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# Spectrum of twisted Dirac operators on the complex projective space $\mathbb{P}^{2q+1}(\mathbb{C})$

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*Abstract.* In this paper, we explicitly determine the spectrum of Dirac operators acting on smooth sections of twisted spinor bundles over the complex projective space $\mathbb{P}^{2q+1}(\mathbb{C})$ for $q \geq 1$.

*Keywords:* complex projective space, Dirac operator, spectral theory

*Classification:* 53C35, 53C27, 58C40

## 1. Introduction

For spin Riemannian symmetric spaces, the spectrum of the Dirac operator is explicitly known only in few cases (see, e.g., [6], [7], [9], [10]).

Consider the complex projective space $\mathbb{P}^n(\mathbb{C}) = SU(n+1)/S(U(n)\times U(1))$ with $n = 2q+1$, equipped with the metric induced by the negative of the Killing form of $SU(n+1)$. As a compact simply connected Riemannian symmetric space, the manifold $M = \mathbb{P}^n(\mathbb{C})$ admits a unique homogeneous spin structure (see, e.g., [1]). Let $L = \{([z], w) \in M \times \mathbb{C}^{n+1};\ w \in [z]\}$ be the tautological bundle over $M$. Recall that each complex line bundle over $M$ is (up to isomorphism) of the form $L^m := L^{\otimes m}$ for some $m \in \mathbb{Z}$. Let $S \longrightarrow M$ be the spinor bundle, and let $\nabla^S$ be the spinor connection. We fix $m \in \mathbb{Z}$ and consider the vector bundle $S \otimes L^m$ endowed with the connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^{L^m}$, where $\nabla^{L^m}$ denotes the Levi-Civita connection on $L^m$. Let $\Gamma^\infty(S \otimes L^m)$ be the space of smooth sections of the bundle $S \otimes L^m$. The twisted Dirac operator $D_m$ acting on $\Gamma^\infty(S \otimes L^m)$ is defined by $D_m = \widetilde{\mu} \circ \nabla$, where $\widetilde{\mu} : T^*M \otimes S \otimes L^m \longrightarrow S \otimes L^m$ is the bundle homomorphism induced by the Clifford multiplication. Being an elliptic operator, $D_m$ has discrete (real) eigenvalues with finite multiplicities. The aim of this paper is to establish the following result:

**Theorem 1.** *On the complex projective space of dimension $n = 2q+1$ $(q \geq 1)$, the spectrum of the twisted Dirac operator $D_m$ acting on smooth sections of the spinor bundle tensored with $L^m$ $(m \in \mathbb{Z})$ is the union of the following sets:*

(1) $\Big\{ \pm \sqrt{a_{k,m}(0,0)};\ k \geq \max\big\{0, -\frac{n+1}{2} - m\big\}\Big\}$;

(2) $\Big\{ \pm \sqrt{a_{k,m}(\varepsilon,l)};\ \varepsilon \in \{0,1\},\ 1 \leq l \leq n-1,\ k \geq \max\big\{\varepsilon,\ l - m - \frac{n-1}{2}\big\}\Big\}$;

(3) $\Big\{ \pm \sqrt{a_{k,m}(1,n)};\ k \geq \max\big\{0,\, \frac{n+1}{2} - m\big\}\Big\}$,

*where*

$$a_{k,m}(\varepsilon, l) = \frac{1}{2(n+1)}\big\{(2k+2m-2\varepsilon+n+1)(k+n-l)\big\}$$

$$\textit{for}\ \ \varepsilon \in \{0,1\}\ \ \textit{and}\ \ 0 \leq l \leq n.$$

For $m = 0$, the above theorem gives the spectrum of the classical Dirac operator on $\mathbb{P}^{2q+1}(\mathbb{C})$ which has already been computed by Seifarth and Semmelmann [9]. Observing that each $\mathrm{Spin}^{\mathbb{C}}$-bundle over $\mathbb{P}^{2q+1}(\mathbb{C})$ is (up to isomorphism) of the form $S \otimes L^m$, $m \in \mathbb{Z}$, with $S$ and $L$ being as above (see [4] for generalities on $\mathrm{Spin}^{\mathbb{C}}$ Riemannian manifolds), one obtains from Theorem 1 the spectrum of each $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator on $\mathbb{P}^{2q+1}(\mathbb{C})$, $q \geq 1$.

## 2. Preliminaries

### 2.1 Twisted Dirac operator on a Riemannian symmetric space

Let $M = G/K$ be a compact, simply connected $n$-dimensional irreducible Riemannian symmetric space with $G$ compact and simply connected. We assume that $G$ and $K$ have the same rank and that $M$ has a spin structure (which is necessarily unique). Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie algebras of $G$ and $K$. We denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $B$ of $\mathfrak{g}$. We identify canonically $\mathfrak{m}$ with the tangent space to $M$ at the point $eK$, $e$ being the neutral element of $G$. The fixed Riemannian metric on $M$ is the metric induced by the restriction to $\mathfrak{m}$ of the scalar product $\langle\cdot,\cdot\rangle := -B$ on $\mathfrak{g}$.

The isotropy representation $\mathrm{Ad} : K \longrightarrow SO(\mathfrak{m})$ lifts to a homomorphism $\widetilde{\mathrm{Ad}} : K \longrightarrow \mathrm{Spin}(\mathfrak{m})$ satisfying $\widetilde{\mathrm{Ad}} \circ \lambda = \mathrm{Ad}$, where $\lambda : \mathrm{Spin}(\mathfrak{m}) \longrightarrow SO(\mathfrak{m})$ is the usual 2-fold covering. Let $\kappa : \mathrm{Spin}(\mathfrak{m}) \longrightarrow \mathrm{Aut}(\Delta)$ be the spin representation. The spinor bundle $S$ over $M$ is naturally identified with the homogeneous vector bundle $G \times_{K,\chi} \Delta$, where $\chi := \kappa \circ \widetilde{\mathrm{Ad}}$.

Let $\tau$ be an irreducible unitary representation of $K$ on a finite dimensional complex vector space $V$. Let us set $E = G \times_{K,\tau} V$, the associated homogeneous vector bundle over $M$, and denote by $\nabla^E$ the Levi-Civita connection on $E$. We consider the spinor bundle twisted by $E$, i.e. the vector bundle $S \otimes E$, endowed with the connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^E$, where $\nabla^S$ denotes the spinor connection. The Clifford multiplication $\mu : \mathrm{Cliff}(\mathfrak{m}) \longrightarrow \mathrm{End}(\Delta)$ induces a bundle homomorphism $\widetilde{\mu} : T^*M \otimes S \otimes E \longrightarrow S \otimes E$. Let $\Gamma^\infty(S \otimes E)$ be the space of smooth sections of $S \otimes E$. The twisted Dirac operator acting on $\Gamma^\infty(S \otimes E)$ is defined by $D_E = \widetilde{\mu} \circ \nabla$.

The main goal of this section is to describe the spectrum of $D_E$. To this end, we next recall a remarkable formula of its square $D_E^2$ which is essentially due to Parthasarathy [8].

### 2.2 A formula for the square of $D_E$

Let us introduce the following vector space:

$$\big(C^\infty(G)\otimes\Delta\otimes V\big)^K := \big\{\varphi\in C^\infty(G)\otimes\Delta\otimes V;\ (r\otimes\chi\otimes\tau)(k)\varphi=\varphi,\quad \forall\, k\in K\big\},$$

where $r$ denotes the right regular representation of $G$ on $C^\infty(G)$, the space of complex valued smooth functions on $G$. Obviously, $(C^\infty(G)\otimes\Delta\otimes V)^K$ is isomorphic to $\Gamma^\infty(S\otimes E)$. In what follows, we will identify these two spaces in the canonical fashion.

Fix an orthonormal basis $\{X_1,\ldots,X_d,Y_1,\ldots,Y_{d'}\}$ of $\mathfrak{g}$ with respect to the scalar product $\langle\cdot,\cdot\rangle$ in such a way that $\{X_1,\ldots,X_d\}$ forms a basis of $\mathfrak{m}$ and $\{Y_1,\ldots,Y_{d'}\}$ a basis of $\mathfrak{k}$. The Casimir operators of $G$ and $K$ (relative to $\langle\cdot,\cdot\rangle$) are respectively:

$$\Omega_G=-\sum_{i=1}^{d}X_i^2-\sum_{p=1}^{d'}Y_p^2\quad\text{and}\quad\Omega_K=-\sum_{p=1}^{d'}Y_p^2.$$

Denote by $\nu$ the representation of $\mathfrak{g}$ on $C^\infty(G)$ given by

$$\nu(X)f=Xf\quad (X\in\mathfrak{g}: f\in C^\infty(G)).$$

We have ([8, Proposition 1.1])

$$D_E=\sum_{i=1}^{d}\nu(X_i)\otimes\mu(X_i)\otimes 1.$$

Thus the square of $D_E$ satisfies

$$\begin{aligned}D_E^2&=\Big(\sum_i \nu(X_i)\otimes\mu(X_i)\otimes 1\Big)^2\\&=-\sum_i\nu(X_i)^2\otimes 1\otimes 1+\frac{1}{2}\sum_{i,j}\nu([X_i,\,X_j])\otimes\mu(X_i)\mu(X_j)\otimes 1.\end{aligned}$$

Using the facts that $[X_i,\,X_j]=\sum_p\langle[X_i,\,X_j],\,Y_p\rangle Y_p$ for $1\le i,j\le d$, and that the action of $Y_p\in\mathfrak{k}$ on $\Delta$ is given by $\chi(Y_p)=\frac{1}{4}\sum_{i,j}\langle[Y_p,\,X_i],\,X_j\rangle\,\mu(X_i)\mu(X_j)$, we obtain

$$D_E^2=-\sum_i\nu(X_i)^2\otimes 1\otimes 1+2\sum_p\nu(Y_p)\otimes\chi(Y_p)\otimes 1.$$

By a direct computation, one has

$$\sum_p (\nu\otimes\chi)(Y_p)^2 \otimes 1 - \sum_p \nu(Y_p)^2 \otimes 1\otimes 1 - \sum_p 1\otimes\chi(Y_p)^2 \otimes 1 = 2\sum_p \nu(Y_p)\otimes\chi(Y_p)\otimes 1.$$

Since $(\nu\otimes\chi)(\Omega_K)\otimes 1 = 1\otimes 1\otimes\tau(\Omega_K)$, it follows that

$$D_E^2 = \nu(\Omega_G)\otimes 1\otimes 1 + 1\otimes\chi(\Omega_K)\otimes 1 - 1\otimes 1\otimes\tau(\Omega_K).$$

Let $T$ be a common maximal torus of $G$ and $K$, and let $\mathfrak{h}$ be its Lie algebra. The Killing form of $\mathfrak{g}$ induces a natural scalar product on the dual $(\mathfrak{h}_{\mathbb{R}})^* := (i\mathfrak{h})^*$ which we also denote by $\langle\cdot,\cdot\rangle$. Let $R_G$ be the root system of $G$ with respect to $T$. Let $R_G^+$ be the system of positive roots of $G$, $R_K^+$ be the system of positive roots of $K$, with respect to a fixed lexicographic ordering in $R_G$. Let $\delta_G$ (resp. $\delta_K$) be the half-sum of the positive roots of $G$ (resp. $K$).

Note that $\chi(\Omega_K) = \frac{R}{8}\,\mathrm{Id}$ with $R$ the (constant) scalar curvature of the Riemannian space $M$ (see [4, Chapter 3] for details). Furthermore, if $\eta$ is the highest weight of the representation $\tau$, then we have $\tau(\Omega_K) = \langle\eta+2\delta_K,\eta\rangle\,\mathrm{Id}$ (see, e.g., [11, Lemma 5.6.4]). Consequently, we get

$$D_E^2 = \nu(\Omega_G)\otimes 1\otimes 1 + \left(\frac{R}{8} - \langle\eta+2\delta_K,\eta\rangle\right).1.$$

For simplicity, we write

$$D_E^2 = \Omega_G + \frac{R}{8} - \langle\eta+2\delta_K,\eta\rangle.$$

### 2.3 The spectrum of $D_E$

Let us fix a $K$-invariant Hermitian scalar product $(\cdot,\cdot)$ on the space $\Delta\otimes V$. Let $L^2(G,\Delta\otimes V)^K$ denote the completion of the vector space

$$C^\infty(G,\Delta\otimes V)^K := \big\{\psi\in C^\infty(G,\Delta\otimes V);\ \psi(gk) = (\chi\otimes\tau)(k^{-1})\psi(g),\ \forall\, g\in G,\ k\in K\big\}$$

with respect to the scalar product $(\cdot,\cdot)_{L^2}$ defined by

$$(\psi,\psi')_{L^2} = \int_G (\psi(g),\psi'(g))\,dg,$$

where $dg$ is a Haar measure on $G$.

Let $\widehat{G}$ be the unitary dual of $G$. For $\gamma \in \widehat{G}$, let $(\pi_\gamma,\, V_\gamma)$ be a fixed representative of the equivalence class $\gamma$. Applying the Peter-Weyl theorem, we obtain

$$L^2(G,\, \Delta \otimes V)^K \cong \widehat{\bigoplus_{\gamma \in \widehat{G}}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma,\, \Delta \otimes V),$$

where $\mathrm{Hom}_K(V_\gamma,\, \Delta \otimes V)$ is the vector space of $K$-equivariant linear homomorphisms from $V_\gamma$ to $\Delta \otimes V$. Let $\Delta \otimes V = \bigoplus_{\delta \in \Lambda} m_\delta\, W_\delta$ be the irreducible decomposition of the $K$-module $\Delta \otimes V$ into irreducible $K$-submodules, where $\Lambda \subset \widehat{K}$ and $m_\delta \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ for each $\delta \in \Lambda$. Then

$$\begin{aligned} L^2(G,\, \Delta \otimes V)^K &\cong \widehat{\bigoplus_{\gamma \in \widehat{G}}} \bigoplus_{\delta \in \Lambda} m_\delta\, (V_\gamma \otimes \mathrm{Hom}_K(V_\gamma,\, W_\delta)) \\ &\cong \widehat{\bigoplus_{\gamma \in \widehat{G}}} \bigoplus_{\delta \in \Lambda} m_\delta\, m_{\gamma|_K}(\delta)\, V_\gamma, \end{aligned}$$

where $m_{\gamma|_K}(\delta) := \dim_{\mathbb{C}} \mathrm{Hom}_K(V_\gamma,\, W_\delta)$.

The Casimir operator $\Omega_G$ acts on the $G$-module $V_\gamma$ as a scalar multiple of the identity:

$$\Omega_G\big|_{V_\gamma} = \langle \lambda_\gamma + 2\delta_G,\, \lambda_\gamma \rangle\ \mathrm{Id},$$

where $\lambda_\gamma$ is the highest weight of the representation $\pi_\gamma$. Using the identifications $\Gamma^\infty(S \otimes E) = (C^\infty(G) \otimes \Delta \otimes V)^K = C^\infty(G,\, \Delta \otimes V)^K$, one can deduce that the spectrum of the operator $D_E^2$ is given by

$$\begin{aligned} \mathrm{Spec}(D_E^2,\, M) = \Big\{ \langle \lambda_\gamma + 2\delta_G,\, \lambda_\gamma \rangle - \langle \eta + 2\delta_K,\, \eta \rangle + \frac{R}{8}\,; \\ \gamma \in \widehat{G},\, \delta \in \Lambda,\, m_{\gamma|_K}(\delta) \neq 0 \Big\}. \end{aligned}$$

Since $M = G/K$ is a Riemannian symmetric space, the spectrum of the twisted Dirac operator $D_E$ is symmetric with respect to the origin (see, e.g., [3]). Hence it is completely determined by the spectrum of $D_E^2$.

## 3. Spectrum of twisted Dirac operators on $\mathbb{P}^{2q+1}(\mathbb{C})$

Throughout this section, let $G = SU(n+1)$ and $K = S(U(n) \times U(1))$ with $n \geq 2$. The Riemannian symmetric space $G/K = \mathbb{P}^n(\mathbb{C})$ admits a spin structure (which is, of course, unique) if and only if $n$ is odd. In the sequel, we assume that $n = 2q+1$ with $q \geq 1$. Let $S$ be the spinor bundle associated to the homogeneous spin structure of $M = G/K$. Since $M$ is a spin Kähler manifold, one has ([4]) the following isomorphism:

$$S \cong S_0 \oplus \cdots \oplus S_n,$$

where $S_n$ is a complex line bundle satisfying $S_n^2 = K_M := \bigwedge^{n,0}(T^*M)^{\mathbb{C}}$, and $S_{n-l} = \bigwedge^{0,l}(T^*M)^{\mathbb{C}} \otimes S_n$ for $l = 0, \ldots, n$.

Fix the maximal torus

$$T = \{A = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\sum_{j=1}^{n}\theta_j}); \theta_j \in \mathbb{R} \ \text{ for } \ j = 1, \ldots, n\}$$

of $G$ and $K$, and denote by $\mathfrak{h}$ its Lie algebra. For $j \in \{1, \ldots, n+1\}$, we define the linear functional

$$e_j : \mathfrak{h}^{\mathbb{C}} \longrightarrow \mathbb{C}, \quad H = \mathrm{diag}(h_1, \ldots, h_{n+1}) \longmapsto h_j.$$

Considering the root spaces decomposition of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G$ under the action of $T$, we can choose the following systems of positive roots

$$\begin{aligned} &R_G^+ = \{e_i - e_j;\ 1 \leq i < j \leq n\} \cup \{e_1 + \cdots + 2\,e_i + \cdots + e_n;\ 1 \leq i \leq n\} \ \text{ and} \\ &R_K^+ = \{e_i - e_j;\ 1 \leq i < j \leq n\}, \end{aligned}$$

respectively of $G$ and $K$ relative to $T$. Then

$$\delta_G = \sum_{j=1}^{n}(n+1-j)e_j \quad \text{and} \quad \delta_K = \frac{1}{2}\sum_{j=1}^{n}(n-2j+1)e_j.$$

Recall that finite-dimensional irreducible representations of $K$ are classified by their highest weights which are of the form $\eta = \eta_1 e_1 + \cdots + \eta_n e_n$ with $\eta_j \in \mathbb{Z}$ for all $1 \leq j \leq n$, and $\eta_1 \geq \cdots \geq \eta_n$. Observe that $(S_n)_{eK}$ is an irreducible (one-dimensional!) $K$-module with highest weight

$$\xi = \tfrac{n+1}{2}\,(e_1 + \cdots + e_n).$$

It follows that $(S_{n-l})_{eK}$ is also an irreducible $K$-module whose highest weight $\vartheta_l$ is easily calculated from the observation that $(TM)_{eK} \cong (\mathbb{C}^n)^* \otimes \mathbb{C}$ as a $U(n) \otimes U(1)$-module:

$$\vartheta_l = \begin{cases} \tfrac{n+1}{2}\,(e_1 + \cdots + e_n) & \text{if } \ l = 0, \\ (\tfrac{n+1}{2} - l)(e_1 + \cdots + e_{n-l}) & \\ +(\tfrac{n-1}{2} - l)(e_{n-l+1} + \cdots + e_n) & \text{if } \ 1 \leq l \leq n-1, \\ -\tfrac{n+1}{2}\,(e_1 + \cdots + e_n) & \text{if } \ l = n. \end{cases}$$

Let $L = \{([z], w) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1};\, w \in [z]\}$ be the tautological bundle over $\mathbb{P}^n(\mathbb{C})$. For $m \in \mathbb{Z}$, we set $L^m := L^{\otimes m}$. As a homogeneous vector bundle, $L^m$ is associated to the irreducible $K$-representation $\tau^m$ defined by

$$\tau^m\left(\begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}\right) = (\det A)^m \quad (A \in U(n)),$$

whose highest weight is $\eta^m = m(e_1 + \cdots + e_n)$. Consequently, $(S_{n-l} \otimes L^m)_{eK}$ is an irreducible $K$-module with highest weight

$$\sigma_{l,m} = \begin{cases} (\frac{n+1}{2} + m)(e_1 + \cdots + e_n) & \text{if } \ l = 0, \\ (\frac{n+1}{2} - l + m)(e_1 + \cdots + e_{n-l}) & \\ +(\frac{n-1}{2} - l + m)(e_{n-l+1} + \cdots + e_n) & \text{if } \ 1 \leq l \leq n-1, \\ (-\frac{n+1}{2} + m)(e_1 + \cdots + e_n) & \text{if } \ l = n. \end{cases}$$

We shall denote by $\tau_{\sigma_{l,m}}$ the unique (up to equivalence) irreducible representation of $K$ with highest weight $\sigma_{l,m}$.

To compute the spectrum of the twisted Dirac operator $D_m := D_{L^m}$, we need the following well known result (compare [2], [5]):

**Proposition 1.** *Let $\rho_\lambda$ (resp. $\tau_\sigma$) be an irreducible representation of $SU(n+1)$ (resp. $S(U(n) \times U(1))$) with highest weight $\lambda = \sum_{j=1}^n \lambda_j e_j$ (resp. $\sigma = \sum_{j=1}^n \sigma_j e_j$). Then the multiplicity $m(\lambda, \sigma) := m_{\rho_\lambda|_{S(U(n)\times U(1))}}(\tau_\sigma)$ is either 0 or 1, and is equal to 1 if and only if $\sigma$ is of the form*

$$\sigma = \sum_{j=1}^n (\xi_j - a) e_j$$

*with $\lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \cdots \geq \lambda_n \geq \xi_n \geq 0$ and $a = \sum_{j=1}^n (\lambda_j - \xi_j)$.*

Applying directly this proposition, we get:

**Proposition 2.** *Let $\rho_\lambda$ be an irreducible representation of $SU(n+1)$ with highest weight $\lambda = \sum_{j=1}^n \lambda_j e_j$, and let $\tau_{\sigma_{l,m}}$ be as before with $0 \leq l \leq n$ and $m \in \mathbb{Z}$. Then the multiplicity $m(\lambda, \sigma_{l,m})$ is non-zero (and hence equal to 1) if and only if,*

(i) *for $l = 0$, $\lambda$ is of the form*

$$\lambda = \left(\frac{n+1}{2} + 2\,k + m\right) e_1 + \left(\frac{n+1}{2} + k + m\right)(e_2 + \cdots + e_n)$$

*with $k \geq \max\{0, -\frac{n+1}{2} - m\}$;*

(ii) *for* $1 \le l \le n-1$, $\lambda$ *is of the form*

$$\lambda = \left(\frac{n+1}{2} + 2\,k + m - l - \varepsilon\right) e_1 + \left(\frac{n+1}{2} + k + m - l\right)(e_2 + \cdots + e_{n-l}) + \left(\frac{n-1}{2} + k + m - l + \varepsilon\right) e_{n-l+1} + \left(\frac{n-1}{2} + k + m - l\right)(e_{n-l+2} + \cdots + e_n)$$

*with* $\varepsilon \in \{0,1\}$ *and* $k \ge \max\{\varepsilon,\, l - m - \frac{n-1}{2}\}$;

(iii) *for* $l = n$, $\lambda$ *is of the form*

$$\lambda = \left(-\frac{n+1}{2} + 2\,k + m\right) e_1 + \left(-\frac{n+1}{2} + k + m\right)(e_2 + \cdots + e_n)$$

*with* $k \ge \max\{0, \frac{n+1}{2} - m\}$.

On the Lie algebra $\mathfrak{g} = \mathfrak{su}(n+1)$, we fix the scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\langle X, Y \rangle = -B(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g},$$

where $B$ is the Killing form of $\mathfrak{g}$. As usual, we extend $\langle \cdot, \cdot \rangle$ to a scalar product on the vector space of real valued linear forms on $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$. Observe that

$$\langle e_i,\, e_i \rangle = \frac{n}{2\,(n+1)^2} \quad \text{and} \quad \langle e_i,\, e_j \rangle = \frac{-1}{2\,(n+1)^2}$$

for $1 \le i \ne j \le n$.

The scalar curvature of the Riemannian space $\mathbb{P}^n(\mathbb{C}) = G/K$ is $R = n$. Thus the spectrum of the twisted Dirac operator $D_m$ is given by

$$\mathrm{Spec}(D_m,\, \mathbb{P}^n(\mathbb{C})) = \Big\{ \pm \sqrt{c(\lambda)};\ \ \exists\, l \in \{0, \ldots, n\}, : m(\lambda,\, \sigma_{l,m}) \ne 0 \Big\} \ \ \text{with}$$
$$c(\lambda) := \langle \lambda + 2\delta_G,\, \lambda \rangle - \langle \eta^m + 2\delta_K,\, \eta^m \rangle + \frac{n}{8}\,.$$

Computing the eigenvalues of $D_m$, we can easily derive Theorem 1.

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