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Positive solutions for systems of generalized three-point nonlinear boundary value problems

J. Henderson, S.K. Ntouyas, I.K. Purnaras

Abstract. Values of λ are determined for which there exist positive solutions of the system of three-point boundary value problems, 
\[ \begin{align*}
  u''(t) + \lambda a(t)f(v(t)) &= 0, \quad 0 < t < 1, \\
  v''(t) + \lambda b(t)g(u(t)) &= 0, \quad 0 < t < 1,
\end{align*} \]
for \( 0 < t < 1 \), and satisfying, 
\[ \begin{align*}
  u(0) &= \beta u(\eta), & u(1) &= \alpha u(\eta), \\
  v(0) &= \beta v(\eta), & v(1) &= \alpha v(\eta).
\end{align*} \]
A Guo-Krasnosel’skii fixed point theorem is applied.

Keywords: generalized three-point boundary value problem, system of differential equations, eigenvalue problem

Classification: 34B18, 34A34

1. Introduction

We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions for the system of three-point boundary value problems,

\( \begin{align*}
  u''(t) + \lambda a(t)f(v(t)) &= 0, \quad 0 < t < 1, \\
  v''(t) + \lambda b(t)g(u(t)) &= 0, \quad 0 < t < 1,
\end{align*} \)

where \( 0 < \eta < 1, \ 0 < \alpha < 1/\eta, \ 0 < \beta < \frac{1-\alpha\eta}{1-\eta} \) and

(A) \( f, g \in C([0, \infty), [0, \infty)) \),

(B) \( a, b \in C([0, 1], [0, \infty)) \), and each does not vanish identically on any subinterval,

(C) all of 
\[ \begin{align*}
  f_0 := \lim_{x \to 0^+} \frac{f(x)}{x}, & \quad g_0 := \lim_{x \to 0^+} \frac{g(x)}{x}, \\
  f_\infty := \lim_{x \to \infty} \frac{f(x)}{x} & \quad \text{and} \quad g_\infty := \lim_{x \to \infty} \frac{g(x)}{x},
\end{align*} \]
exist as positive real numbers.
For several years now, there has been a great deal of activity in studying positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from both a theoretical sense [3], [5], [8], [11], [18] and as applications for which only positive solutions are meaningful [1], [4], [12], [13]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [9], [10], [15], [17], [19]. The existence of positive solutions for three-point boundary value problems has been studied extensively in recent years. For some appropriate references we refer the reader to [15], [16]. Recently in [14], the existence of positive solutions was studied for the following generalized second order three-point boundary value problem

\[ y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < T, \]
\[ y(0) = \beta y(\eta), \quad y(T) = \alpha y(\eta). \]

When \( \beta = 0 \), the conditions (4) reduce to the usual three-point boundary conditions

\[ y(0) = 0, \quad y(T) = \alpha y(\eta). \]

Recently Benchohra et al. [2] and Henderson and Ntouyas [6] studied the existence of positive solutions for systems of nonlinear eigenvalue problems. Also Henderson and Ntouyas [7] studied the existence of positive solutions for systems of nonlinear eigenvalue problems for three-point boundary conditions of the form (5) with \( T = 1 \). Here we extend these results to eigenvalue problems for the systems of generalized three-point boundary value problems (1), (2). The main tool in this paper is an application of the Guo-Krasnosel’skii fixed point theorem for operators leaving a Banach space cone invariant [5]. A Green’s function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Some preliminaries

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel’skii fixed point theorem.

**Lemma 2.1** ([14]). Let \( \beta \neq \frac{1-\alpha \eta}{1-\eta} \); then for any \( y \in C[0,1] \), the boundary value problem

\[ u''(t) + y(t) = 0, \quad 0 < t < 1 \]
\[ u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta), \]

has the unique solution

\[ u(t) = \int_0^1 k(t,s)y(s) \, ds. \]
where \( k(t, s) : [0, 1] \times [0, 1] \to \mathbb{R}^+ \) is defined by

\[
(8) \quad k(t, s) = \begin{cases} 
\frac{[(1-\beta)t+\beta \eta](1-s)}{1-\alpha \eta - \beta(1-\eta)} 
\quad + \frac{[(\beta-\alpha)t-\beta](s-\eta)}{1-\alpha \eta - \beta(1-\eta)} - (t-s), & 0 \leq s \leq t \leq 1 \text{ and } s \leq \eta, \\
\frac{[(1-\beta)t+\beta \eta](1-s)}{1-\alpha \eta - \beta(1-\eta)} 
\quad + \frac{[(\beta-\alpha)t-\beta](s-\eta)}{1-\alpha \eta - \beta(1-\eta)} - (t-s), & 0 \leq t \leq s \leq 1 \text{ and } s \geq \eta, \\
\frac{[(1-\beta)t+\beta \eta](1-s)}{1-\alpha \eta - \beta(1-\eta)} 
\quad + \frac{[(\beta-\alpha)t-\beta](s-\eta)}{1-\alpha \eta - \beta(1-\eta)} - (t-s), & \eta \leq s \leq t \leq 1.
\end{cases}
\]

Notice that by Lemma 2.1 it follows that

\[
(9) \quad u(t) = \frac{(1-\beta)t+\beta \eta}{1-\alpha \eta - \beta(1-\eta)} \int_0^1 (1-s)y(s) \, ds 
\quad + \frac{(\beta-\alpha)t-\beta}{1-\alpha \eta - \beta(1-\eta)} \int_0^{\eta} (\eta-s)y(s) \, ds - \int_0^t (t-s)y(s) \, ds.
\]

If \( y \geq 0 \) and \( 0 < \beta < \frac{1-\alpha \eta}{1-\eta} \), from (9) we have that

\[
(10) \quad u(t) \leq \frac{(1-\beta)t+\beta \eta}{1-\alpha \eta - \beta(1-\eta)} \int_0^1 (1-s)y(s) \, ds,
\]

and

\[
(11) \quad u(\eta) \geq \frac{\eta}{1-\alpha \eta - \beta(1-\eta)} \int_\eta^1 (1-s)y(s) \, ds.
\]

**Lemma 2.2** ([14]). Let \( 0 < \alpha < 1/\eta \), \( 0 < \beta < \frac{1-\alpha \eta}{1-\eta} \) and assume that (A) and (B) hold. Then, the unique solution of (1)–(2) satisfies

\[
\inf_{t \in [0, 1]} u(t) \geq \gamma \|u\|,
\]

where \( \gamma = \min \left\{ \alpha \eta, \frac{\alpha(1-\eta)}{1-\alpha \eta}, \beta \eta, \beta(1-\eta) \right\} \).

We note that a pair \((u(t), v(t))\) is a solution of the eigenvalue problem (1), (2) if, and only if,

\[
u(t) = \lambda \int_0^1 k(t, s)a(s)f \left( \lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \right) \, ds, \quad 0 \leq t \leq 1,
\]

and

\[
v(t) = \lambda \int_0^1 k(t, s)b(s)g(u(s)) \, ds, \quad 0 \leq t \leq 1.
\]
Values of \( \lambda \) for which there are positive solutions (positive with respect to a cone) of (1), (2) will be determined via applications of the following fixed point theorem, which is now commonly called the Guo-Krasnosel’skii fixed point theorem.

**Theorem 1.** Let \( \mathcal{B} \) be a Banach space, and let \( \mathcal{P} \subset \mathcal{B} \) be a cone in \( \mathcal{B} \). Assume that \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( \mathcal{B} \) with \( 0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \), and let

\[
T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}
\]

be a completely continuous operator such that, either

(i) \( \|Tu\| \leq \|u\|, \ u \in \mathcal{P} \cap \partial \Omega_1 \), and \( \|Tu\| \geq \|u\|, \ u \in \mathcal{P} \cap \partial \Omega_2 \), or

(ii) \( \|Tu\| \geq \|u\|, \ u \in \mathcal{P} \cap \partial \Omega_1 \), and \( \|Tu\| \leq \|u\|, \ u \in \mathcal{P} \cap \partial \Omega_2 \).

Then \( T \) has a fixed point in \( \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

3. Positive solutions in a cone

In this section, we apply Theorem 1 to obtain positive solution pairs of (1), (2). For our construction, let \( \mathcal{B} = C[0,1] \) be equipped with the usual supremum norm, \( \| \cdot \| \), and define a cone \( \mathcal{P} \subset \mathcal{B} \) by

\[
\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [0,1], \text{ and } \min_{t \in [\eta, 1]} x(t) \geq \gamma \|x\| \right\}.
\]

For our first result, we define the positive numbers \( L_1 \) and \( L_2 \) by

\[
L_1 := \max \left\{ \frac{\gamma \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - r)a(r)f_\infty \, dr \right\}^{-1},
\]

\[
\left[ \frac{\gamma \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - r)b(r)g_\infty \, dr \right]^{-1},
\]

and

\[
L_2 := \min \left\{ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)a(r)f_0 \, dr \right\}^{-1},
\]

\[
\left[ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)b(r)g_0 \, dr \right]^{-1} \right\}.
\]

**Theorem 2.** Assume that conditions (A), (B) and (C) hold. Then, for each \( \lambda \) satisfying

\[
L_1 < \lambda < L_2,
\]

there exists a pair \( (u, v) \) satisfying (1), (2) such that \( u(x) > 0 \) and \( v(x) > 0 \) on \( (0,1) \).
Proof: Let \( \lambda \) be as in (12), and let \( \epsilon > 0 \) be chosen such that
\[
\max \left\{ \left[ \frac{\gamma \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - r)(f_{\infty} - \epsilon) \, dr \right]^{-1}, \right.
\left. \frac{\gamma \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - r)(g_{\infty} - \epsilon) \, dr \right\} \leq \lambda
\]
and
\[
\lambda \leq \min \left\{ \left[ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)(f_{0} + \epsilon) \, dr \right]^{-1}, \right. \left. \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)(g_{0} + \epsilon) \, dr \right\}.
\]

Define an integral operator \( T : \mathcal{P} \to \mathcal{B} \) by
\[
(13) \quad Tu(t) := \lambda \int_{0}^{1} k(t, s)a(s)f \left( \lambda \int_{0}^{1} k(s, r)b(r)g(u(r)) \, dr \right) \, ds, \quad u \in \mathcal{P}.
\]

We seek suitable fixed points of \( T \) in the cone \( \mathcal{P} \). By Lemma 2.2, \( T \mathcal{P} \subset \mathcal{P} \). In addition, standard arguments show that \( T \) is completely continuous. Now, from the definitions of \( f_{0} \) and \( g_{0} \), there exists an \( H_{1} > 0 \) such that
\[
f(x) \leq (f_{0} + \epsilon)x \quad \text{and} \quad g(x) \leq (g_{0} + \epsilon)x, \quad 0 < x \leq H_{1}.
\]

Let \( u \in \mathcal{P} \) with \( \|u\| = H_{1} \). First, from (10) and the choice of \( \epsilon \), we have
\[
\lambda \int_{0}^{1} k(s, r)b(r)g(u(r)) \, dr \leq \lambda \frac{(1 - \beta) + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)b(r)g(u(r)) \, dr
\]
\[
\leq \lambda \frac{(1 - \beta) + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)b(r)(g_{0} + \epsilon)u(r) \, dr
\]
\[
\leq \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{0}^{1} (1 - r)b(r) \, dr(g_{0} + \epsilon)\|u\|
\leq \|u\| = H_{1}.
\]
As a consequence, in view of (10), and the choice of $\epsilon$, we obtain
\[
Tu(t) = \lambda \int_0^1 k(t, s)a(s)f \left( \lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \right) \, ds \\
\leq \lambda \frac{(1 - \beta)t + \beta\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)f \left( \lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \right) \, ds \\
\leq \lambda \frac{(1 - \beta)t + \beta\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)(f_0 + \epsilon)\lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \, ds \\
\leq \lambda \frac{1 - \beta + \beta\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)(f_0 + \epsilon)H_1 \, ds \\
\leq \lambda \left[ 1 - \beta + \beta\eta \right] \int_0^1 (1 - s)a(s)(f_0 + \epsilon)H_1 \, ds \\
\leq \lambda \left[ 1 - \beta + \beta\eta \right] \int_0^1 (1 - s)a(s)(g_\infty - \epsilon)u(r) \, dr \\
\geq \lambda \left[ 1 - \beta + \beta\eta \right] \int_0^1 (1 - s)a(s)(g_\infty - \epsilon)u(r) \, dr \gamma \| u \| \\
\geq \| u \| \\
= \| u \|.
\]

So, $\| Tu \| \leq \| u \|$ for every $u \in \mathcal{P}$ with $\| u \| = H_1$. Hence if we set
\[\Omega_1 = \{ x \in \mathcal{B} \mid \| x \| < H_1 \},\]
then
\[\| Tu \| \leq \| u \|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial \Omega_1.\]
\[\text{(14)}\]

Next, by the definitions of $f_\infty$ and $g_\infty$, there exists an $\overline{H}_2 > 0$ such that
\[f(x) \geq (f_\infty - \epsilon)x \quad \text{and} \quad g(x) \geq (g_\infty - \epsilon)x, \quad x \geq \overline{H}_2.\]

Let
\[H_2 = \max \left\{ 2H_1, \frac{\overline{H}_2}{\gamma} \right\}.\]

Then, for $u \in \mathcal{P}$ and $\| u \| = H_2$,
\[\min_{t \in [\eta, 1]} u(t) \geq \gamma \| u \| \geq \overline{H}_2.\]

Consequently, from (11) and the choice of $\epsilon$, we find
\[
\lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \geq \lambda \frac{\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_\eta^1 (1 - r)b(r)g(u(r)) \, dr \\
\geq \lambda \frac{\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_\eta^1 (1 - r)b(r)g(u(r)) \, dr \\
\geq \lambda \frac{\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_\eta^1 (1 - r)b(r)(g_\infty - \epsilon)u(r) \, dr \\
\geq \lambda \frac{\eta}{1 - \alpha\eta - \beta(1 - \eta)} \int_\eta^1 (1 - r)b(r)(g_\infty - \epsilon) \, dr \gamma \| u \| \\
\geq \| u \| \\
= H_2.
\]
And so, we have from (11) and the choice of $\epsilon$,

$$Tu(\eta) \geq \lambda \frac{\eta}{1 - \alpha \eta - \beta (1 - \eta)} \left( \lambda \int_{\eta}^{1} (1 - s) a(s) f \left( \lambda \int_{0}^{1} k(s, r) b(r) g(u(r)) \, dr \right) \, ds \right)$$

$$\geq \lambda \frac{\eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - s) a(s) f_\infty \lambda \int_{0}^{1} k(s, r) b(r) g(u(r)) \, dr \, ds$$

$$\geq \lambda \frac{\eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - s) a(s) (f_\infty - \epsilon) H_2 \, ds$$

$$\geq \lambda \frac{\eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_{\eta}^{1} (1 - s) a(s) (f_\infty - \epsilon) H_2 \, ds$$

$$\geq H_2 \geq \|u\|.$$}

Hence, $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{ x \in \mathcal{B} \mid \|x\| < H_2 \},$$

then

(15) $$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial \Omega_2.$$]

In view of (14) and (15), applying Theorem 1 we obtain that $T$ has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$. As such, and with $v$ defined by

$$v(t) = \lambda \int_{0}^{1} k(t, s) b(s) g(u(s)) \, ds,$$

the pair $(u, v)$ is a desired solution of (1), (2) for the given $\lambda$. The proof is complete. $\square$

Prior to our next result, we define positive numbers $L_3$ and $L_4$ by

$$L_3 := \max \left\{ \left[ \frac{\gamma \eta}{1 - \alpha \eta - \beta (1 - \eta)} \right] \int_{\eta}^{1} (1 - r) a(r) f_0 \, dr \right\}^{-1},$$

and

$$L_4 := \min \left\{ \left[ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \right] \int_{0}^{1} (1 - r) a(r) f_\infty \, dr \right\}^{-1},$$

$$\left[ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \right] \int_{0}^{1} (1 - r) b(r) g_\infty \, dr \right\}^{-1}.$$
**Theorem 3.** Assume that conditions (A)–(C) hold. Then, for each \( \lambda \) satisfying (16) \( L_3 < \lambda < L_4 \), there exists a pair \((u, v)\) satisfying (1), (2) such that \( u(x) > 0 \) and \( v(x) > 0 \) on \((0, 1)\).

**Proof:** Let \( \lambda \) be as in (16) and \( \epsilon > 0 \) be chosen such that

\[
\max \left\{ \left[ \frac{\gamma \eta}{1 - \alpha \eta - \beta(1 - \eta)} \right] \int_{\eta}^{1} (1 - r) a(r) (f_0 - \epsilon) \, dr \right\}^{-1}, \quad \left[ \frac{\gamma \eta}{1 - \alpha \eta - \beta(1 - \eta)} \right] \int_{\eta}^{1} (1 - r) b(r) (g_0 - \epsilon) \, dr \right\}^{-1} \leq \lambda
\]

and

\[
\lambda \leq \min \left\{ \left[ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta(1 - \eta)} \right] \int_{0}^{1} (1 - r) a(r) (f_\infty + \epsilon) \, dr \right\}^{-1}, \quad \left[ \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta(1 - \eta)} \right] \int_{0}^{1} (1 - r) b(r) (g_\infty + \epsilon) \, dr \right\}^{-1} \right\}.
\]

Let \( T \) be the cone preserving, completely continuous operator defined by (13). By the definitions of \( f_0 \) and \( g_0 \), there exists an \( \overline{H}_3 > 0 \) such that

\[
f(x) \geq (f_0 - \epsilon) x \quad \text{and} \quad g(x) \geq (g_0 - \epsilon) x, \quad 0 < x \leq \overline{H}_3.
\]

Also, from the definition of \( g_0 \) it follows that \( g(0) = 0 \) and so there exists \( 0 < H_3 < \overline{H}_3 \) such that

\[
\lambda g(x) \leq \frac{\overline{H}_3}{1 - \beta + \beta \eta} \int_{0}^{1} (1 - r) b(r) \, dr, \quad 0 \leq x \leq H_3.
\]

Let \( u \in \mathcal{P} \) with \( \|u\| = H_3 \). Then

\[
\lambda \int_{0}^{1} k(s, r) b(r) g(u(r)) \, dr \leq \lambda \int_{0}^{1} (1 - r) b(r) g(u(r)) \, dr \leq \lambda \int_{0}^{1} (1 - r) b(r) g(u(r)) \, dr \leq \frac{\overline{H}_3}{1 - \beta + \beta \eta} \int_{0}^{1} (1 - s) b(s) \, ds \leq \overline{H}_3.
\]
Then, by (11)

\[ Tu(\eta) \geq \lambda \frac{\eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s) \times \]

\[ \times f \left( \lambda \frac{\eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - r)b(r)g(u(r)) dr \right) ds \]

\[ \geq \lambda \frac{\eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s) \times \]

\[ \times (f_0 - \epsilon) \lambda \frac{\eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - r)b(r)g(u(r)) dr ds \]

\[ \geq \lambda \frac{\eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s) \times \]

\[ \times (f_0 - \epsilon) \lambda \frac{\gamma \eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - r)b(r)(g_0 - \epsilon)\|u\| dr ds \]

\[ \geq \lambda \frac{\eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)(f_0 - \epsilon)\|u\| ds \]

\[ \geq \lambda \frac{\gamma \eta}{1 - \alpha \eta - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)(f_0 - \epsilon)\|u\| ds \]

\[ \geq \|u\|. \]

So, \( \|Tu\| \geq \|u\| \). If we put

\[ \Omega_1 = \{ x \in \mathcal{B} \mid \|x\| < H_3 \}, \]

then

(17) \[ \|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial \Omega_3. \]

Next, by the definitions of \( f_\infty \) and \( g_\infty \), there exists an \( \overline{H}_4 \) such that

\[ f(x) \leq (f_\infty + \epsilon)x \text{ and } g(x) \leq (g_\infty + \epsilon)x, \quad x \geq \overline{H}_4. \]

Clearly, since \( g_\infty \) is assumed to be a positive real number, it follows that \( g \) is unbounded at \( \infty \), and so, there exists an \( \widetilde{H}_4 > \max\{2H_3, \overline{H}_4\} \) such that \( g(x) \leq g(\widetilde{H}_4) \), for \( 0 < x \leq \widetilde{H}_4 \).

Set

\[ f^*(t) = \sup_{0 \leq s \leq t} f(s), \quad g^*(t) = \sup_{0 \leq s \leq t} g(s), \quad \text{for } t \geq 0. \]
Clearly $f^*$ and $g^*$ are nondecreasing real valued function for which it holds

\[
\lim_{x \to \infty} \frac{f^*(x)}{x} = f_\infty, \quad \lim_{x \to \infty} \frac{g^*(x)}{x} = g_\infty.
\]

Hence, there exists an $H_4$ such that $f^*(x) \leq f^*(H_4)$, $g^*(x) \leq g^*(H_4)$ for $0 < x \leq H_4$. For $u \in \mathcal{P}$ with $\|u\| = H_4$, we have

\[
Tu(t) = \lambda \int_0^1 k(t, s)a(s)f \left( \lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \right) \, ds \\
\leq \lambda \int_0^1 k(t, s)a(s)f^* \left( \lambda \int_0^1 k(s, r)b(r)g(u(r)) \, dr \right) \, ds \\
\leq \lambda \int_0^1 k(t, s)a(s)f^* \left( \lambda \int_0^1 k(s, r)b(r)g^*(u(r)) \, dr \right) \, ds \\
\leq \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_0^1 (1 - s)a(s) \times \\
\quad \times f^* \left( \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_0^1 (1 - r)b(r)g^*(H_4) \, dr \right) \, ds \\
\leq \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_0^1 (1 - s)a(s) \times \\
\quad \times f^* \left( \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_0^1 (1 - r)b(r)(g_\infty + \epsilon H_4) \, dr \right) \, ds \\
\leq \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_0^1 (1 - s)a(s) f^*(H_4) \, ds \\
\leq \lambda \frac{1 - \beta + \beta \eta}{1 - \alpha \eta - \beta (1 - \eta)} \int_0^1 (1 - s)a(s) ds (f_\infty + \epsilon) H_4 \\
\leq H_4 \\
= \|u\|,
\]

and so $\|Tu\| \leq \|u\|$. For this case, if we set

\[
\Omega_2 = \{ x \in \mathcal{B} \mid \|x\| < H_4 \},
\]

then

\[
(18) \quad \|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial \Omega_4.
\]
Application of part (ii) of Theorem 1 yields a fixed point $u$ of $T$ belonging to $\mathcal{P} \cap (\Omega_4 \setminus \Omega_3)$, which in turn yields a pair $(u, v)$ satisfying (1), (2) for the chosen value of $\lambda$. The proof is complete.

4. Examples

In this section we give some examples illustrating our results. For the sake of simplicity we take $a(t) = b(t)$ and $f(t) = g(t)$.

Example 1. Consider the three-point boundary value problem

$$u''(t) + \frac{1}{10} \lambda t \frac{kve^{2v}}{c + e^v + e^{2v}} = 0, \quad 0 < t < 1,$$
$$v''(t) + \frac{1}{10} \lambda t \frac{ku^{2u}}{c + e^u + e^{2u}} = 0, \quad 0 < t < 1,$$
$$u(0) = \frac{1}{4} u \left( \frac{1}{3} \right), \quad u(1) = 2u \left( \frac{1}{3} \right),$$
$$v(0) = \frac{1}{4} v \left( \frac{1}{3} \right), \quad v(1) = 2v \left( \frac{1}{3} \right).$$

Here: $a(t) = b(t) = \frac{1}{10} t$, $k = 500$, $c = 1000$, $\alpha = 2$, $\beta = \frac{1}{4}$, $\eta = \frac{1}{3}$, $f(v) = \frac{kve^{2v}}{c + e^v + e^{2v}}$, $f(u) = \frac{ku^{2u}}{c + e^u + e^{2u}}$. By simple calculations we find: $\gamma = \frac{1}{12}$, $f_0 = g_0 = \frac{k}{c + 2} = 500 \div 1002$, $f_\infty = g_\infty = k = 500$, $L_1 = \frac{486}{500} \approx 0.972$, $L_2 = \frac{12024}{500} = 24.048$. By Theorem 2 it follows that for every $\lambda$ such that $0.972 < \lambda < 24.048$ the three-point boundary value problem has at least one positive solution.

Example 2. Consider the system of three-point boundary value problem

$$u''(t) + \lambda tv \left( 1 + \frac{c}{1 + v^2} \right) = 0, \quad 0 < t < 1,$$
$$v''(t) + \lambda tu \left( 1 + \frac{c}{1 + u^2} \right) = 0, \quad 0 < t < 1,$$
$$u(0) = \frac{1}{2} u \left( \frac{1}{4} \right), \quad u(1) = 2u \left( \frac{1}{4} \right),$$
$$v(0) = \frac{1}{2} v \left( \frac{1}{4} \right), \quad v(1) = 2v \left( \frac{1}{4} \right).$$

Here: $a(t) = b(t) = t$, $c = 100$, $\alpha = 2$, $\beta = \frac{1}{2}$, $\eta = \frac{1}{3}$, $f(v) = v \left( 1 + \frac{c}{1 + v^2} \right)$, $f(u) = u \left( 1 + \frac{c}{1 + u^2} \right)$. We find: $\gamma = \frac{1}{8}$, $f_0 = g_0 = 1 + c$, $f_\infty = g_\infty = 1$, $L_3 = \frac{768}{27277} \approx 0.28$, $L_4 = \frac{6}{5} = 1.2$. Therefore Theorem 3 holds for every $\lambda$ such that $0.28 < \lambda < 1.2$. 

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References


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