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## Directoids with an antitone involution

I. CHAJDA, M. KOLAŘÍK

*Abstract.* We investigate  $\sqcap$ -directoids which are bounded and equipped by a unary operation which is an antitone involution. Hence, a new operation  $\sqcup$  can be introduced via De Morgan laws. Basic properties of these algebras are established. On every such an algebra a ring-like structure can be derived whose axioms are similar to that of a generalized boolean quasiring. We introduce a concept of symmetrical difference and prove its basic properties. Finally, we study conditions of direct decomposability of directoids with an antitone involution.

*Keywords:* directoid, antitone involution, D-quasiring, symmetrical difference, direct decomposition

*Classification:* 06A12, 06A06, 06E20, 16Y99

### 1. Bounded directoids with an antitone involution

The concept of directoid was introduced by J. Ježek and R. Quackenbush [6] and independently by V.M. Kopytov and Z.I. Dimitrov [7] and B.J. Gardner and M.M. Parmenter [5]. Recall that a *directoid* is an algebra  $\mathcal{D} = (D; \sqcap)$  of type (2) satisfying the identities

$$(D1) \quad x \sqcap x = x;$$

$$(D2) \quad (x \sqcap y) \sqcap x = x \sqcap y;$$

$$(D3) \quad y \sqcap (x \sqcap y) = x \sqcap y;$$

$$(D4) \quad x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z.$$

Putting  $x \leq y$  if and only if  $x \sqcap y = x$ , the relation  $\leq$  is an order on  $D$ , the so-called *induced order* of directoid  $\mathcal{D}$ . It was shown in [6] that  $x \sqcap y$  is a common lower bound of  $x, y$ . Also conversely, if  $(D; \leq)$  is an ordered set where for each  $x, y \in D$  their lower bound set  $L(x, y) = \{d \in D; d \leq x \text{ and } d \leq y\}$  is non-void, one can pick up freely an element  $d \in L(x, y)$  with only one constrain: if  $x \leq y$  then  $d$  must be equal to  $x$ . Then, putting  $x \sqcap y = d$ , the algebra  $(D; \sqcap)$  is a directoid. We do not assume the commutativity  $x \sqcap y = y \sqcap x$  throughout the paper.

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**Lemma 1.** *A directoid  $\mathcal{D} = (D; \sqcap)$  is a semilattice if and only if it satisfies the condition*

$$(S) \quad (x \leq a \text{ and } x \leq b) \Rightarrow x \leq a \sqcap b.$$

PROOF: Of course, (S) is satisfied in every  $\wedge$ -semilattice. Conversely, let a directoid  $\mathcal{D} = (D; \sqcap)$  satisfy (S), let  $a, b \in D$  and  $x \in L(a, b)$ . Then, by (S),  $x \leq a \sqcap b$  and hence,  $a \sqcap b$  is the greatest lower bound of  $a, b$ , i.e.  $a \sqcap b = \inf(a, b)$ . Thus  $(D; \sqcap)$  is a  $\wedge$ -semilattice.  $\square$

In what follows, we will deal with directoids having a least element 0 and a greatest element 1. This fact will be expressed by the notation  $\mathcal{D} = (D; \sqcap, 0, 1)$ . By an *antitone involution* on  $\mathcal{D} = (D; \sqcap, 0, 1)$  is meant a mapping  $x \mapsto x'$  of  $D \rightarrow D$  such that  $x'' = x$  and  $x \leq y \Rightarrow y' \leq x'$  where  $\leq$  is the induced order of  $\mathcal{D}$ . If  $\mathcal{D} = (D; \sqcap, 0, 1)$  has an antitone involution, we will write  $\mathcal{D} = (D; \sqcap, ', 0, 1)$ . Of course,  $0' = 1$  and  $1' = 0$  is valid in every bounded directoid with an antitone involution. Due to [7], the operations  $\sqcup$  and  $\sqcap$  are connected by the absorption laws.

Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$ . The term operation  $\sqcup$  defined via  $x \sqcup y = (x' \sqcap y')'$  will be called an *assigned operation* of  $\mathcal{D}$ .

**Theorem 1.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution, let  $\sqcup$  be the assigned operation. Then:*

- (i)  $x \sqcap y = (x' \sqcup y')'$ ;
- (ii)  $x \sqcup x = x$ ,  
 $(x \sqcup y) \sqcup x = x \sqcup y$ ,  
 $y \sqcup (x \sqcup y) = x \sqcup y$ ,  
 $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$ ;
- (iii)  $x \sqcap (x \sqcup y) = x$ ,  $x \sqcup (x \sqcap y) = x$ ,  $x \sqcap (y \sqcup x) = x$ ,  $x \sqcup (y \sqcap x) = x$ ,  
 $(x \sqcup y) \sqcap x = x$ ,  $(x \sqcap y) \sqcup x = x$ ,  $(y \sqcup x) \sqcap x = x$ ,  $(y \sqcap x) \sqcup x = x$ .

PROOF: (i)  $(x' \sqcup y')' = (x'' \sqcap y'')'' = x \sqcap y$ .

(ii)  $x \sqcup x = (x' \sqcap x')' = x'' = x$ ,

$(x \sqcup y) \sqcup x = (x' \sqcap y')' \sqcup x = ((x' \sqcap y') \sqcap x')' = (x' \sqcap y')' = x \sqcup y$ ,

$y \sqcup (x \sqcup y) = y \sqcup (x' \sqcap y')' = (y' \sqcap (x' \sqcap y'))' = (x' \sqcap y')' = x \sqcup y$ ,

$x \sqcup ((x \sqcup y) \sqcup z) = (x' \sqcap ((x' \sqcap y') \sqcap z'))' = ((x' \sqcap y') \sqcap z')' = (x \sqcup y) \sqcup z$ .

(iii) The absorption laws were proved in [7]. For the reader's convenience, we present an easy proof as follows. By using (ii), we compute

$$x \sqcup (x \sqcup y) = x \sqcup ((x \sqcup y) \sqcup x) = (x \sqcup y) \sqcup x = x \sqcup y$$

thus  $x \leq x \sqcup y$  whence  $x \sqcap (x \sqcup y) = x$ . Similarly we can prove the remaining absorption laws.  $\square$

**Remark 1.** The identities  $x \sqcup y = (x' \sqcap y)'$  and  $x \sqcap y = (x' \sqcup y)'$  will be referred under the name De Morgan laws because they are formally the same as De Morgan laws in lattices.

Due to De Morgan laws,  $(D; \sqcup)$  is a directoid again for any  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  with the assigned operation  $\sqcup$ . Clearly  $x \leq y$  if and only if  $x \sqcup y = y$ .

**Example 1.** Consider the directed set whose diagram is drawn in Figure 1

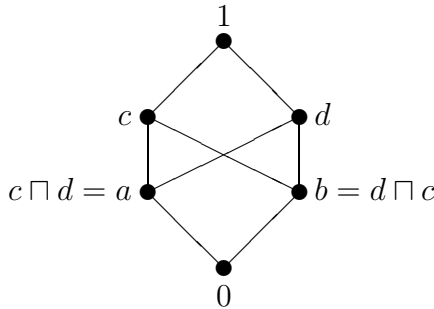


FIGURE 1

Let us pick up  $c \sqcap d = a$  and  $d \sqcap c = b$ . Then  $\mathcal{D} = (D; \sqcap, 0, 1)$  for  $D = \{0, a, b, c, d, 1\}$  is a bounded  $\sqcap$ -directoid. Further, define  $x \mapsto x'$  on  $D$  as follows

$$\frac{x \parallel \begin{array}{cccccc} 0 & a & b & c & d & 1 \end{array}}{x' \parallel \begin{array}{cccccc} 1 & d & c & b & a & 0 \end{array}} .$$

It is clearly an antitone involution on  $D$ . For the assigned operation  $\sqcup$  we have:

$$\begin{aligned} a \sqcup b &= (a' \sqcap b')' = (d \sqcap c)' = b' = c, \\ b \sqcup a &= (b' \sqcap a')' = (c \sqcap d)' = a' = d. \end{aligned}$$

◇

The following example gives an answer to the question whether is it possible to define an antitone involution on every  $\sqcap$ -directoid:

**Example 2.** Consider the  $\sqcap$ -directoid  $\mathcal{D} = (\{0, x, y, z, 1\}; \sqcap)$  depicted in Figure 2 where for binary operation  $\sqcap$  we have:  $x \sqcap y = 0, y \sqcap x = z$  (and trivially for comparable elements).

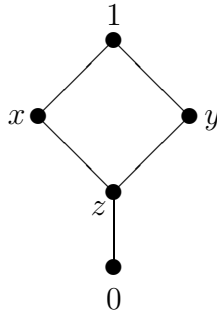


FIGURE 2

We show that on this  $\sqcap$ -directoid it is not possible to define an antitone involution  $'$ : Clearly,  $0' = 1$  and  $1' = 0$ . If we put  $x' = z$ , then  $y'$  must be equal to  $y$  but  $z \leq y$  implies  $y = y' \leq z' = x$ , a contradiction. If we pick  $x' = y$ , then  $z' = z$  and  $z \leq x$  implies  $y = x' \leq z' = z$ , a contradiction. Finally, if  $x' = x$  then for  $z' = z$  or  $z' = y$  we have  $x \leq z$  or  $x \leq y$  which is a contradiction again.

Note, that if a  $\sqcap$ -directoid is not commutative, it needs to have at least 2 non-comparable elements  $x, y$  such that  $|L(x, y)| \geq 2$ . Thus, the directoid from Figure 2 is the smallest one which cannot have an antitone involution and hence also the assigned operation  $\sqcup$ .  $\diamond$

It can be proved dually as in Lemma 1 that a  $\sqcup$ -directoid  $(D; \sqcup)$  is a  $\vee$ -semilattice if and only if it satisfies the condition

$$(S') \quad (a \leq x \text{ and } b \leq x) \Rightarrow a \sqcup b \leq x.$$

Lemma 1 enables us to show that when  $\sqcup$  and  $\sqcap$  are connected by a stronger identity such as modularity or distributivity then the resulting structure is a lattice. A similar result was already shown by J. Nieminen [9] for the so-called  $\chi$ -lattices.

**Theorem 2.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. If  $\mathcal{D}$  satisfies the modularity laws*

$$\begin{aligned} x \sqcup (y \sqcap (x \sqcup z)) &= (x \sqcup y) \sqcap (x \sqcup z), \\ x \sqcap (y \sqcup (x \sqcap z)) &= (x \sqcap y) \sqcup (x \sqcap z) \end{aligned}$$

then  $(D; \sqcup, \sqcap)$  is a lattice.

PROOF: Suppose  $x, y, a \in D, x, y \leq a$ . Then  $x = a \sqcap x, y = a \sqcap y$  and hence  $x \sqcup y = (a \sqcap x) \sqcup (a \sqcap y) = a \sqcap (x \sqcup (a \sqcap y)) = a \sqcap (x \sqcup y)$  thus  $x \sqcup y \leq a$ . In other words, it satisfies (S') and hence  $(D; \sqcup)$  is a  $\vee$ -semilattice. Dually it can be shown

that also  $(D; \sqcap)$  is a  $\wedge$ -semilattice. Due to Theorem 1,  $\sqcap$  and  $\sqcup$  are connected with the absorption laws, i.e.  $(D; \sqcup, \sqcap)$  is a lattice. □

Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  its assigned operation. If  $\sqcap$  is commutative, i.e.  $x \sqcap y = y \sqcap x$  then also  $\sqcup$  is commutative and  $(D; \sqcup, \sqcap)$  is the so-called  $\lambda$ -lattice as defined in [10]. Moreover, every  $\chi$ -lattice (defined in [9], [8]) is a particular case of  $\lambda$ -lattice. In our investigation we do not assume commutativity of  $\sqcap$  and hence our algebras are more general. Nevertheless, we are still able to prove a result which holds for lattices, i.e.:

**Theorem 3.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Then*

$$m(x, y, z) = ((x \sqcap y) \sqcup (z \sqcap y)) \sqcup (x \sqcap z)$$

is the majority term on  $\mathcal{D}$  and hence the congruence lattice  $\text{Con } \mathcal{D}$  is distributive.

PROOF:  $m(x, x, y) = ((x \sqcap x) \sqcup (y \sqcap x)) \sqcup (x \sqcap y) = (x \sqcup (y \sqcap x)) \sqcup (x \sqcap y) = x \sqcup (x \sqcap y) = x,$   
 $m(x, y, x) = ((x \sqcap y) \sqcup (x \sqcap y)) \sqcup (x \sqcap x) = (x \sqcap y) \sqcup x = x,$   
 $m(y, x, x) = ((y \sqcap x) \sqcup (x \sqcap x)) \sqcup (y \sqcap x) = ((y \sqcap x) \sqcup x) \sqcup (y \sqcap x) = x \sqcup (y \sqcap x) = x.$  □

## 2. Derived quasirings

The concept of a (boolean) quasiring was introduced firstly for orthomodular lattices and ortholattices and then for bounded lattices with an antitone involution in [4], [1], [2]. We are going to introduce similar ring-like structures for directoids with an antitone involution.

By a *D-quasiring* is meant an algebra  $\mathcal{R} = (R; +, \cdot, 0, 1)$  of type  $(2, 2, 0, 0)$  satisfying the identities

- (Q1)  $(x \cdot y) \cdot x = x \cdot y;$
- (Q2)  $y \cdot (x \cdot y) = x \cdot y;$
- (Q3)  $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z;$
- (Q4)  $x \cdot 0 = 0;$
- (Q5)  $x \cdot 1 = x;$
- (Q6)  $x + 0 = x;$
- (Q7)  $1 + (1 + x \cdot y) \cdot (1 + y) = y.$

**Remark 2.** Due to (Q3) with  $y = z = 1$  and (Q5), we obtain immediately that a *D-quasiring* satisfies the identity

(I)  $x \cdot x = x.$

Hence, for every  $D$ -quasiring  $\mathcal{R} = (R; +, \cdot, 0, 1)$ ,  $(R; \cdot, 0, 1)$  is a bounded directoid with 0 and 1, thus  $\mathcal{R}$  may be considered as a partially ordered set  $(R; \leq)$  with smallest element 0 and greatest element 1 where  $\leq$  is the induced order of  $(R; \cdot, 0, 1)$  i.e. for every  $x, y \in R$ , the order  $\leq$  is defined by  $x \leq y$  if and only if  $x \cdot y = x$ .

**Lemma 2.** *Let  $(R; +, \cdot, 0, 1)$  be a  $D$ -quasiring. Then  $x \mapsto 1 + x$  is an antitone involution on  $R$ .*

PROOF: Denote by  $x' = x + 1$ . If we put  $x = y$  in (Q7) and apply (I), we obtain the identity

$$(N) \quad 1 + (1 + x) = x$$

proving that  $x'' = x$ . Suppose  $x \leq y$ , i.e.  $x = x \cdot y$ . Then, from (Q7), we have

$$1 + (1 + x) \cdot (1 + y) = y,$$

whence

$$(1 + (1 + x) \cdot (1 + y))' = y',$$

i.e.

$$1 + (1 + (1 + x) \cdot (1 + y)) = 1 + y.$$

By (N) we obtain

$$(1 + x) \cdot (1 + y) = 1 + y$$

which yields  $(1 + y) \leq (1 + x)$ , i.e.  $y' \leq x'$ . Thus the operation  $'$  is an antitone involution on  $R$ . □

**Theorem 4.** *Let  $\mathcal{R} = (R; +, \cdot, 0, 1)$  be a  $D$ -quasiring. Define*

$$x \sqcap y = x \cdot y, \quad x' = 1 + x \quad \text{and} \quad x \sqcup y = 1 + (1 + x) \cdot (1 + y).$$

Then  $\mathcal{D}(R) = (R; \sqcap, \sqcup, 0, 1)$  is a bounded directoid with an antitone involution where  $\sqcup$  is the assigned operation.

PROOF: As mentioned in Remark 2,  $(R; \sqcap, 0, 1)$  is a bounded directoid. By Lemma 2,  $'$  is an antitone involution on  $R$ . Further, using (N), we compute

$$x' \sqcup y' = 1 + (1 + x') \cdot (1 + y') = 1 + x \cdot y = (x \sqcap y)'$$

and

$$x' \sqcap y' = (1 + x) \cdot (1 + y) = 1 + (1 + (1 + x) \cdot (1 + y)) = (x \sqcup y)',$$

thus  $\mathcal{D}(R)$  satisfies De Morgan laws and hence  $\sqcup$  is the assigned operation. □

**Theorem 5.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Define*

$$x + y = (x \sqcup y) \sqcap (x \sqcap y)' \quad \text{and} \quad x \cdot y = x \sqcap y.$$

*Then  $\mathcal{R}(\mathcal{D}) = (D; +, \cdot, 0, 1)$  is a  $D$ -quasiring. Moreover,  $\mathcal{R}(\mathcal{D})$  satisfies the following correspondence identity*

$$(Cor1) \quad (1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = x + y.$$

PROOF: Since  $(D; \sqcap, 0, 1)$  is a bounded  $\sqcap$ -directoid, the identities (Q1)–(Q5) hold. The identity (Q6) is evident. Evidently,  $1 + x = (1 \sqcup x) \sqcap (1 \sqcap x)' = 1 \sqcap x' = x'$ . For (Q7) we use the properties of an antitone involution to compute

$$1 + (1 + x \cdot y) \cdot (1 + y) = ((x \sqcap y)' \sqcap y')' = y'' = y.$$

Using the De Morgan laws we obtain

$$\begin{aligned} (1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) &= (x' \sqcap y')' \sqcap (x \sqcap y)' \\ &= (x \sqcup y) \sqcap (x \sqcap y)' = x + y \end{aligned}$$

which is just the identity (Cor1). □

**Theorem 6.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a bounded directoid with an antitone involution. Then  $\mathcal{D}(\mathcal{R}(\mathcal{D})) = \mathcal{D}$ .*

*Let  $\mathcal{R} = (R; +, \cdot, 0, 1)$  be a  $D$ -quasiring satisfying the correspondence identity (Cor1). Then  $\mathcal{R}(\mathcal{D}(\mathcal{R})) = \mathcal{R}$ .*

PROOF: Evidently, the operation meet coincides in both  $\mathcal{D}(\mathcal{R}(\mathcal{D}))$  and  $\mathcal{D}$ . Hence, it remains to prove  $\cup = \sqcup$  and  $x^* = x'$  where  $\cup$  is the binary operation and  $*$  the antitone involution of  $\mathcal{D}(\mathcal{R}(\mathcal{D}))$ . We have

$$x^* = 1 + x = (1 \sqcup x) \sqcap (1 \sqcap x)' = 1 \sqcap x' = x'$$

and

$$x \cup y = 1 + (1 + x) \cdot (1 + y) = (x' \sqcap y')' = x \sqcup y.$$

Analogously, the multiplicative operations coincide in the both  $\mathcal{R}(\mathcal{D}(\mathcal{R}))$  and  $\mathcal{R}$ . To prove  $\mathcal{R}(\mathcal{D}(\mathcal{R})) = \mathcal{R}$  we need only to show that also  $\oplus = +$  where  $\oplus$  is the additive operation in  $\mathcal{R}(\mathcal{D}(\mathcal{R}))$ . Applying (Cor1) we compute

$$x \oplus y = (x \sqcup y) \sqcap (x \sqcap y)' = (1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = x + y.$$

□



**Example 3.** Consider the  $\sqcap$  directoid  $\mathcal{D}$  with an antitone involution  $'$  and assigned operation  $\sqcup$  from Example 1 (see Figure 1).

The operation tables of the  $D$ -quasiring  $\mathcal{R}(D)$  corresponding to  $\mathcal{D}$  are as follows (see Theorem 5):

$\cdot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	a	c
d	0	a	b	b	d	d
1	0	a	b	c	d	1

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	a	d	d
b	b	d	b	c	b	c
c	c	a	c	b	d	b
d	d	d	b	c	a	a
1	1	d	c	b	a	0

Note that  $\cdot$  and  $+$  are not commutative. ◇

**Remark 3.** Let us consider the directoid  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  of Example 1. One can pick  $a \sqcup b = d$  and  $b \sqcup a = c$  (and trivially for comparable elements). The resulting structure  $(D; \sqcup)$  is clearly a  $\sqcup$ -directoid again but  $\sqcup$  is not the assigned operation of  $\mathcal{D}$ . Evidently, the De Morgan laws are not satisfied. On the contrary the structure  $\mathcal{L} = (D; \sqcup, \sqcap, ', 0, 1)$  still induces a  $D$ -quasiring  $\mathcal{R}(\mathcal{L})$  via  $x \cdot y = x \sqcap y$  and  $x + y = (x \sqcup y) \sqcap (x \sqcap y)'$ . However, (Cor1) is not satisfied and hence  $\mathcal{R} \neq \mathcal{R}(\mathcal{L}(R))$ .

### 3. Symmetrical difference

**Definition 1.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Let  $a, b \in D$ . The element  $a$  is called a *complement* of  $b$  if  $a \sqcap b = 0$  and  $a \sqcup b = 1$ .

**Remark 4.** If  $a$  is a complement of  $b$  then  $b$  need not be a complement of  $a$ ; see the following

**Example 4.** A bounded  $\sqcap$ -directoid with an antitone involution  $'$  is depicted in Figure 3 where  $c \sqcap d = a$ ,  $d \sqcap c = 0$  and  $0' = 1$ ,  $a' = d$ ,  $b' = c$ .

Then  $a$  is a complement of  $b$  but  $b$  is not a complement of  $a$  since

$$a \sqcup b = (a' \sqcap b')' = (d \sqcap c)' = 0' = 1,$$

but

$$b \sqcup a = (b' \sqcap a')' = (c \sqcap d)' = a' = d.$$

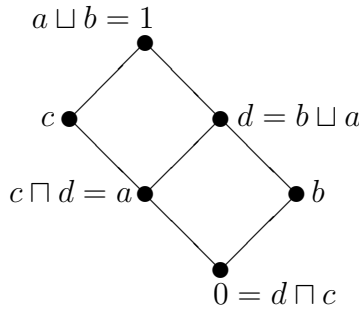


FIGURE 3

Analogously,  $d$  is a complement of  $c$  but not vice versa. On the other hand,  $b$  is a complement of  $c$  and  $c$  is a complement of  $b$ . Of course,  $0$  is a complement of  $1$  and  $1$  is a complement of  $0$ .  $\diamond$

**Lemma 3.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Let  $\mathcal{R}(\mathcal{D}) = (D; +, \cdot, 0, 1)$  be the induced  $D$ -quasiring. Then

- (a)  $a + b = 1$  if and only if  $a$  is a complement of  $b$ ;
- (b)  $a + b = a \sqcup b$  if and only if  $a \sqcup b \leq a' \sqcup b'$ ;
- (c) if  $a \leq b$  then  $a + b = b \sqcap a'$ .

PROOF: (a) Assume  $a + b = 1$ . Then  $(a \sqcup b) \sqcap (a \sqcap b)' = 1$ , i.e.  $a \sqcup b = 1$  and  $(a \sqcap b)' = 1$ , hence  $a \sqcap b = 0$  thus  $a$  is a complement of  $b$ . The converse is trivial.

(b) If  $a \sqcup b = a + b = (a \sqcup b) \sqcap (a \sqcap b)'$  then  $a \sqcup b \leq (a \sqcap b)' = a' \sqcup b'$ . The converse is evident.

(c) If  $a \leq b$  then  $a + b = (a \sqcup b) \sqcap (a \sqcap b)' = b \sqcap a'$ .  $\square$

**Definition 2.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. By a *symmetrical difference* of  $x, y$  is meant the term function

$$x \Delta y = (x' \sqcap y) \sqcup (x \sqcap y').$$

We can get a mutual relationship between the symmetrical difference and the operation  $+$  of the induced  $D$ -quasiring as follows:

**Lemma 4.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Then  $x \Delta y = (x + y)'$  and  $x + y = (x \Delta y)'$ .

PROOF: Using the De Morgan laws, we infer directly

$$(x \Delta y)' = ((x' \sqcap y) \sqcup (x \sqcap y))' = (x \sqcup y) \sqcap (x \sqcap y)' = x + y$$

and

$$\begin{aligned}(x + y)' &= ((x \sqcup y') \sqcap (x \sqcap y'))' = (x \sqcup y')' \sqcup (x \sqcap y') \\ &= (x' \sqcap y) \sqcup (x \sqcap y') = x \Delta y.\end{aligned}$$

□

**Lemma 5.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Then*

- (a)  $x \Delta y = 0$  if and only if  $x'$  is a complement of  $y$ ;
- (b)  $x \Delta x = 0$  if and only if  $x' \Delta x' = 0$  if and only if  $x'$  is a complement of  $x$ ;
- (c)  $1 \Delta x = x \Delta 1 = x'$ .

PROOF: (a) Assume  $x \Delta y = 0$ . Then  $(x' \sqcap y) \sqcup (x \sqcap y') = 0$  thus also  $x' \sqcap y = 0$  and  $x \sqcap y' = 0$ , whence  $x' \sqcup y = (x \sqcap y')' = 0' = 1$ , i.e.  $x'$  is a complement of  $y$ . Conversely, if  $x'$  is a complement of  $y$  then  $x' \sqcap y = 0$  and  $x' \sqcup y = 1$ , i.e.  $x \sqcap y' = (x' \sqcup y)' = 1' = 0$  and hence  $x \Delta y = 0$ .

(b) The first implication follows directly from the definition of symmetrical difference and (a) immediately yields the second.

- (c)  $1 \Delta x = (1' \sqcap x) \sqcup (1 \sqcap x') = x'$ ; analogously  $x \Delta 1 = x'$ . □

We are able to show that the symmetrical difference can also serve as an additive operation in a certain induced  $D$ -quasiring.

**Theorem 7.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Let  $\Delta$  be the symmetric difference. Then  $\mathcal{R}^*(D) = (D; \Delta, \sqcap, 0, 1)$  is a  $D$ -quasiring.*

PROOF: It is trivial to verify the axioms (Q1)–(Q5). For (Q6) we have

$$x \Delta 0 = (x' \sqcap 0) \sqcup (x \sqcap 0') = 0 \sqcup x = x.$$

It remains to prove (Q7). By Lemma 5 (c) we have

$$1 \Delta (1 \Delta (x \sqcap y)) \sqcap (1 \Delta y) = ((x \sqcap y)' \sqcap y')' = y'' = y.$$

□

**Lemma 6.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. The  $D$ -quasiring  $\mathcal{R}^*(D) = (D; \Delta, \cdot, 0, 1)$  with  $x \cdot y = x \sqcap y$  satisfies the identity*

$$\text{(Cor2)} \quad 1 \Delta (1 \Delta (1 \Delta x) \cdot y) \cdot (1 \Delta x \cdot (1 \Delta y)) = x \Delta y.$$

PROOF: By using Lemma 5 (c) and the De Morgan laws we compute

$$\begin{aligned} 1\Delta(1\Delta(1\Delta x) \cdot y) \cdot (1\Delta x \cdot (1\Delta y)) &= ((x' \sqcap y)') \sqcap (x \sqcap y')' \\ &= (x' \sqcap y) \sqcup (x \sqcap y') = x\Delta y. \end{aligned}$$

□

The following result is a counterpart of Theorem 6 and can be proved analogously:

**Theorem 8.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution,  $\sqcup$  the assigned operation and  $\Delta$  the symmetrical difference. Then  $\mathcal{D}(\mathcal{R}^*(D)) = \mathcal{D}$ . Let  $\mathcal{R} = (R; \Delta, \cdot, 0, 1)$  be a  $D$ -quasiring satisfying (Cor2). Then  $\mathcal{R}^*(\mathcal{D}(R)) = \mathcal{R}$ .*

#### 4. A decompositions of directoids

Define  $aCb$  if  $b = (b \sqcap a) \sqcup (b \sqcap a')$ . An element  $a \in D$  is called **central** if  $aCx$  and  $a'Cx$  for each  $x \in D$ . Denote by  $C(D)$  the set of all central elements of a directoid  $\mathcal{D} = (D; \sqcap, ', 0, 1)$ . Hence,

$$(C) \quad a \in C(D) \quad \text{iff} \quad x = (x \sqcap a) \sqcup (x \sqcap a') = (x \sqcap a') \sqcup (x \sqcap a)$$

for each  $x \in D$ .

**Lemma 7.** *Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Then*

- (a) if  $b \leq a$  then  $aCb$ ;
- (b)  $0, 1 \in C(D)$ ;
- (c) if  $a \in C(D)$  then  $a'$  is a complement of  $a$  and  $a$  is a complement of  $a'$ ;
- (d) if  $a \in C(D)$  then

$$(x \sqcup a') \sqcap (x \sqcup a) = x = (x \sqcup a) \sqcap (x \sqcup a')$$

for each  $x \in D$ .

PROOF: (a) If  $b \leq a$  then  $(b \sqcap a) \sqcup (b \sqcap a') = b \sqcup (b \sqcap a') = b$ .

(b) Of course,  $x = (x \sqcap 1) \sqcup (x \sqcap 0) = (x \sqcap 0) \sqcup (x \sqcap 1)$  for each  $x \in D$ .

(c) Take  $x = 1$  in (C). Then

$$1 = (1 \sqcap a) \sqcup (1 \sqcap a') = a \sqcup a'$$

and

$$1 = (1 \sqcap a') \sqcup (1 \sqcap a) = a' \sqcup a.$$

Due to De Morgan laws, we have that  $a'$  is a complement of  $a$  and vice versa.

(d) We compute

$$(x \sqcup a') \sqcap (x \sqcup a) = (x' \sqcap a)' \sqcap (x' \sqcap a')' = ((x' \sqcap a) \sqcup (x' \sqcap a'))' = x'' = x.$$

The second equation can be shown analogously.

□

**Definition 3.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Denote by  $\text{Is}(D)$  the set of all elements  $a \in D$  such that

- (i)  $(x \sqcap y) \sqcap a = (x \sqcap a) \sqcap (y \sqcap a), (x \sqcap y) \sqcap a' = (x \sqcap a') \sqcap (y \sqcap a')$ ;
- (ii)  $(x \sqcup y) \sqcap a = (x \sqcap a) \sqcup (y \sqcap a), (x \sqcup y) \sqcap a' = (x \sqcap a') \sqcup (y \sqcap a')$ .

It is clear that  $0, 1 \in \text{Is}(D)$  in any case.

**Remark 5.** It is immediate that  $a \in \text{Is}(D)$  if and only if  $a' \in \text{Is}(D)$  and  $a \in C(D)$  if and only if  $a' \in C(D)$ .

**Lemma 8.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Then

- (a) if  $a \in \text{Is}(D)$  then  $x \leq y \Rightarrow x \sqcap a \leq y \sqcap a$ ;
- (b) if  $a \in C(D) \cap \text{Is}(D)$  then

$$(x \sqcap a)' \sqcap a = x' \sqcap a \quad \text{and} \quad (x \sqcap a')' \sqcap a' = x' \sqcap a'.$$

**PROOF:** (a) If  $x \leq y$  then  $x \sqcap y = x$  and, by (i) of Definition 3,  $x \sqcap a = (x \sqcap y) \sqcap a = (x \sqcap a) \sqcap (y \sqcap a)$  thus  $x \sqcap a \leq y \sqcap a$ .

(b) Of course,  $(x \sqcap a)' \sqcap a = (x' \sqcup a') \sqcap a$ . By (ii) of Definition 3, we have  $(x' \sqcup a') \sqcap a = (x' \sqcap a) \sqcup (a' \sqcap a)$  and, due to Lemma 7(c),  $a' \sqcap a = 0$ . Hence  $(x \sqcap a)' \sqcap a = x' \sqcap a$ . The second equality is established similarly.  $\square$

**Theorem 9.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be a directoid with an antitone involution and  $\sqcup$  the assigned operation. Let  $a \in C(D) \cap \text{Is}(D)$ . Define

$$x^* = x' \sqcap a \quad \text{and} \quad x^+ = x' \sqcap a'.$$

Then  $\mathcal{D}_1 = ((a]; \sqcap, *, 0, a)$  and  $\mathcal{D}_2 = ((a']; \sqcap, +, 0, a')$  are bounded directoids with an antitone involution and  $\mathcal{D}$  is isomorphic to  $\mathcal{D}_1 \times \mathcal{D}_2$  where the isomorphism is defined by  $\varphi(x) = (x \sqcap a, x \sqcap a')$ .

Conversely, let  $\mathcal{D}$  be isomorphic with  $\mathcal{D}_1 \times \mathcal{D}_2$  where  $\mathcal{D}_1, \mathcal{D}_2$  are directoids with an antitone involution. Then there exists  $a \in C(D) \cap \text{Is}(D)$  such that  $\mathcal{D}_1 \cong ((a], \sqcap, *, 0, a)$  and  $\mathcal{D}_2 \cong ((a'], \sqcap, +, 0, a')$ .

**PROOF:** Evidently, if  $x, y \in (a]$  then  $x \sqcap y \leq x \leq a$  thus also  $x \sqcap y \in (a]$ , i.e.  $((a]; \sqcap)$  is a directoid as well as  $((a']; \sqcap)$ .

Let  $x \in (a]$ . Then  $x \leq a$ , i.e.  $x \sqcup a = a$  and, by Lemma 7(d),

$$x^{**} = (x' \sqcap a)' \sqcap a = (x \sqcup a') \sqcap (x \sqcup a) = x.$$

Thus  $\mathcal{D}_1 = ((a]; \sqcap, *, 0, a)$  is a bounded directoid with the involution  $*$ . Since  $x \leq y$  implies  $y' \leq x'$  and  $a \in \text{Is}(D)$ , also

$$y^* = y' \sqcap a \leq x' \sqcap a = x^*$$

by (a) of Lemma 8, thus this involution is antitone. Similarly it can be shown for  $\mathcal{D}_2 = ((a'); \sqcap, ^+, 0, a')$ .

Now, define  $\varphi : D \rightarrow D_1 \times D_2$  by  $\varphi(x) = (x \sqcap a, x \sqcap a')$ . Moreover, define  $\psi : D_1 \times D_2 \rightarrow D$  by  $\psi((x, y)) = x \sqcup y$ . Since  $a \in C(D)$ , we infer

$$\psi(\varphi(x)) = (x \sqcap a) \sqcup (x \sqcap a') = x,$$

i.e.,  $\varphi$  is an injective mapping. Suppose  $(x, y) \in D_1 \times D_2$ . Then  $x \leq a, y \leq a'$  and by (ii) of Definition 3, we have

$$\begin{aligned} \varphi(\psi((x, y))) &= \varphi(x \sqcup y) = ((x \sqcup y) \sqcap a, (x \sqcup y) \sqcap a') \\ &= ((x \sqcap a) \sqcup (y \sqcap a), (x \sqcap a') \sqcup (y \sqcap a')) = (x \sqcup (y \sqcap a), (x \sqcap a') \sqcup y). \end{aligned}$$

Since  $a, a' \in \text{Is}(D)$ ,  $y \leq a'$  we obtain (according to (a) of Lemma 8) that

$$y \sqcap a \leq a' \sqcap a = 0$$

and therefore  $y \sqcap a = 0$ . Analogously,  $x \sqcap a' = 0$ . Hence,  $\varphi(\psi((x, y))) = (x \sqcup 0, 0 \sqcup y) = (x, y)$ . Thus,  $\varphi$  is a bijection and  $\psi = \varphi^{-1}$ .

It remains to prove that  $\varphi$  is a homomorphism. Clearly,

$$\begin{aligned} \varphi(b) \sqcap \varphi(c) &= (b \sqcap a, b \sqcap a') \sqcap (c \sqcap a, c \sqcap a') \\ &= ((b \sqcap a) \sqcap (c \sqcap a), (b \sqcap a') \sqcap (c \sqcap a')) = ((b \sqcap c) \sqcap a, (b \sqcap c) \sqcap a') = \varphi(b \sqcap c) \end{aligned}$$

according to (i) of Definition 3. Further, using of Lemma 8(b), we obtain

$$\begin{aligned} \varphi(b)' &= (b \sqcap a, b \sqcap a')' = ((b \sqcap a)^*, (b \sqcap a')^+) \\ &= ((b \sqcap a)' \sqcap a, (b \sqcap a')' \sqcap a') = (b' \sqcap a, b' \sqcap a') = \varphi(b'). \end{aligned}$$

Hence,  $\varphi$  is an isomorphism of  $\mathcal{D}$  onto  $\mathcal{D}_1 \times \mathcal{D}_2$ .

Conversely, let  $\mathcal{D}_1 = (D; \sqcap, *, 0_1, 1_1)$  and  $\mathcal{D}_2 = (D; \sqcap, ^+, 0_2, 1_2)$  be directoids with antitone involutions and  $\mathcal{D}$  is isomorphic to  $\mathcal{D}_1 \times \mathcal{D}_2$ . It is an easy exercise to verify that elements  $a = (1_1, 0_2)$  and  $(0_1, 1_2)$  belong to  $C(D_1 \times D_2) \cap \text{Is}(D_1 \times D_2)$  and  $(0_1, 1_2) = a'$  in  $\mathcal{D}_1 \times \mathcal{D}_2$ . Of course,  $\mathcal{D}_1 \cong \overline{\mathcal{D}_1} = ((a); \sqcap, ^*, (0_1, 0_2), a)$  and  $\mathcal{D}_2 \cong \overline{\mathcal{D}_2} = ((a'); \sqcap, ^+, (0_1, 0_2), a')$  and hence also  $\mathcal{D} \cong \overline{\mathcal{D}_1} \times \overline{\mathcal{D}_2}$ .  $\square$

**Remark 6.** If  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  is a semilattice with an antitone involution then every element satisfies (i) of Definition 3 and (a) of Lemma 8.

**Example 5.** Let  $\mathcal{D} = (D; \sqcap, ', 0, 1)$  be the  $\sqcap$ -directoid with an antitone involution as shown in Example 4 (see Figure 3). Let  $\sqcup$  be its assigned operation. Then  $b \notin C(D)$  and  $c \notin C(D)$ , because

$$d \neq (d \sqcap b) \sqcup (d \sqcap b') = b \sqcup 0 = b$$

and

$$d \neq (d \sqcap c) \sqcup (d \sqcap c') = 0 \sqcup b = b.$$

Due to Lemma 7(c) also  $a \notin C(D)$ ,  $d \notin C(D)$ . Further, elements  $c$  and  $d$  do not belongs to  $\text{Is}(D)$ , since

$$a = a \sqcap c = (a \sqcap d) \sqcap c \neq (a \sqcap c) \sqcap (d \sqcap c) = a \sqcap 0 = 0$$

and

$$d = 1 \sqcap d = (a \sqcup b) \sqcap d \neq (a \sqcap d) \sqcup (b \sqcap d) = a \sqcup b = 1.$$

Hence also  $b = c' \notin \text{Is}(D)$  and  $a = d' \notin \text{Is}(D)$ . Thus  $C(D) = \text{Is}(D) = \{0, 1\}$ .

On the contrary, let Figure 3 be now the Hasse diagram of the lattice  $\mathcal{L} = (L; \wedge, \vee)$  with a two binary operations join and meet. Then  $\mathcal{L}$  is as a direct product of the two-element and three-element chains.

For the non-trivial decomposition of directoid let us see the following

**Example 6.** Consider the  $\sqcap$ -directoid  $\mathcal{D} = (D; \sqcap)$  whose diagram is drawn in Figure 4 where  $m \sqcap n = k$ ,  $n \sqcap m = l$ ,  $s \sqcap t = q$ ,  $t \sqcap s = r$  and trivially for the other couples.

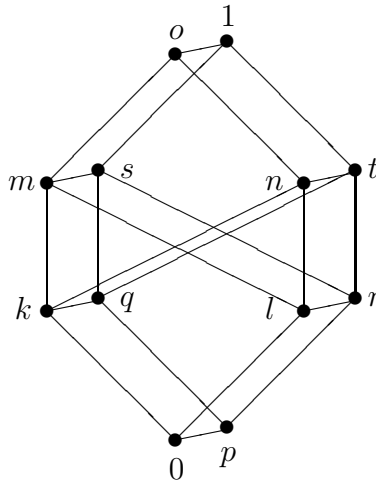


FIGURE 4

Define an antitone involution  $x \mapsto x'$  on  $D$  as follows

$$\begin{array}{c|cccccc} x & 0 & k & l & p & q & r \\ \hline x' & 1 & t & s & o & n & m \end{array}.$$

One can easily check that  $a = p$ ,  $a' = o \in C(D) \cap \text{Is}(D)$ . Therefore,  $\mathcal{D} \cong \mathcal{D}_1 \times \mathcal{D}_2$  for  $\mathcal{D}_1 = ((a], \sqcap, *, 0, a)$  and  $\mathcal{D}_2 = ((a'], \sqcap, +, 0, a')$ .

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