_\varphi(x, y) = \varphi^{-1}(C(\varphi(x), \varphi(y))).$$

It is well known that only if $\varphi$ is concave, $C_\varphi$ is a copula whenever $C$ is a copula.

Now we prove that if $\varphi$ is concave, then also quasi-concavity of copulas is preserved.

**Theorem 4.** If $C$ is a quasi-concave copula and $\varphi$ is concave, then $C_\varphi$ is a quasi-concave copula.

**Proof:** We already know that under our assumptions, $C_\varphi$ is a copula. Assume, by contradiction, that $C_\varphi$ is not quasi-concave. This means that there exist two pairs $(x_1, y_1)$ and $(x_2, y_2)$ and some $\lambda \in [0,1]$ such that for the point $(x_\lambda, y_\lambda)$, where $x_\lambda = \lambda x_1 + (1-\lambda)x_2$; $y_\lambda = \lambda y_1 + (1-\lambda)y_2$, we have:

$$C_\varphi(x_\lambda, y_\lambda) < C_\varphi(x_1, y_1); C_\varphi(x_\lambda, y_\lambda) < C_\varphi(x_2, y_2);$$
since $\phi$ is increasing, these are equivalent to

$$C(\phi(x_\lambda), \phi(y_\lambda)) < C(\phi(x_1), \phi(y_1)); C(\phi(x_\lambda), \phi(y_\lambda)) < C(\phi(x_2), \phi(y_2)).$$

Now set, for $\lambda \in [0, 1]$:

$$x_\lambda' = \lambda \phi(x_1) + (1 - \lambda) \phi(x_2); y_\lambda' = \lambda \phi(y_1) + (1 - \lambda) \phi(y_2).$$

The fact that $\phi$ is concave implies

$$\phi(x_\lambda) \geq x_\lambda', \phi(y_\lambda) \geq y_\lambda';$$

therefore we obtain:

$$C(x_\lambda', y_\lambda') \leq C(\phi(x_\lambda), \phi(y_\lambda)) < C(\phi(x_1), \phi(y_1)), $$

and similarly,

$$C(x_\lambda', y_\lambda') < C(\phi(x_2), \phi(y_2)),$$

against the quasi-concavity of $C$. This contradiction proves the theorem. □

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