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Some results on $L\Sigma(\kappa)$-spaces

F. Casarrubias Segura, O. Okunev, C.G. Paniagua Ramírez

Abstract. We present several results related to $L\Sigma(\kappa)$-spaces where $\kappa$ is a finite cardinal or $\omega$; we consider products and some constructions that lead from spaces of these classes to other spaces of similar classes.

Keywords: upper semicontinuous mappings, products, Lindelöf $\Sigma$-spaces

Classification: 54D20, 54C60, 54B10

All spaces in this article are assumed to be Tychonoff (= completely regular Hausdorff). We use terminology and notation as in [Eng2]. For multivalued mappings we do not require that images of points all be nonempty; if $p: X \to Y$ is a multivalued mapping and $A \subset X$, then $p(A)$ is defined as $\bigcup \{ p(x) : x \in A \}$. The composition of two multivalued mappings $p: X \to Y$ and $q: Y \to Z$ is defined by the rule $(q \circ p)(x) = q(p(x))$. A multivalued mapping $p: X \to Y$ is upper semicontinuous if for every open set $V$ in $Y$ the set $\{ x \in X : p(x) \subset V \}$ is open in $X$, or, equivalently, if for every point $x$ in $X$ and every neighborhood $V$ of $p(x)$ in $Y$ there is a neighborhood $U$ of $x$ in $X$ such that $p(U) \subset V$.

It is well-known that the composition of compact-valued upper semicontinuous mappings is compact-valued upper semicontinuous. In fact, it is easy to prove that a mapping is compact-valued upper semicontinuous iff it is the composition of a continuous single-valued function, the inverse of a perfect mapping and the inverse of a closed embedding (see, e.g., [KOS]).

The symbol $\mathfrak{c}$ denotes the cardinality of the continuum. If $\kappa$ is an infinite cardinal, $A(\kappa)$ denotes the one-point compactification of a discrete space of cardinality $\kappa$. The symbol $I$ stands for the closed interval $[0,1]$.

Let $\mathcal{K}$ be a cover of a space $X$. A family $\mathcal{N}$ of subsets of $X$ is called a network modulo $\mathcal{K}$ if for every element $K$ of $\mathcal{K}$ and a neighborhood $U$ of $K$, there is an element $N$ of $\mathcal{N}$ such that $K \subset N \subset U$ [Nag].

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Given a cardinal $\kappa$, finite or infinite, a space $X$ is called an $L\Sigma(<\kappa)$-space \cite{KOS} if it satisfies one of the following equivalent conditions:

There is a second-countable space $M$ and a compact-valued upper semicontinuous mapping $p : M \to X$ such that $p(M) = X$ and $w(p(z)) < \kappa$ for each $z \in M$;

or

There are a compact cover $K$ of $X$ such that $w(K) < \kappa$ for every $K \in K$ and a countable network modulo $K$ in $X$.

$X$ is an $L\Sigma(\leq \kappa)$-space if it is an $L\Sigma(<\kappa^+)$-space. $X$ is an $L\Sigma(\kappa)$-space if it is an $L\Sigma(\leq \kappa)$-space and is not an $L\Sigma(<\kappa)$-space; this concept is especially important in the case of finite cardinals $\kappa$. Of course, for finite $\kappa$, the weights of images of points and of the elements of the compact covers in the above characterizations can be replaced by the cardinalities.

The classes of $L\Sigma(<\kappa)$-spaces are invariant with respect to closed subspaces, continuous images and countable unions. Obviously, all $L\Sigma(\kappa)$-spaces are Lindelöf $\Sigma$-spaces in the sense of \cite{Nag}; it is easy to see that $L\Sigma(\leq 1)$-spaces are exactly the spaces of countable network weight. The class of $L\Sigma(2)$-spaces includes the Double Arrow space, one-point compactifications of uncountable discrete spaces of cardinality less or equal to the continuum, and the one-point compactifications of $\Psi$-like spaces (that is, the spaces of the form $\Psi(A)$, where $A$ is an almost disjoint family on $\omega$; see Section 2 for a detailed description). Assuming $\text{MA}(\omega_1)$, all scattered compact spaces of height 3 and cardinality $\omega_1$ are in $L\Sigma(\leq 3)$ \cite{KOS}.

If $\kappa \geq c$, then $L\Sigma(\leq \kappa)$-spaces are exactly Lindelöf $\Sigma$-spaces of network weight $\leq \kappa$.

1. Products of $L\Sigma(n)$-spaces

It is easy to see, using the fact that the product of compact-valued upper semicontinuous mappings is upper semicontinuous, that the product of an $L\Sigma(\kappa)$-space with an $L\Sigma(\lambda)$-space is an $L\Sigma(\leq \lambda \cdot \kappa)$-space. However, if $\lambda$ and $\kappa$ are finite, it turns out that the product may be of the “type” lower than $\lambda \cdot \kappa$. For example, the one-point compactification $A(\omega_1)$ of the discrete space of cardinality $\omega_1$ is an $L\Sigma(2)$-space; as shown in \cite{KOS}, for every $n \in \omega$, $A(\omega_1)^n$ is $L\Sigma(n + 1)$. On the other hand, if $\omega_2 \leq c$, then the space $A(\omega_2)$ is also an $L\Sigma(2)$-space, but its square is in $L\Sigma(4)$. Thus, it may be interesting to find the exact $L\Sigma$-classes for products of some $L\Sigma(n)$-spaces. Several problems of this type were posed in \cite{KOS} and \cite{Oku}; here we present solutions to some of these problems.

**Theorem 1.1.** Suppose $m, n \in \omega$, $X$ is an $L\Sigma(m)$-space and $Y$ is an $L\Sigma(n)$-space. Then $X \times Y$ is an $L\Sigma(k)$-space, where $n + m - 1 \leq k \leq mn$.

**Proof:** Let $p_1 : M_1 \to X$ and $p_2 : M_2 \to Y$ be upper semicontinuous mappings from second countable spaces $M_1$ and $M_2$ onto $X$ and $Y$ such that $p_1$ is at most $m$-valued and $p_2$ is at most $n$-valued. Then the mapping $p_1 \times p_2 : M_1 \times M_2 \to X \times Y$
(defined by the rule \((p_1 \times p_2)(m_1, m_2) = p_1(m_1) \times p_2(m_2)\)) is upper semicontinuous and onto \(X \times Y\). This proves that \(X \times Y\) is an \(L\Sigma(\leq mn)\)-space; therefore, \(X \times Y\) is an \(L\Sigma(k)\)-space for some \(k \leq mn\).

To prove the second part of the inequality, suppose for contradiction that \(X \times Y \in L\Sigma(k)\) and \(k \leq n + m - 2\). Fix a second countable space \(M\) and an at most \(k\)-valued upper semicontinuous mapping \(p: M \to X \times Y\) such that \(p(M) = X \times Y\). Let \(\pi_X, \pi_Y\) be the projections of the product \(X \times Y\); put

\[
A = \{ z \in M : |\pi_X(p(z))| \leq m - 1 \}.
\]

Since the composition \(\pi_X \circ p\) is upper semicontinuous and \(X \notin L\Sigma(\leq m - 1)\), there is a point \(x_0 \in X\) such that \(x_0 \notin \pi_X(p(A))\), hence \(\{x_0\} \times Y \cap p(A) = \emptyset\).

Let \(B = M \setminus A\) and \(q: B \to Y\) be the multivalued mapping defined by the rule:

\[
q(z) = \pi_Y(p(z) \cap \{x_0\} \times Y).
\]

Since \(\{x_0\} \times Y\) is closed in \(X \times Y\), the mapping \(q\) is upper semicontinuous, and from \(p(M) = X \times Y\) and \(\{x_0\} \times Y \cap p(A) = \emptyset\) it follows that \(q(B) = Y\).

For every \(z \in B\), \(p(z)\) has at most \(n + m - 2\) points, and at least \(m - 1\) of these points have their projections on \(X\) different from \(x_0\). Hence, \(q(z)\) contains at most \(n - 1\) points. Thus, \(q\) is an upper semicontinuous, at most \((n - 1)\)-valued mapping from the second countable space \(B\) onto the space \(Y\), a contradiction with the assumption that \(Y\) is an \(L\Sigma(n)\)-space.

**Corollary 1.2.** If \(X\) is an \(L\Sigma(n)\)-space for some \(n \in \omega\), then \(X^m\) is an \(L\Sigma(k)\)-space for some \(k \geq mn - m + 1\).

In particular,

**Corollary 1.3.** If there is an \(n \in \omega\) such that \(X^m\) is an \(L\Sigma(\leq n)\)-space for every \(m \in \omega\), then \(X\) has a countable network.

It was shown in [KOS] that if \(X^\omega\) is an \(L\Sigma(\leq \omega)\)-space, then there is an \(n \in \omega\) such that \(X^m\) is an \(L\Sigma(\leq n)\)-space for every \(m \in \omega\); it was also shown that, consistently, this implies that \(X\) has a countable network. Corollary 1.3 now allows to prove this in ZFC (thus giving an answer to Question 7.4 in [KOS]):

**Corollary 1.4.** If \(X^\omega\) is an \(L\Sigma(< \omega)\)-space, then \(X\) has a countable network (and hence \(X^\omega\) is in fact an \(L\Sigma(\leq 1)\)-space).

Another interesting corollary of Theorem 1.1 is

**Corollary 1.5.** If \(X\) is an \(L\Sigma(m)\)-space for some \(m \in \omega\), and \(Y\) is an \(L\Sigma(n)\)-space for some \(n \in \omega\), \(n \geq 2\), then \(X \times Y\) is not homeomorphic to \(X\).

In particular, if \(X\) is an \(L\Sigma(m)\)-space for some \(m \in \omega\), \(m \geq 2\), then all finite powers of \(X\) are pairwise non-homeomorphic.
Since the classes of $L\Sigma(\leq n)$-spaces are invariant with respect to closed subspaces and continuous images, we may further strengthen Corollary 1.5.: 

**Corollary 1.6.** If $X$ is an $L\Sigma(m)$-space for some $m \in \omega$, and $Y$ is an $L\Sigma(n)$-space for some $n \in \omega$, $n \geq 2$, then $X \times Y$ is not homeomorphic to a continuous image of any closed subspace of $X$.

**Corollary 1.7.** If $X$ is an $L\Sigma(\leq n)$-space for some $n \in \omega$, and there are natural $k$ and $m$ such that $k < m$ and $X^m$ is a continuous image of a closed subspace of $X^k$, then $X$ has a countable network.

For example,

**Corollary 1.8.** Let $X$ be the Double Arrow space. If $m, n \in \omega$ and $n > m$, then $X^n$ cannot be embedded into a continuous image of $X^m$.

For some individual spaces, in particular, for products of given spaces, finding the exact $L\Sigma(k)$-class where they belong appears a non-trivial task. For example, it is still not clear whether the square of the Double Arrow space is in $L\Sigma(3)$ or $L\Sigma(4)$ (Problem 1(132) in [Oku]).

The next theorem solves Problem 3(134) in [Oku].

Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\omega$. Recall that the space $\Psi(\mathcal{A})$ is defined as the union $\omega \cup \mathcal{A}$ with the topology in which the points of $\omega$ are isolated, and basic neighborhoods of the points $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus F$ where $F \subset A$ is finite. Clearly, $\Psi(\mathcal{A})$ is a Hausdorff zero-dimensional (hence Tychonoff) locally compact space. Let $\alpha \Psi(\mathcal{A})$ be its one-point compactification. Then $\alpha \Psi(\mathcal{A})$ is an $L\Sigma(2)$-space, because it is a countable union of singletons (points of $\omega$) and the subspace homeomorphic to $A(|\mathcal{A}|)$; the latter space is in $L\Sigma(2)$, and the class $L\Sigma(2)$ is invariant with respect to countable unions (see [KOS]). Problem 3(134) in [Oku] was whether the square of a space $\alpha \Psi(\mathcal{A})$ can be an $L\Sigma(3)$-space and whether it can be an $L\Sigma(4)$-space.

**Theorem 1.9.** Let $\mathcal{A}, \mathcal{B}$ be uncountable almost disjoint families of infinite subsets of $\omega$, and let $X = \alpha \Psi(\mathcal{A}) \times \alpha \Psi(\mathcal{B})$. Then

- $X$ is an $L\Sigma(3)$-space iff both $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\omega_1$;
- $X$ is an $L\Sigma(4)$-space iff one of the families $\mathcal{A}, \mathcal{B}$ has cardinality greater than $\omega_1$.

**Proof:** Since both $\alpha \Psi(\mathcal{A})$ and $\alpha \Psi(\mathcal{B})$ are $L\Sigma(2)$-spaces, their product is an $L\Sigma(\leq 4)$-space. By Theorem 1.1, $X$ is not $L\Sigma(2)$, so it is either $L\Sigma(3)$ or $L\Sigma(4)$.

If one of the families $\mathcal{A}, \mathcal{B}$ has cardinality greater or equal to $\omega_2$, then the one-point compactification of the corresponding $\Psi$-space contains a closed copy of $A(\omega_2)$ while the other contains a closed copy of $A(\omega_1)$. Hence, the product $X$ contains a closed subspace homeomorphic to $A(\omega_2) \times A(\omega_1)$, which is not an $L\Sigma(\leq 3)$-space by (the remark after the proof of) Theorem 4.7 in [KOS]. Since
the class of $L\Sigma(\leq 3)$-spaces is hereditary with respect to closed subspaces, this proves that $X$ cannot be an $L\Sigma(3)$-space.

On the other hand, if both $A$ and $B$ have cardinality $\omega_1$, then each of them is the union of a countable space and the space $A(\omega_1)$. It follows that $X$ is the union of a countable set, countably many copies of $A(\omega_1)$, and a copy of $A(\omega_1) \times A(\omega_1)$. Since each of these spaces is in $L\Sigma(\leq 3)$, the space $X$ is in $L\Sigma(\leq 3)$. □

Corollary 1.10. If $\mathfrak{c} = \omega_1$, then for any uncountable almost disjoint families $A, B$ on $\omega$, the product $\alpha\Psi(A) \times \alpha\Psi(B)$ is an $L\Sigma(3)$-space.

2. One-point compactifications

In [KOS], the consistently positive answer to Question 7.4 was obtained by showing that a counterexample would have to be a strong $S$-space and an $L\Sigma(n)$-space for some $n \in \omega$. It appears natural to ask if this kind of spaces can exist. In this section we present a construction that shows, in particular, that the answer is “yes”.

Theorem 2.1. Let $X$ be a locally compact space. Suppose that for some $n, m \in \omega$ there exist an $L\Sigma(\leq n)$-space $Y$ and a continuous mapping $j: X \to Y$ such that $j(X) = Y$ and $|j^{-1}(y)| \leq m$ for all $y \in Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(\leq nm + 1)$-space.

Proof: If $X$ is compact, then the mapping $j$ is perfect, so its inverse is upper semicontinuous and at most $m$-valued. If $p: M \to Y$ is an upper semicontinuous at most $n$-valued mapping from a second countable space $M$ onto $Y$, then the composition $j^{-1} \circ p$ is upper semicontinuous, onto $X$, and at most $nm$-valued, so $\alpha X = X$ is an $L\Sigma(\leq nm)$-space.

Thus, we may assume that $X$ is not compact. Let $\infty$ be the point such that $\{\infty\} = \alpha X \setminus X$.

Let $p: M \to Y$ be an upper semicontinuous mapping from a second-countable space $M$ onto $Y$ such that $|p(z)| \leq n$ for every $z \in M$. Define a multivalued mapping $q: M \to X$ by putting

$$q(z) = j^{-1}(p(z)) \cup \{\infty\}.$$ 

Obviously, the mapping $q$ is onto $\alpha X$ and is at most $(nm + 1)$-valued, so to complete the proof, it remains to verify that $q$ is upper semicontinuous.

Let $z_0$ be a point of $M$ and $U$ an open neighborhood of $q(z_0)$ in $\alpha X$; we need to find a neighborhood $V$ of $z_0$ in $M$ so that $q(V) \subset U$. 
Since $\infty \in U$, the set $K = X \setminus U$ is compact. Put $W = Y \setminus j(K)$. The set $W$ is open in $Y$ and contains $p(z_0)$, so by the upper semicontinuity of $p$, there is a neighborhood $V$ of $z_0$ in $M$ such that $p(V) \subset W$. Then $q(V) = \{\infty\} \cup j^{-1}(p(V)) \subset \{\infty\} \cup j^{-1}(W) \subset U$, and the proof is complete. \[\square\]

**Corollary 2.2.** If $X$ is a locally compact space, and $X$ admits a continuous bijection onto a second-countable space, then $\alpha X$ is an $L\Sigma(2)$-space.

The Kunen Line and the Todorčević line [Todor] are locally compact, admit weaker second-countable topologies, and are strong $S$-spaces. Since the Todorčević line is constructed assuming $b = \omega_1$, we arrive at the following.

**Corollary 2.3.** Assume $b = \omega_1$. Then there exists a strong $S$-space which is an $L\Sigma(2)$-space.

Arguments similar to that of the proof of Theorem 2.1 lead to the following versions:

**Theorem 2.4.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(<\omega)$-space $Y$ and a continuous finite-to-one mapping $j : X \to Y$ such that $j(X) = Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(<\omega)$-space.

**Theorem 2.5.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(<\omega)$-space $Y$ and a continuous mapping $j : X \to Y$ such that $j(X) = Y$ and $j^{-1}(y)$ is compact and metrizable for every $y \in Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(\leq \omega)$-space.

Recall that a mapping $j : X \to Y$ is called compact-covering if for every compact set $K$ in $Y$ there is a compact set $F$ in $X$ such that $j(F) = K$.

**Theorem 2.6.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous compact-covering bijection $j : X \to Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(\leq \omega)$-space.

In all three latter theorems the mapping $q$ is defined in the same way as in the proof of Theorem 2.1, and the upper semicontinuity of $q$ is verified by the same argument. In Theorem 2.4, $q$ is trivially finite-valued, and in Theorem 2.5, $q$ has compact metrizable images of points because finite unions of metrizable compacta are metrizable compacta. In Theorem 2.6, the compactness and metrizability of images of points under $q$ are verified as follows: there is a compact subset $C$ of $X$ such that $p(z) \subset j(C)$; since $j$ is a continuous bijection, the restriction of $j$ to $C$ is a homeomorphism. Thus, $q(z)$ is the union of the set $j^{-1}(p(z))$, homeomorphic to $p(z)$, and a singleton, hence compact metrizable.

It is not clear if it is possible to omit the requirement that $j$ be compact-covering in Theorem 2.6. Hence,
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**Problem 2.7.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous bijection $j: X \to Y$. Must the one-point compactification $\alpha X$ of $X$ be an $L\Sigma(\leq \omega)$-space?

It is also not clear whether Theorem 2.6 remains true if we require that $j$ be finite-to-one instead of being a bijection. The reason of course is that the preimage of a compact metrizable space under a perfect finite-to-one mapping need not be metrizable, so the argument as above does not work. Hence,

**Problem 2.8.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous finite-to-one compact-covering mapping $j: X \to Y$. Must the one-point compactification $\alpha X$ of $X$ be an $L\Sigma(\leq \omega)$-space?

**Problem 2.9.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous finite-to-one mapping $j: X \to Y$ such that $j(X) = Y$. Must the one-point compactification $\alpha X$ of $X$ be an $L\Sigma(\leq \omega)$-space?

3. The Alexandroff duplicates

One of intriguing questions in the theory of $L\Sigma(\leq \omega)$-spaces is the following (Question 7.5 in [KOS]; also Problem 13(144) in [Oku]): Let $X$ be an $L\Sigma(\leq \omega)$-space and let $p: X \to Y$ be a finite-valued upper semicontinuous mapping such that $p(X) = Y$. Must $Y$ be an $L\Sigma(\leq \omega)$-space?

Below we prove that the answer is positive for a particular case of the Alexandroff duplicate of an $L\Sigma(\leq \omega)$-space; this gives a positive answer to Problem 15(146) in [Oku].

Recall that the *Alexandroff duplicate* $AD(X)$ of a space $X$ is $X \times 2$ with the topology defined as follows: the points of $X \times \{1\}$ are isolated, and basic neighborhoods of the points $(x,0)$ are of the form $(U \times 2) \setminus \{(x,1)\}$ where $U$ is a neighborhood of $x$ in $X$ (see [Eng1] for a discussion of this construction). It is easy to see that the mapping $\pi: AD(X) \to X$ defined by the rule $\pi((x,i)) = x$ is two-to-one and perfect, so its inverse is 2-valued upper semicontinuous.

**Theorem 3.1.** If $X$ is an $L\Sigma(\leq \omega)$-space, then so is $AD(X)$.

**Proof:** Fix a second-countable space $M$ and an upper semicontinuous compact-valued mapping $p: M \to X$ so that $p(M) = X$ and $w(p(z)) \leq \omega$ for every $z \in M$. Since the cardinalities of $M$ and of $p(z)$, $z \in M$, are at most $\mathfrak{c}$, we have $|X| \leq \mathfrak{c}$, and we may fix a one-to-one function (not necessarily continuous) $j: X \to I = [0,1]$. Define a multivalued mapping $q: M \times I \to AD(X)$ by the rule:

$$q(z,t) = (p(z) \times \{0\}) \cup ((p(z) \cap j^{-1}(t)) \times \{1\}).$$

Since for every $(z,t) \in M \times I$ the set $j^{-1}(t)$ contains at most one point, the images of points under $q$ are compact and metrizable. Let us verify that $q$ is upper semicontinuous.
Let \((z_0, t_0) \in M \times I\), and let \(U\) be a neighborhood of \(q(z_0, t_0)\); we need to find a neighborhood \(V\) of \((z_0, t_0)\) so that \(q(V) \subset U\). Since \(p(z_0)\) is compact, there is a neighborhood \(W\) of \(p(z_0)\) in \(X\) and a finite set \(F \subset X\) such that \(F \cap j^{-1}(t_0) = \emptyset\) and \(U \supset (W \times 2) \setminus (F \times \{1\})\). Indeed, for every point \(x \in p(z_0)\) we can fix a standard open neighborhood \((W_x \times 2) \setminus \{(x, 1)\}\) of \((x, 0)\) contained in \(U\); choose a finite subfamily \(W_{x_1}, \ldots, W_{x_n}\) of the family \(\{W_x : x \in p(z_0)\}\) so that \(p(z_0) \subset \bigcup_{i=1}^n W_{x_i}\), and put \(W = \bigcup_{i=1}^n W_{x_i}\) and \(F = \{x_1, \ldots, x_n\} \setminus j^{-1}(t_0)\).

Let \(S = j(F)\); then \(S\) is finite and \(t_0 \notin S\). By the upper semicontinuity of \(p\), there is an open neighborhood \(G\) of \(z_0\) in \(M\) such that \(p(G) \subset W\). Put \(V = G \times (I \setminus S)\). Now if \((z, t) \in V\), then \(p(z) \subset W\) and \(p(z) \cap j^{-1}(t) \subset W \setminus F\), so \(q(z, t) \subset (W \times 2) \setminus (F \times \{1\}) \subset U\), and \(V\) is as required.

Let us now verify that \(q\) is onto \(AD(X)\). If \(x \in X\), then there is \(z_0 \in M\) such that \(x \in p(z_0)\). Put \(t_0 = j(x)\). Then both \((x, 0)\) and \((x, 1)\) are in \(q(z_0, t_0)\).

Thus, there is an upper semicontinuous compact-valued mapping with metrizable images of points from a second-countable space \(M \times I\) onto \(AD(X)\), and the proof is complete.

Theorem 3.1 gives the positive answer to Problem 15(146) in [Oku].

A space \(X\) is called a \(KL\Sigma(\leq \omega)\)-space if there is a \(compact\) second-countable space \(M\) and a compact-valued upper semicontinuous mapping \(p: M \to X\) such that \(p(M) = X\) and \(w(p(z)) \leq \omega\) for all \(z \in M\) [KOS]. It is observed in [KOS] that a compact \(L\Sigma(\leq \omega)\)-space need not be a \(KL\Sigma(\leq \omega)\)-space. The same argument as in the proof of Theorem 3.1 can be used to prove the following:

**Theorem 3.2.** If \(X\) is a \(KL\Sigma(\leq \omega)\)-space, then so is \(AD(X)\).

Of course, the same argument works for the next statement:

**Theorem 3.3.** Let \(\kappa\) be an infinite cardinal. If \(|X| \leq \mathfrak{c}\) and \(X\) is an \(L\Sigma(\leq \kappa)\)-space (\(KL\Sigma(\leq \kappa)\)-space), then so is \(AD(X)\).

The condition “\(|X| \leq \mathfrak{c}\)” in Theorem 3.3 cannot be omitted unless \(2^\kappa \leq \mathfrak{c}\). Indeed, if \(2^\kappa > \mathfrak{c}\), let \(X = 2^\kappa\) (with the product topology). Trivially, \(X \in KL\Sigma(\leq \kappa)\). On the other hand, every \(L\Sigma(\leq \kappa)\)-space is a union of at most \(\mathfrak{c}\) subspaces of weight at most \(\kappa\), so its network weight is at most \(\kappa \cdot \mathfrak{c}\). The network weight of \(AD(2^\kappa)\) is \(2^\kappa\), so it cannot be an \(L\Sigma(\leq \kappa)\)-space.

**References**


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