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Abstract. A function f mapping the topological space X to the space Y is called a *z-open* function if for every cozeroset neighborhood H of a zeroset Z in X, the image $f(H)$ is a neighborhood of $\mathrm{cl}_Y(f(Z))$ in Y. We say f has the *z-separation property* if whenever U, V are cozerosets and Z is a zeroset of X such that $U \subseteq Z \subseteq V$, there is a zeroset Z' of Y such that $f(U) \subseteq Z' \subseteq f(V)$. A surjective function is z-open if and only if it maps cozerosets to cozerosets and has the z-separation property. We investigate z-open functions and other functions that map cozerosets to cozerosets. We show that if f is a continuous z-open function, then the Stone extension of f is an open function. This is used to show several properties of topological spaces related to F-spaces are preserved under continuous z-open functions.

Keywords: open function, cozeroset preserving function, z-open function, F-space, SV space, finite rank

Classification: Primary 54C30, 54C10

1. Introduction

Given a continuous open function f mapping a completely regular space X to a completely regular space Y, the Stone extension of the function mapping βX to βY is not necessarily open as is shown in [L5]. An F-space is a space in which every cozeroset is C^*-embedded. The image under a continuous open function of an F-space is not necessarily an F-space (see [L5]). However, the image under a continuous open function of a compact F-space is an F-space. A similar situation holds for several other related types of spaces. Motivated by these facts, we investigate functions that map cozerosets to cozerosets, investigate when a function's Stone extension will be open, and show which of these functions will preserve various topological properties related to the F-space property.

A function $f : X \to Y$ is said to be a

(i) *cozeroset preserving function* if for every cozeroset U of X, $f(U)$ is a cozeroset of Y;

(ii) *z-open function* if for every cozeroset neighborhood H of a zeroset Z in X, the image $f(H)$ is a neighborhood of $\mathrm{cl}_Y(f(Z))$ in Y;

(iii) *strongly open function* if it is open and whenever U, V are open sets in X with $\mathrm{cl}_X(U) \subseteq V$, then $\mathrm{cl}_Y(f(U)) \subseteq f(V)$.

Continuous z-open functions were studied in [W]. We describe the relationship between the types of functions just defined and give basic properties of these functions. In Section 4, we show that if $f : X \to Y$ is a continuous z-open function, then the Stone extension of f is an open function mapping βX to βY and give a condition on a continuous function under which the Stone extension is open if and only if the function is z-open. In our last section, we use the fact that continuous open functions defined on compact spaces preserve several properties related to the F-space property and our results from Section 4 to show that continuous z-open functions preserve F-spaces, spaces that are finitely an F-space, SV spaces of rank at most 2, spaces of finite rank, and quasi-F spaces. It is known that continuous open functions do not necessarily preserve F-spaces, spaces that are finitely an F-space, SV spaces of rank at most 2, and spaces of finite rank.

2. Preliminaries

Throughout, we assume all topological spaces are completely regular. Let X denote a completely regular topological space, and $C(X)$ denote the f-ring of real-valued continuous functions defined on X. Also, let $C^*(X)$ denote the f-ring of all bounded real-valued continuous functions defined on X. Recall that in X, a *zeroset* is a set of the form $\{x \in X : f(x) = 0\}$ for some function $f \in C(X)$, and a *cozeroset* is the complement of a zeroset. Given a function $f \in C(X)$, we let $Z(f)$ (resp. $\operatorname{coz}(f)$) denote the zeroset (resp. cozeroset) determined by f. In a normal space, an open F_σ-set is a cozeroset (see 3D in [GJ]). A subspace S of X is said to be *C^*-embedded* in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$. For a completely regular space, we let βX denote the Čech-Stone compactification of X. A fundamental property of the Čech-Stone compactification is that given a continuous function $f : X \to Y$ there is a unique continuous extension of f mapping βX into βY. This extension is called the *Stone extension* of f.

An *F-space* is a space in which every cozeroset is C^*-embedded. A number of conditions, both topological conditions on X, and algebraic conditions on $C(X)$, are equivalent to X being an F-space and appear in 14.25, [GJ]; 1, [MW]; and 2.4, [L1]. We will have occasion to use the following well known fact.

Theorem 1 (14.25, [GJ]). *A normal space X is an F-space if and only if for each $x \in X$, the maximal ℓ-ideal $M_x = \{f \in C(X) : f(x) = 0\}$ contains a single minimal prime ideal.*

When it is necessary to explicitly indicate that we are considering a topological property with respect to the space X, we use "X-" as a prefix to the topological property. For example, when we say a set is X-open, we simply mean that it is an open set with respect to the topology on X.

In several examples, we will make use of $W(\alpha)$, the set of all ordinals less than the ordinal α.

3. Basic properties

In this section, we study cozeroset preserving, z-open, and strongly open functions and give basic properties of these functions. We show the relationships between the various types of functions can be summarized as follows, with none of the implications being reversible.

open and closed function $\Rightarrow$ *strongly open function* $\Rightarrow$ *z-open function* $\Rightarrow$ *cozeroset preserving function* $\Rightarrow$ *open function*

All of these types of functions, except open functions, map cozerosets to cozerosets. Because of the importance of cozerosets in studying rings of continuous functions, it is natural to investigate functions that map cozerosets to cozerosets. In this section, we will not assume that all functions considered are continuous; however in later sections we will.

The proof of our first lemma is straightforward and omitted.

Lemma 2. *If $f : X \to Y$ is a cozeroset preserving function, then f is an open function.*

An example showing that an open function need not be cozeroset preserving is given in 3.17, [HW]. An example of an (open) projection mapping that is not cozeroset preserving is the projection of $X \times Y$ onto X, where $X = \mathbb{R} \cup \{\alpha\}$ under the topology in which each element of $\mathbb{R}$ is an isolated point while neighborhoods of α are co-countable subsets of $\mathbb{R}$ and Y denotes $\mathbb{R}$ under the discrete topology.

A topological space X is said to be *cozero complemented* if for every cozeroset V of X, there is a disjoint cozeroset V' such that $V \cup V'$ is dense in X. These spaces are studied in [HW]. It is straightforward to show that if X is a cozero complemented space and $f : X \to Y$ is a cozeroset preserving map of X onto Y, then Y must also be a cozero complemented space.

A proof of the first of the following two properties of z-open functions can be found in 15.3, [W]. There, a z-open function is assumed to be continuous, but continuity is not used in the proof.

Lemma 3. *Let $f : X \to Y$ be a z-open function.*

(1) *f is an open function.*
(2) *If U is a clopen set in X, then $f(U)$ is clopen in Y.*

Proof of (2): Suppose f is z-open. Note that since U is clopen, it is both a cozeroset and a zeroset and $U \subseteq U$. By hypothesis, $f(U)$ is a Y-neighborhood of $\mathrm{cl}_Y(f(U))$. Since $f(U) \subseteq \mathrm{cl}_Y(f(U))$, we have $\mathrm{cl}_Y(f(U)) = f(U)$. So $f(U)$ is closed and by (1), $f(U)$ is open. $\square$

The next example demonstrates that a continuous cozeroset preserving function need not be a z-open function.

Example 4. Let $\mathbb{N}^*$ denote the one-point compactification of $\mathbb{N}$, with β denoting the point "at infinity". Consider the function $f : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N}^*$ defined by $f(x,y) = x$. Then f is continuous and open since it is a projection function. First we show f is a cozeroset preserving function. Suppose U is a $(\mathbb{N}^* \times \mathbb{N})$-cozeroset. Then for each $n \in \mathbb{N}$, $U \cap (\mathbb{N}^* \times \{n\})$ is a cozeroset of the subspace $\mathbb{N}^* \times \{n\}$, which is isomorphic to $\mathbb{N}^*$. It follows that $f(U) = \bigcup_{n=1}^{\infty} f(U \cap (\mathbb{N}^* \times \{n\}))$ is the union of countably many cozerosets in $\mathbb{N}^*$ and hence is an $\mathbb{N}^*$-cozeroset. So f is a cozeroset preserving function. Next we show that f is not a z-open function. Let $U = \{(n,n) : n \in \mathbb{N}\}$. It is not hard to see that U is a clopen set. Now $f(U) = \mathbb{N}^* - \{\beta\}$ is not clopen and hence by Lemma 3(2), f is not a z-open function.

Lemma 5. *Suppose $f : X \to Y$ is a z-open function mapping X onto Y and $h \in C^*(X)$. The function $\ell : Y \to \mathbb{R}$ defined by $\ell(y) = \sup\{h(x) : x \in f^{-1}(y)\}$ is continuous.*

PROOF: Let $y_0 \in Y$, $\epsilon > 0$, and $a = \ell(y_0)$. Let $U_1 = h^{-1}((a + \frac{\epsilon}{4}, \infty))$ and $U_2 = h^{-1}([a + \frac{\epsilon}{2}, \infty))$. Then $U_2 \subseteq U_1$ and U_1 is an X-cozeroset while U_2 is an X-zeroset. Since f is z-open, $f(U_1)$ is a neighborhood of $\mathrm{cl}_Y(f(U_2))$ in Y. Note that $y_0 \in Y - f(U_1) \subseteq Y - \mathrm{cl}_Y(f(U_2))$.

Now let $V = h^{-1}((a - \frac{\epsilon}{2}, \infty))$. Then $f(V)$ is open (by Lemma 3(1)) and $y_0 \in f(V)$. So $(Y - \mathrm{cl}_Y(f(U_2))) \cap f(V)$ is a Y-open neighborhood of y_0. We will show that $\ell((Y - \mathrm{cl}_Y(f(U_2))) \cap f(V)) \subseteq (a - \epsilon, a + \epsilon)$. Let $z \in (Y - \mathrm{cl}_Y(f(U_2))) \cap f(V)$. Then $z = f(v)$ for some $v \in V$. So $\ell(z) = \sup\{h(x) : x \in f^{-1}(z)\} \geq h(v) > a - \epsilon$. Since $z \in Y - \mathrm{cl}_Y(f(U_2))$, $z \notin f(U_2)$. It follows that $f^{-1}(z) \subseteq X - U_2$, which implies $h(f^{-1}(z)) \subseteq (-\infty, a + \frac{\epsilon}{2}]$. Then $\ell(z) = \sup\{h(x) : x \in f^{-1}(z)\} \leq a + \frac{\epsilon}{2}$. We have, $a - \epsilon < \ell(z) < a + \epsilon$. This shows that $\ell \in C(Y)$. □

A function is *z-closed* if it maps zerosets onto closed sets. In [CH], Comfort and Hager prove that the projection mapping from the product space $Y \times X$ onto Y is z-closed if and only if every function defined on Y of the form $\ell(y) = \sup\{h(y,x) : x \in X\}$ for some $h \in C^*(Y \times X)$ is continuous. They also give a large number of other conditions on a product space that are equivalent to the projection mapping being z-closed. In the previous lemma, if $f : Y \times X \to Y$ is the projection mapping and $h \in C^*(Y \times X)$, the function ℓ is of the form considered by Comfort and Hager. Using their result, it is easy to show that a projection mapping from $Y \times X$ onto Y is z-open if and only if it is z-closed.

We say a function $f : X \to Y$ has the *z-separation property* if whenever U, V are cozerosets and Z is a zeroset of X such that $U \subseteq Z \subseteq V$, there is a zeroset Z' of Y such that $f(U) \subseteq Z' \subseteq f(V)$.

Theorem 6. *Suppose $f : X \to Y$ is a function mapping X onto Y. The following are equivalent.*

(1) *f is z-open.*
(2) *f is cozeroset preserving and has the z-separation property.*

PROOF: $1 \Rightarrow 2$: Suppose f is z-open. We will first show f is cozeroset preserving. Let U be an X-cozeroset and $h \in C^*(X)$ such that $\mathrm{coz}(h) = U$. Define a function $\ell : Y \to \mathbb{R}$ by $\ell(y) = \sup\{h(x) : x \in f^{-1}(y)\}$. By the previous lemma, ℓ is continuous. Let $y \in \mathrm{coz}(\ell)$. Then there is an $x_1 \in f^{-1}(y)$ such that $h(x_1) > 0$. Then $x_1 \in \mathrm{coz}(h) = U$ and $y = f(x_1) \in f(U)$. Thus $\mathrm{coz}(\ell) \subseteq f(U)$. A similar argument shows that $f(U) \subseteq \mathrm{coz}(\ell)$ and hence $\mathrm{coz}(\ell) = f(U)$. Thus f is cozeroset preserving.

Now suppose $U \subseteq Z \subseteq V$ for X-cozerosets U, V and X-zeroset Z. Then Z and $X - V$ are disjoint X-zerosets. Hence they are completely separated (see 1.15 of [GJ]) and there is a $g \in C(X)$ such that $g(X - V) = \{0\}$, $g(Z) = \{1\}$, and $0 \leq g \leq 1$. Let $h \in C(X)$ be a function such that $\mathrm{coz}(h) = U$, and $0 \leq h \leq 1$. Note that $(g + h)(x) > 1$ for every $x \in U$ and that $(g + h)(X - V) = \{0\}$. Define a function $\ell : Y \to \mathbb{R}$ by $\ell(y) = \sup\{(g + h)(x) : x \in f^{-1}(y)\}$. By the previous lemma, ℓ is continuous. Let $Z' = \ell^{-1}([1, \infty))$. Then Z' is a Y-cozeroset. Now $f(U) \subseteq Z'$ since if $u \in U$, then $\ell(f(u)) = \sup\{(g + h)(x) : x \in f^{-1}(f(u))\} \geq (g + h)(u) > 1$. We now show that $Z' \subseteq f(V)$. Suppose $y' \in Z'$. Then $\ell(y') = \sup\{(g + h)(x) : x \in f^{-1}(y')\} \geq 1$. So there is an $x' \in f^{-1}(y')$ such that $(g + h)(x') > \frac{1}{2}$. But then $x' \in V$ and $y' = f(x') \in f(V)$. We have, $f(U) \subseteq Z' \subseteq f(V)$.

$2 \Rightarrow 1$: Let Z be an X-zeroset and H an X-cozeroset such that $Z \subseteq H$. Then Z and $X - H$ are disjoint X-zerosets. Hence they are completely separated (see 1.15 of [GJ]) and there must be disjoint X-cozerosets H_1, H_2 such that $Z \subseteq H_1$ and $X - H \subseteq H_2$. Then $Z \subseteq H_1 \subseteq X - H_2 \subseteq H$. By (2), there is a Y-zeroset Z' such that $f(H_1) \subseteq Z' \subseteq f(H)$. Then $\mathrm{cl}_Y(f(Z)) \subseteq \mathrm{cl}_Y(f(H_1)) \subseteq Z' \subseteq f(H)$. Since f is cozeroset preserving, $f(H)$ is a Y-cozeroset. So $f(H)$ is a Y-neighborhood of $\mathrm{cl}_Y(f(Z))$ and hence f is z-open. □

Theorem 7. *Suppose $f : X \to Y$ is a strongly open function mapping X onto Y. Then f is a z-open function.*

PROOF: Suppose Z is an X-zeroset and H is an X-cozeroset with $Z \subseteq H$. Now Z and $X - H$ are disjoint zerosets and so are completely separated. It follows that there is an X-open set U such that $Z \subseteq U \subseteq \mathrm{cl}_X(U)$ and $\mathrm{cl}_X(U)$ is disjoint from $X - H$. Then $Z \subseteq \mathrm{cl}_X(U) \subseteq H$. Since f is a strongly open function, $\mathrm{cl}_Y(f(U)) \subseteq f(H)$. So $\mathrm{cl}_Y(f(Z)) \subseteq \mathrm{cl}_Y(f(U)) \subseteq f(H)$ and $f(H)$ is open. So f is z-open. □

Functions that are z-open are not necessarily strongly open functions as our next example demonstrates. In this example and a later example, we let $W(\alpha)$

denote the set of all ordinals less than the ordinal α.

Example 8. Let W^* denote the space of ordinals less than or equal to ω_1, the first uncountable ordinal, and let $X = W^* \times W^* - \{(\omega_1, \omega_1)\}$. Define $f : X \to W^*$ by $f((\alpha, \beta)) = \alpha$. Then, since f is the restriction of a projection function to an open set, it is a continuous and open function. We will show that f is a z-open function, but fails to be a strongly open function.

Suppose Z is an X-zeroset, H is an X-cozeroset and $Z \subseteq H$. We begin by showing that $f(Z)$ is closed by considering two cases. First suppose there is an ordinal $\beta < \omega_1$ such that for all $(z_1, z_2) \in Z$, either $z_1 < \beta$ or $z_2 < \beta$. The subspaces $W^* \times W(\beta+1)$ and $W(\beta+1) \times W^*$ are compact and the restricted functions $f|_{W^* \times W(\beta+1)}$ and $f|_{W(\beta+1)\times W^*}$ are closed functions. So $f(Z) = f|_{W^*\times W(\beta+1)}(Z \cap (W^* \times W(\beta+1))) \cup f|_{W(\beta+1)\times W^*}(Z \cap (W(\beta+1) \times W^*))$ is the union of two closed sets and hence is closed. Next, suppose for all ordinals $\beta < \omega_1$, there is a $(z_1, z_2) \in Z$ such that $z_1, z_2 > \beta$. Since Z is a zeroset and every continuous function defined on X is constant on a set of the form $[W^* - W(\alpha)] \times [W^* - W(\alpha)] - \{(\omega_1, \omega_1)\}$, there must be an ordinal $\gamma < \omega_1$ such that for all $z_1, z_2 \geq \gamma$ with z_1, z_2 not both equal to ω_1, the point (z_1, z_2) is in Z. Then $f(Z) = f|_{W(\gamma+1)\times W^*}(Z \cap (W(\gamma+1) \times W^*)) \cup f(Z \cap ([W^* - W(\gamma+1)] \times W^*)) = f|_{W(\gamma+1)\times W^*}(Z \cap (W(\gamma+1) \times W^*)) \cup (W^* - W(\gamma+1))$, which is the union of two closed sets and hence is closed. In either case, $f(Z)$ is closed. Now $\mathrm{cl}_Y(f(Z)) = f(Z) \subseteq f(H)$ and $f(H)$ is open since f is a projection function. This implies f is a z-open function.

Finally, we show that f is not a strongly open function. Let $U = \{(u_1, u_2) \in X : u_1 < u_2\}$ and $V = \{(u_1, u_2) \in X : u_1 < \omega_1\}$. Then U, V are X-open sets. Also, $\mathrm{cl}_X(U) \subseteq \{(u_1, u_2) \in X : u_1 \leq u_2\} \subseteq V$. Now $f(U) = W^* - \{\omega_1\} = f(V)$. However, $\mathrm{cl}_{W^*}(f(U)) = W^* \not\subseteq f(V)$. So f is not a strongly open function.

Lemma 9. *Suppose $f : X \to Y$ is an open and closed function mapping X onto Y. Then f is a strongly open function.*

Proof: Suppose $f : X \to Y$ is an open and closed function mapping X onto Y. Suppose U, V are X-open sets with $\mathrm{cl}_X(U) \subseteq V$. Then $f(\mathrm{cl}_X(U))$ is a closed set and $\mathrm{cl}_Y(f(U)) \subseteq f(\mathrm{cl}_X(U)) \subseteq f(V)$. So f is a strongly open function. $\square$

Corollary 10. *A continuous open function defined on a compact space is a strongly open function.*

The next example shows that the converse to Lemma 9 does not hold: a continuous strongly open function is not necessarily closed.

Example 11. Let L_1 denote the space of all ordinals $\leq \omega_1$ under the topology in which neighborhoods of ω_1 are as in the interval topology and all other points are isolated. Let L_2 denote the space of all ordinals $\leq \omega_2$ under the topology in which neighborhoods of ω_2 are as in the interval topology and all other points are isolated. Let $X = L_2 \times L_1 - \{(\omega_2, \omega_1)\}$. Let $f : X \to L_2$ be defined by $f((a,b)) = a$. Then f is a continuous open function since it is a restriction of a projection function to an open subspace. We will show that f is a strongly open function. Let U, V be X-open sets such that $\mathrm{cl}_X(U) \subseteq V$. Suppose first that for all ordinals $\alpha < \omega_1$, the point $(\omega_2, \alpha) \notin \mathrm{cl}_X(U)$. Then for each $\alpha < \omega_1$, there is a $\beta_\alpha < \omega_2$ such that $[L_2 - W(\beta_\alpha)] \times \{\alpha\}$ is disjoint from U. Since there are at most ω_1 distinct values of the $\beta_\alpha{}'s$, there must be a $\beta < \omega_2$ such that $\beta > \beta_\alpha$ for all α. Since every continuous function defined on X is constant on a set of the form $[L_2 - W(\gamma_2)] \times [L_1 - W(\gamma_1)] - \{(\omega_2, \omega_1)\}$, it follows that $([L_2 - W(\beta+1)] \times L_1 - \{(\omega_2, \omega_1)\}) \cap U = \emptyset$. So every element of $f(U)$ is less that β. Hence $\mathrm{cl}_{L_2}(f(U)) = f(U) \subseteq f(V)$.

Next suppose there is an $\alpha < \omega_1$ such that $(\omega_2, \alpha) \in \mathrm{cl}_X(U)$. Then $(\omega_2, \alpha) \in V$ and since V is an open set, there must exist an ordinal $\gamma < \omega_2$ such that $[L_2 - W(\gamma)] \times \{\alpha\} \subseteq V$. Then $L_2 - W(\gamma) \subseteq f(V)$. Then $\mathrm{cl}_{L_2}(f(U)) = \mathrm{cl}_{L_2}(f(U) \cap W(\gamma)) \cup \mathrm{cl}_{L_2}(f(U) \cap [L_2 - W(\gamma)]) \subseteq [f(U) \cap W(\gamma)] \cup [L_2 - W(\gamma)] \subseteq f(V)$. Hence f is a strongly open function.

However, f is not a closed function since the set $A = \{(b, \omega_1) : b < \omega_2\}$ is closed in X, yet $f(A) = L_2 - \{\omega_2\}$ is not closed in L_2.

One of the hypotheses in the next lemma is a condition on a continuous function that is similar to, but apparently weaker than, the condition that the function is z-open as judged by Theorem 6.

Lemma 12. *Suppose $f : X \to Y$ is a continuous open function mapping X onto Y with the z-separation property. If X is a normal space, then Y is a normal space.*

Proof: Let A, B be disjoint closed sets in Y. Then $f^{-1}(A), f^{-1}(B)$ are disjoint closed sets in X. Since X is normal, there is a $g \in C(X)$ such that $g(f^{-1}(A)) = \{1\}, g(f^{-1}(B)) = \{0\}$, and $0 \le g \le 1$. Let $U = g^{-1}((\frac{3}{4}, 1])$, $V = g^{-1}((\frac{1}{4}, 1])$, and $Z = g^{-1}([\frac{1}{2}, 1])$. Then U, V are X-cozerosets, Z is an X-zeroset and $U \subseteq Z \subseteq V$ in X. By hypothesis, there is a Y-zeroset Z' such that $f(U) \subseteq Z' \subseteq f(V)$. Now $f^{-1}(A) \subseteq U$; so $A \subseteq f(U)$. We will show that if $x \in f^{-1}(B)$, then $f(x) \in Y - f(V)$. Suppose not — that is, suppose $x \in f^{-1}(B)$ and $f(x) \in f(V)$. Let $v \in V$ such that $f(x) = f(v)$. Then $v \in f^{-1}(B)$ and $g(v) = 0$. But then $v \notin g^{-1}((\frac{1}{4}, 1]) = V$, and we have a contradiction. So, if $x \in f^{-1}(B)$, then $f(x) \in Y - f(V)$. It then follows that $f(f^{-1}(B)) \subseteq Y - f(V)$. Thus $B \subseteq Y - f(V) \subseteq Y - Z'$. Since f is open, $f(U)$ is Y-open and certainly $Y - Z'$ is Y-open. We have, $A \subseteq f(U)$, $B \subseteq Y - Z'$, and $f(U), Y - Z'$ are disjoint open sets. Hence Y is normal. □

When the domain of a continuous function f is a normal space, saying f is a strongly open function is equivalent to saying f is a z-open function as the next theorem shows.

Theorem 13. *Suppose $f : X \to Y$ is a continuous function mapping X onto Y. Consider the following conditions. If either X or Y is normal, then (1) and (2) are equivalent and if X is normal, then all four conditions are equivalent.*

(1) *f is open and has the z-separation property.*
(2) *f is a z-open function.*
(3) *f is a strongly open function.*
(4) *f is an open and closed function.*

Proof: Assume X or Y is normal. We first show (1) $\Rightarrow$ (2). Suppose f is open and has the z-separation property. We need only show that f is a cozeroset preserving function by Theorem 6. By the previous lemma, if X is normal, then Y is normal. So we may assume Y is normal. Let W be an X-cozeroset and $g \in C^+(X)$ such that $\mathrm{coz}(g) = W$. For each $n \in \mathbb{N}$, we let W_n be the X-cozeroset $W_n = g^{-1}((\frac{1}{n}, \infty))$, and let Z_n be the X-zeroset $Z_n = g^{-1}([\frac{1}{n}, \infty))$. Then $W_n \subseteq Z_n \subseteq W$ for all n and $\bigcup_n W_n = W$. By hypothesis, $f(W)$ is open and for each n, there are Y-zerosets Z'_n such that $f(W_n) \subseteq Z'_n \subseteq f(W)$. Then $f(W) = f(\bigcup_n W_n) = \bigcup_n f(W_n) \subseteq \bigcup_n Z'_n \subseteq f(W)$. So $f(W) = \bigcup_n Z'_n$ is an open F_σ set in Y. Since Y is normal, $f(W)$ is a cozeroset (see 3D of [GJ]).

That (2) implies (1) follows from Lemma 3(1) and Theorem 6.

Now suppose that X is normal. (2) $\Rightarrow$ (3): Suppose f is a z-open function. Then f is open by Lemma 3(1). Now suppose $\mathrm{cl}_X(U) \subseteq V$ for some X-open sets U, V. Since $\mathrm{cl}_X(U)$ and $X - V$ are disjoint closed sets, there are X-cozerosets U', V' and an X-zeroset Z such that $\mathrm{cl}_X(U) \subseteq U' \subseteq Z \subseteq V' \subseteq V$. By hypothesis there is a Y-zeroset Z' such that $f(U') \subseteq Z' \subseteq f(V')$. It follows that $\mathrm{cl}_Y(f(U)) \subseteq Z' \subseteq f(V)$.

(3) implies (2): Theorem 7.

(3) $\Rightarrow$ (4): Suppose X is normal and f is strongly open. We need only show that f is closed. Suppose A is X-closed. For each $y \in Y - f(A)$, A and $f^{-1}(y)$ are disjoint closed sets in X. Since X is normal, there is an X-open set U_y such that $A \subseteq U_y \subseteq \mathrm{cl}_X(U_y)$ and $\mathrm{cl}_X(U_y)$ is disjoint from $f^{-1}(y)$. So $\mathrm{cl}_X(U_y) \subseteq X - f^{-1}(y)$ and since f is strongly open, $\mathrm{cl}_Y(f(U_y)) \subseteq f(X - f^{-1}(y))$. This implies $y \notin \mathrm{cl}_Y(f(U_y))$. Hence $f(A) = \bigcap_{y \in Y - f(A)} \mathrm{cl}_Y(f(U_y))$. We have written $f(A)$ as the intersection of a collection of closed sets and hence $f(A)$ is a Y-closed set and f is a closed function.

(4) $\Rightarrow$ (2): Lemma 9 and Theorem 7. $\square$

A number of basic properties analogous to properties of open functions hold for cozeroset preserving, z-open, and strongly open functions. The following lemma records these. The proofs are straightforward and left to the reader.

Lemma 14. *Let $f : X \to Y$ and $g : Y \to Z$. Then*

(1) *if f, g are cozeroset preserving (z-open, strongly open) functions, so also is $g \circ f$;*

(2) *if* $g \circ f$ *is a cozeroset preserving* (*z-open, strongly open*) *function and* f *is continuous and surjective, then* g *is a cozeroset preserving* (*z-open, strongly open*) *function;*
(3) *if* $g \circ f$ *is a cozeroset preserving* (*z-open, strongly open*) *function and* g *is continuous and injective, then* f *is a cozeroset preserving* (*z-open, strongly open*) *function;*
(4) *if* f *is a cozeroset preserving function and* U *is a cozeroset contained in* X, *then* $f|_U : U \to Y$ *is also a cozeroset preserving function.*

It is well known that the projection mapping of a product space onto one of its coordinates is an open map. The examples in this section also serve to show that a projection mapping defined on a product space is not necessarily a cozeroset preserving, z-open, or strongly open function. As noted earlier, a projection mapping is z-open if and only if it is z-closed.

4. Open Stone extensions of continuous functions

For the remainder of this paper, all functions considered are assumed to be continuous. Suppose $f : X \to Y$ is a continuous function. In what follows, we let f^* denote the Stone extension of f mapping βX to βY. An example was given in 2.9, [L5], showing that when f is a continuous open function, f^* is not necessarily an open function. In [I], T. Isiwata introduces the notion of a WZ-mapping while studying when a continuous image of a realcompact space is realcompact. A continuous function mapping $X \to Y$ is a *WZ-mapping* if $\mathrm{cl}_{\beta X}(f^{-1}(y)) = f^{*-1}(y)$ for every $y \in Y$. A closed continuous function is a WZ-mapping and a continuous function that maps zerosets to closed sets is a WZ-mapping as Isiwata shows in [I]. In 4.4(ii), [I], it is shown that if $f : X \to Y$ is a continuous WZ-mapping, then f is open if and only if f^* is open. We begin this section by showing that if $f : X \to Y$ is a continuous open WZ-mapping, then f is z-open, and not conversely. We will then show that if f is a continuous z-open function mapping X onto Y, then f^* is also open.

In the proof of our first theorem, we will make use of the following result from Isiwata's paper (4.4(i), [I]). A continuous mapping $f : X \to Y$ is a WZ-mapping if and only if $f(U \cap X) = f^*(U) \cap Y$ for every open set U of βX.

Theorem 15. *Suppose* $f : X \to Y$ *is a continuous open WZ-mapping. Then* f *is a z-open function.*

Proof: Suppose f is a continuous open WZ-mapping. By Isiwata's result (4.4(ii) of [I]), f^* is open. Now let U be an X-cozeroset. Then $U = U' \cap X$ for some βX-cozeroset U'. By Corollary 10 and Lemma 7, f^* is a z-open function and by Theorem 6, $f^*(U')$ is a βY-cozeroset. Since f is a WZ-mapping, $f(U) = f^*(U') \cap Y$. So $f(U)$ is a Y-cozeroset. Hence f is a cozeroset preserving function.

Now suppose $U \subseteq Z \subseteq V$ for some X-cozerosets U, V and X-zeroset Z. There is a βX-cozeroset U' such that $U = U' \cap X$. Now $Z, X - V$ are disjoint X-zerosets and so there are disjoint βX-zerosets Z', Z'' such that $Z' \cap X = Z$, and $Z'' \cap X = X - V$. Now suppose $u \in U'$. Then every βX-neighborhood of u meets U and so $u \in \mathrm{cl}_{\beta X}(U) \subseteq \mathrm{cl}_{\beta X}(Z) \subseteq Z'$. So $U' \subseteq Z'$. Let $V' = \beta X - Z''$. Then V' is a βX-cozeroset and $V' \cap X = V \cap X = V$. Now $Z' \subseteq V'$, for if not, there would have to exist a $z \in Z' - V'$ and so $z \in Z' \cap (\beta X - V') = Z' \cap Z'' = \emptyset$, a contradiction. We now know $U' \subseteq Z' \subseteq V'$.

Since f^* is a z-open function, there exists a βY-zeroset $\hat{Z}$ such that $f^*(U') \subseteq \hat{Z} \subseteq f^*(V')$. Then $f(U) = f^*(U' \cap X) = f^*(U') \cap Y \subseteq \hat{Z} \cap Y \subseteq f^*(V') \cap Y = f^*(V' \cap X) = f(V)$. Since $\hat{Z} \cap Y$ is a Y-zeroset, we have just what we wanted: $f(U) \subseteq \hat{Z} \cap Y \subseteq f(V)$. By Theorem 6, f is z-open. □

Our next example shows that a continuous z-open function need not be a WZ-mapping.

Example 16. Let W^* denote the space of ordinals less than or equal to ω_1, the first uncountable ordinal and let $\mathbb{N}^*$ denote the one-point compactification of $\mathbb{N}$, with β denoting the point "at infinity". For every ordinal $\alpha < \omega_1$, let $X_\alpha = W^* \times \mathbb{N}^* - \{(\omega_1, \beta)\}$. Let X be the free union of the X_α. For each X_α, let $f_\alpha : X_\alpha \to W^*$ be defined by $f_\alpha(x, y) = x$ and let $f : X \to W^*$ be defined by $f(x, y) = f_\alpha(x, y)$ whenever $(x, y) \in X_\alpha$. As shown in 15.17, [W], each f_α is continuous and z-open. Since X is the free union of the X_α, it follows that f is continuous and z-open.

We will show that f is not a WZ-mapping. We begin by defining two zerosets in X. For each $\alpha < \omega_1$, let $Z_\alpha = (W^* - W(\alpha + 1)) \times \mathbb{N}^* - \{(\omega_1, \beta)\}$ in X_α. Then each Z_α is a zeroset in X_α and hence $Z = \bigcup_{\alpha < \omega_1} Z_\alpha$ is a zeroset in X. For each $\alpha < \omega_1$, let $Z'_\alpha = W(\alpha + 1) \times \{\beta\}$ in X_α. Then each Z'_α is a zeroset in X_α and hence $Z' = \bigcup_{\alpha < \omega_1} Z_\alpha$ is a zeroset in X. Now Z, Z' are disjoint X-zerosets, and so $\mathrm{cl}_{\beta X}(Z) \cap \mathrm{cl}_{\beta X}(Z') = \emptyset$. Since W^* is compact, the Stone extension f^* maps βX onto W^*. Because f^* is a closed mapping, $f^*(\mathrm{cl}_{\beta X}(Z'))$ is W^*-closed. Then since $W^* - \{\omega_1\} \subseteq f^*(Z') \subseteq f^*(\mathrm{cl}_{\beta X}(Z'))$, it follows that $W^* = f^*(\mathrm{cl}_{\beta X}(Z'))$. Let $x' \in \mathrm{cl}_{\beta X}(Z')$ such that $f^*(x') = \omega_1$. Now $\mathrm{cl}_{\beta X}(f^{-1}(\omega_1)) \subseteq \mathrm{cl}_{\beta X}(Z)$ and $\mathrm{cl}_{\beta X}(Z)$ is disjoint from $\mathrm{cl}_{\beta X}(Z')$, and so $x' \notin \mathrm{cl}_{\beta X}(f^{-1}(\omega_1))$. In particular, $\mathrm{cl}_{\beta X}(f^{-1}(\omega_1)) \neq f^{*-1}(\omega_1)$ and hence f is not a WZ-mapping.

Our next theorem will generalize one of the implications in 4.4(ii), [I].

Theorem 17. *Suppose $f : X \to Y$ is a continuous z-open function mapping X onto Y. Then $f^* : \beta X \to \beta Y$ is an open function.*

Proof: Let U be a βX-open set and let $y \in f^*(U)$. Then $y = f^*(u)$ for some $u \in U$. There are βX-cozerosets U', V and a βX-zeroset Z such that $u \in V \subseteq Z \subseteq U' \subseteq \mathrm{cl}_{\beta X}(U') \subseteq U$. Then $f(V \cap X)$ and $f(U' \cap X)$ are Y-cozerosets.

Since $V \cap X \subseteq Z \cap X \subseteq U' \cap X$, Theorem 6 implies there exists a Y-zeroset Z' such that $f(V \cap X) \subseteq Z' \subseteq f(U' \cap X)$. Let W be a βY-cozeroset such that $f(U' \cap X) = W \cap Y$. In Y, Z' and $Y - (W \cap Y)$ are disjoint Y-zerosets and hence have disjoint βY-closures. Since βY is normal, there are disjoint βY-zerosets Z_1, Z_2 such that $Z' \subseteq Z_1$ and $Y - (W \cap Y) \subseteq Z_2$. Now let $W' = W \cup (\beta Y - Z_2)$. Then W' is a βY-cozeroset and $W' \cap Y = (W \cup (\beta Y - Z_2)) \cap Y = (W \cap Y) \cup (Y - Z_2) \subseteq (W \cap Y) \cup (W \cap Y) = W \cap Y$ and it follows that $W' \cap Y = W \cap Y$.

Next we will show that $f^*(V) \subseteq Z_1$. Let $g \in C^+(\beta X)$ such that $\mathrm{coz}(g) = V$ and for each $i \in \mathbb{N}$, let $V_i = g^{-1}((\frac{1}{i}, \infty))$. Since $f^*(V_i \cap X) = f(V_i \cap X) \subseteq f(V \cap X) \subseteq Z' \subseteq Z_1$, we have $V_i \cap X \subseteq f^{*-1}(Z_1)$. But $f^{*-1}(Z_1)$ is βX-closed, and so $\mathrm{cl}_{\beta X}(V_i \cap X) \subseteq f^{*-1}(Z_1)$ for each i. Hence $\bigcup_{i=1}^{\infty} f^*(\mathrm{cl}_{\beta X}(V_i \cap X)) \subseteq Z_1$. But $\bigcup_{i=1}^{\infty} \mathrm{cl}_{\beta X}(V_i \cap X) \subseteq V$. If $v \in V$, then $v \in V_i$ for some i and it follows that $v \in \mathrm{cl}_{\beta X}(V_i \cap X)$. Hence $\bigcup_{i=1}^{\infty} \mathrm{cl}_{\beta X}(V_i \cap X) = V$. Thus $f^*(V) = f^*(\bigcup_{i=1}^{\infty} \mathrm{cl}_{\beta X}(V_i \cap X)) = \bigcup_{n=1}^{\infty} f^*(\mathrm{cl}_{\beta X}(V_i \cap X)) \subseteq Z_1$. Since Z_1 and Z_2 are disjoint in βY, we have $f^*(V) \subseteq Z_1 \subseteq \beta Y - Z_2 \subseteq W'$.

Next we show $W' \subseteq \mathrm{cl}_{\beta Y}(f(U' \cap X))$. Let $w \in W'$ and A be a βY-neighborhood of w. Then $A \cap W' \cap Y \neq \emptyset$. But $A \cap W' \cap Y = A \cap f(U' \cap X)$. So $w \in \mathrm{cl}_{\beta Y}(f(U' \cap X))$. This shows $W' \subseteq \mathrm{cl}_{\beta Y}(f(U' \cap X))$ and since W' is open, $W' \subseteq \mathrm{int}_{\beta Y}(\mathrm{cl}_{\beta Y}(f(U' \cap X)))$. We now have $y \in f^*(V) \subseteq W' \subseteq \mathrm{int}_{\beta Y}(\mathrm{cl}_{\beta Y}(f(U' \cap X))) = \mathrm{int}_{\beta Y}(\mathrm{cl}_{\beta Y}(f^*(U' \cap X)))$. Now f^* is a closed function and so $\mathrm{cl}_{\beta Y}(f^*(U' \cap X)) \subseteq f^*(\mathrm{cl}_{\beta X}(U' \cap X)) \subseteq f^*(\mathrm{cl}_{\beta X}(U')) \subseteq f^*(U)$. Putting these inequalities together, we see $y \in f^*(V) \subseteq \mathrm{int}_{\beta Y}(\mathrm{cl}_{\beta Y}(f^*(U' \cap X))) \subseteq f^*(U)$. This implies $f^*(U)$ is open and f^* is an open function. □

Since for any continuous function $f : X \to Y$ mapping X onto Y, the Stone extension is a closed function, the previous theorem actually implies f^* is strongly open by Lemma 9.

As we have indicated, Isiwata shows the reverse implication in case f is a WZ-mapping. In particular, if $f : X \to Y$ is a continuous function mapping X onto Y such that f^* is open and $f^{*-1}(Y) = X$, then f is z-open.

5. Consequences

The image of an F-space under a continuous and open function is not necessarily an F-space as is shown in [L5]. There, an example is given of a normal F-space for which there is an open and continuous image that is not an F-space, not an SV space, not finitely an F-space, nor of finite rank. However, the image of a compact F-space under a continuous open function is a (compact) F-space. A similar situation holds for the other classes of spaces mentioned. In this section we use results from the previous section and the fact that continuous open functions defined on compact spaces preserve the F-space property and other related properties to show that the F-space property and other related properties are preserved under the image of a continuous z-open function. We begin by reviewing the definitions of various classes of spaces.

Definitions 18. The space X is

(1) an *F-space* if every cozeroset of X is C^*-embedded;
(2) *finitely an F-space* if βX is a union of finitely many closed F-spaces;
(3) an *SV-space* if $C(X)/P$ is a valuation domain for each prime ring ideal P of $C(X)$;
(4) of *finite rank* if there is an $n \in \mathbb{N}$ such that every maximal ideal of $C(X)$ contains at most n distinct minimal prime ideals. The smallest such n is called the *rank* of X;

and,

(5) if $x \in X$, the *rank of* x is n if the maximal ℓ-ideal $M_x = \{f \in C(X) : f(x) = 0\}$ contains exactly n distinct minimal prime ideals.

F-spaces have been studied extensively and are important because they are the spaces for which every finitely generated ideal is principal. See [GH], [D] and [H] for a sampling of articles on F-spaces. Spaces that are finitely an F-space were introduced in [HW1] and are studied further in [HLMW]. Spaces of finite rank have been studied in [HLMW], and [L2], and SV-rings have been studied in [HW1], [HW2], [HLMW], [L3], [L4], and [L5]. An F-space is finitely an F-space, and is also an SV space of rank 1. It is shown in [HW1] that every space that is finitely an F-space is an SV space and in [L3], an example is given that shows not every SV space is finitely an F-space. It is shown in [HLMW] that all SV spaces are of finite rank and it is an open question as to whether the converse holds.

The proof of our next theorem makes use of the following results.

(i) (14.25, [GJ]) X is an F-space if and only if βX is an F-space.
(ii) (2.3, [HW1]) X is an SV-space if and only if βX is an SV-space.
(iii) (2.7, [L5]) If $f : X \to Y$ is a continuous open function mapping X, a compact SV space of rank at most 2, onto Y, then Y is a compact SV space of rank at most 2.
(iv) (2.1, [L5]) Suppose $f : X \to Y$ is a continuous open function mapping the space X onto Y. If $x \in X$ has X-rank n, then $f(x)$ has Y-rank less than or equal to n.
(v) (3.2, [HLMW]) X has rank n if and only if βX has rank n.

Note that (iv) does not imply that the image under a continuous open function of a space of finite rank is necessarily of finite rank since not every maximal ideal of a non-compact space X has the form M_x for some $x \in X$.

Theorem 19. *Suppose $f : X \to Y$ is a continuous z-open function mapping X onto Y.*

(1) *If X is an F-space, then Y also is an F-space.*
(2) *If X is finitely an F-space, then Y also is finitely an F-space.*

(3) *If X is SV with rank at most 2, then Y also is SV with rank at most 2.*
(4) *If X has finite rank, then Y also has finite rank.*

PROOF: (1) Suppose X is an F-space. Then βX is an F-space and each $x \in \beta X$ has βX-rank 1 by Theorem 1. By Theorem 17, $f^* : \beta X \to \beta Y$ is an open mapping. By (iv) above, every $y \in \beta Y$ has βY-rank 1 and since βY is normal, βY must be an F-space. This implies that Y is an F-space.

(2) Suppose X is finitely an F-space. Then $\beta X = \bigcup_{i=1}^{n} A_i$ for some compact F-spaces $A_1, A_2, \ldots, A_n$. Then $\beta Y = f^*(\beta X) = f^*(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} f^*(A_i)$ and by (1), each $f^*(A_i)$ is a compact F-space. Hence Y is finitely an F-space.

(3) Suppose X is SV with rank at most 2. Then βX is SV with rank at most 2 by (ii) and (v). By Theorem 17, f^* is open. Then by (iii), βY is SV with rank at most 2 and hence Y is SV with rank at most 2.

The proof of (4) is similar to the proof of (3), using the fact that a compact space has finite rank if and only if every point of the space has finite rank. $\square$

A natural question to ask is if (3) of the previous theorem can be generalized to SV spaces of rank n, for $n > 2$. The answer is not known. It is also not known if (iii) above can be generalized to SV spaces of rank n, for $n > 2$.

Recall that the space X is a *quasi-F space* if every dense cozeroset of X is C^*-embedded. Quasi-F spaces have been studied in a number of articles including [DHH], and [HVW]. Every F-space is a quasi-F space and the converse does not hold. We can add the quasi-F property to our list of properties that are preserved under continuous z-open functions after first showing this property is preserved under continuous open functions defined on compact spaces.

Theorem 20. *Suppose $f : X \to Y$ is a continuous open function mapping the compact space X onto Y. If X is a quasi-F space, then Y is also a quasi-F space.*

PROOF: Suppose U is a dense cozeroset in Y and $\ell \in C^*(U)$. We want to show that ℓ can be extended continuously to Y. First, note $f^{-1}(U)$ is an X-cozeroset. If $\mathrm{cl}_X(f^{-1}(U)) \neq X$, then $f(X - \mathrm{cl}_X(f^{-1}(U)))$ is nonempty, open, and disjoint from U, which would imply that $\mathrm{cl}_Y(U) \cap f(X - \mathrm{cl}_X(f^{-1}(U))) = \emptyset$ and that U is not dense. So it must be that $\mathrm{cl}_X(f^{-1}(U)) = X$ and hence $f^{-1}(U)$ is a dense X-cozeroset. Let $h \in C^*(f^{-1}(U))$ be defined by $h = \ell \circ f|_{f^{-1}(U)}$. Since X is a quasi-F space and $f^{-1}(U)$ is a dense cozeroset, there is an $\hat{h} \in C(X)$ such that $\hat{h}|_{f^{-1}(U)} = h$.

Define a function $\hat{\ell} : Y \to \mathbb{R}$ by $\hat{\ell}(y) = \sup\{\hat{h}(x) : x \in f^{-1}(y)\}$. By Corollary 10, f is strongly open and by Theorem 7 and Lemma 5, $\hat{\ell}$ is continuous. Finally, we show that $\hat{\ell}|_U = \ell$. Let $u \in U$. For every $x \in f^{-1}(u)$, $\hat{h}(x) = h(x) = \ell \circ f|_{f^{-1}(U)}(x) = \ell(u)$. Hence $\hat{\ell}(u) = \sup\{\hat{h}(x) : x \in f^{-1}(u)\} = \ell(u)$. This shows that ℓ can be continuously extended to Y and hence Y is a quasi-F space. $\square$

Corollary 21. *Suppose $f : X \to Y$ is a continuous z-open function mapping X onto Y. If X is a quasi-F space, then Y also is a quasi-F space.*

Since a space X is a quasi-F space if and only if βX is a quasi-F space (5.1, [DHH]), the proof of this corollary is easy and similar to that of Theorem 19(3).

Finally, we note that there is a borderline of sorts near F-spaces. As we have seen, the F-space property and related, but weaker, properties generally are not preserved by open continuous functions, but are preserved by the stronger continuous z-open functions. On the other hand, properties stronger than the F-space property tend to be preserved by open functions. For example, consider P-spaces, extremely disconnected spaces, and basically disconnected spaces. A *P-space* is a space in which every G_δ set is open. *Extremely disconnected* spaces are those spaces in which every open set has an open closure and a *basically disconnected* space is a space in which every cozeroset has an open closure. Every P-space, extremely disconnected space, and basically disconnected space is an F-space, but not every F-space is one of these three types of spaces. It is straightforward to show that the image under a continuous open function of a P-space is a P-space. In [AP], it is shown that the image under a continuous open function of an extremally disconnected space is extremally disconnected. We conclude by looking at basically disconnected spaces.

Theorem 22. *Suppose $f : X \to Y$ is a continuous open function mapping X onto Y. If X is basically disconnected, then Y is also basically disconnected.*

Proof: Let U be a Y-cozeroset. Then $f^{-1}(U)$ is an X-cozeroset. Since X is basically disconnected, $\mathrm{cl}_X(f^{-1}(U))$ is open. Since f is open, $f(\mathrm{cl}_X(f^{-1}(U)))$ is open and the proof will be complete if we show $\mathrm{cl}_Y(U) = f(\mathrm{cl}_X(f^{-1}(U)))$. First, suppose $y \in \mathrm{cl}_Y(U)$. There is an $x \in X$ such that $f(x) = y$. Let V be an X-neighborhood of x. Then $f(V)$ is a Y-neighborhood of y, and so there is a $z \in f(V) \cap U$. But $z = f(v)$ for some $v \in V$. Then $v \in V \cap f^{-1}(U)$. In particular, this says $V \cap f^{-1}(U) \neq \emptyset$ and so $x \in \mathrm{cl}_X(f^{-1}(U))$. So $y = f(x) \in f(\mathrm{cl}_X(f^{-1}(U)))$. Hence $\mathrm{cl}_Y(U) \subseteq f(\mathrm{cl}_X(f^{-1}(U)))$.

Now let $y \in f(\mathrm{cl}_X(f^{-1}(U)))$. Then there is an $x \in \mathrm{cl}_X(f^{-1}(U))$ such that $f(x) = y$. Let V be a Y-neighborhood of y. Then $f^{-1}(V)$ is an X-neighborhood of x and so there is a $z \in f^{-1}(V) \cap f^{-1}(U)$. So $f(z) \in V \cap U$ and $y \in \mathrm{cl}_Y(U)$. Hence $f(\mathrm{cl}_X(f^{-1}(U))) \subseteq \mathrm{cl}_Y(U)$. □

References

[AP] Arkhangelskii A.V., Ponomarev V.I., *Fundamentals of General Topology*, Reidel Publishing, Boston, 1983.

[CH] Comfort W., Hager A., *The projection mapping and other continuous functions on a product space*, Math. Scand. **28** (1971), no. 3, 77–90.

[D] Dow A., *On F-spaces and F′-spaces*, Pacific J. Math. **108** (1983), no. 2, 275–284.

[DHH] Dashiell F., Hager A., Henriksen M., *Order-Cauchy completions of rings and vector lattices of continuous functions*, Canad. J. Math. **32** (1980), no. 3, 657–685.

[GH] Gillman L., Henriksen M., *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. **82** (1956), no. 2, 366–391.

[GJ] Gillman L., Jerison M., *Rings of Continuous Functions*, D. Van Nostrand Publishing, New York, 1960.

[H] Hindman N., *The product of F-spaces with P-spaces*, Pacific J. Math. **47** (1973), 473–480.

[HLMW] Henriksen M., Larson S., Martinez, J., Woods R.G., *Lattice-ordered algebras that are subdirect products of valuation domains*, Trans. Amer. Math. Soc. **345** (1994), 193–221.

[HVW] Henriksen M., Vermeer H., Woods R.G., *Quasi F-covers of Tychonoff spaces*, Trans. Amer. Math. Soc. **303** (1987), no. 2, 779–803.

[HW] Henriksen M., Woods R.G., *Cozero complemented spaces; When the space of minimal prime ideals of a $C(X)$ is compact*, Topology Appl. **141** (2004), 147–170.

[HW1] Henriksen M., Wilson R., *When is $C(X)/P$ a valuation ring for every prime ideal P?*, Topology Appl. **44** (1992), 175–180.

[HW2] Henriksen M., Wilson R., *Almost discrete SV-spaces*, Topology Appl. **46** (1992), 89–97.

[I] Isiwata T., *Mappings and spaces*, Pacific J. Math. **20** (1967), no. 3, 455–480.

[L1] Larson S., *Convexity conditions on f-rings*, Canadian J. Math. **38** (1986), 48–64.

[L2] Larson S., *f-Rings in which every maximal ideal contains finitely many minimal prime ideals*, Comm. Algebra **25** (1997), no. 12, 3859–3888.

[L3] Larson S., *Constructing rings of continuous functions in which there are many maximal ideals with nontrivial rank*, Comm. Algebra **31** (2003), no. 5, 2183–2206.

[L4] Larson S., *Rings of continuous functions on spaces of finite rank and the SV property*, Comm. Algebra, to appear.

[L5] Larson S., *Images and open subspaces of SV spaces*, Comm. Algebra, to appear.

[MW] Martinez J, Woodward S., *Bezout and Prüfer f-rings*, Comm. Algebra **20** (1992), 2975–2989.

[W] Weir M., *Hewitt-Nachbin Spaces*, North Holland, Amsterdam, 1975.

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