Ahmed Alsaedi
Approximation of solutions of the forced duffing equation with nonlocal discontinuous type integral boundary conditions

Archivum Mathematicum, Vol. 44 (2008), No. 4, 295--305

Persistent URL: http://dml.cz/dmlcz/119769

Terms of use:

© Masaryk University, 2008

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
APPROXIMATION OF SOLUTIONS OF THE FORCED DUFFING EQUATION WITH NONLOCAL DISCONTINUOUS TYPE INTEGRAL BOUNDARY CONDITIONS

Ahmed Alsaedi

Abstract. A generalized quasilinearization technique is applied to obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the forced Duffing equation with nonlocal discontinuous type integral boundary conditions.

1. Introduction

Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics, see for example [16] [17] [24]. In fact, boundary value problems involving integral boundary conditions have received considerable attention, see for instance, [3] [10], [12] [15], [18] [19] [26] and the references therein. In a recent reference [2], Ahmad, et. al. discussed the existence and uniqueness of the solutions of a boundary value problem with discontinuous type integral boundary conditions.

The monotone iterative technique coupled with the method of upper and lower solutions [5] [8] [20] [23] [25] manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. In general, the convergence of the sequence of approximate solutions given by the monotone iterative technique is at most linear [11] [21]. To obtain a sequence of approximate solutions converging quadratically, we use the method of quasilinearization (QSL) [9]. This method has been developed for a variety of problems [1] [4] [6] [7] [22]. In view of its diverse applications, this approach is quite an elegant and easier for application algorithms. To the best of our knowledge, the method of quasilinearization has not been developed for Duffing equation with nonlocal discontinuous type integral boundary conditions.

In this paper, we apply a quasilinearization technique to obtain the analytic approximation of the solution of the forced Duffing equation with nonlocal discontinuous type integral boundary conditions. In fact, we obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem at hand. The concept of nonlocal discontinuous integral

2000 Mathematics Subject Classification: primary 34B10; secondary 34B15.
Key words and phrases: duffing equation, integral boundary conditions, quasilinearization, quadratic convergence.
Received December 12, 2007. Editor O. Došlý.
boundary conditions corresponds to a situation when some forcing term is present at an arbitrary intermediate point of the boundary segment and thereby generates a discontinuity in the integral boundary conditions.

2. Preliminaries

We consider the following boundary value problem

\[ u''(t) + \sigma u'(t) + f(t, u) = 0, \quad t \in [0, 1], \quad \sigma \in \mathbb{R}\setminus\{0\}, \]

\[ u(0) - \mu_1 u'(0) = g_1(u(\gamma)) + \int_0^{\gamma-} q_1(u(s)) \, ds + \int_{\gamma+}^1 q_1(u(s)) \, ds, \]

\[ u(1) + \mu_2 u'(1) = g_2(u(\gamma)) + \int_0^{\gamma-} q_2(u(s)) \, ds + \int_{\gamma+}^1 q_2(u(s)) \, ds, \]

\[ 0 < \gamma < 1, \]

where \( f: [0, 1] \times \mathbb{R} \to \mathbb{R}, g_i: \mathbb{R} \to \mathbb{R} \) (\( i = 1, 2 \)) are continuous functions, \( q_i \) are continuous functions on \((0, \gamma)\) and \((\gamma, 1)\) and \( \mu_i \) are nonnegative constants.

The quasilinearization technique is applied to obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem (2.1).

Definition 2.1. A function \( \alpha \in C^2[0, 1] \) is a lower solution of (2.1) if

\[ \alpha''(t) + \sigma \alpha'(t) + f(t, \alpha(t)) \geq 0, \quad t \in [0, 1], \]

\[ \alpha(0) - \mu_1 \alpha'(0) \leq g_1(\alpha(\gamma)) + \int_0^{\gamma-} q_1(\alpha(s)) \, ds + \int_{\gamma+}^1 q_1(\alpha(s)) \, ds, \]

\[ \alpha(1) + \mu_2 \alpha'(1) \leq g_2(\alpha(\gamma)) + \int_0^{\gamma-} q_2(\alpha(s)) \, ds + \int_{\gamma+}^1 q_2(\alpha(s)) \, ds. \]

Similarly, \( \beta \in C^2[0, 1] \) is an upper solution of (2.1) if the inequalities in the definition of lower solution are reversed.

Since the associated homogeneous problem of (2.1) has only the trivial solution, therefore, by Green’s function method, the solution \( u(t) \) of (2.1) can be written as

\[ u(t) = \frac{1}{(1 + \sigma \mu_1) - (1 - \sigma \mu_2)e^{-\sigma}} \left[ (-1 + \sigma \mu_2)e^{-\sigma} + e^{-\sigma t} \right] \]

\[ \times \left\{ g_1(u(\gamma)) + \int_0^{\gamma-} q_1(u(s)) \, ds + \int_{\gamma+}^1 q_1(u(s)) \, ds \right\} \]

\[ + ((1 + \sigma \mu_1) - e^{-\sigma t}) \left\{ g_2(u(\gamma)) + \int_0^{\gamma-} q_2(u(s)) \, ds + \int_{\gamma+}^1 q_2(u(s)) \, ds \right\} \]

\[ + \int_0^1 G(t,s) f(s, u(s)) \, ds, \]
Then, there exists a sequence

\[ G(t, s) = \Lambda \left\{ \begin{array}{ll}
(1 - \sigma \mu_2) - e^{\sigma(1-s)} & 0 \leq t \leq s, \\
(1 - \sigma \mu_2) - e^{\sigma(1-t)} & s \leq t \leq 1,
\end{array} \right. \]

where

\[ \Lambda = \frac{e^{\sigma s}}{\sigma((1 - \sigma \mu_2) - (1 + \sigma \mu_1)e^\sigma)}. \]

Observe that \( G(t, s) > 0 \) on \((0, 1) \times (0, 1)\). We state the following results which lay a foundation to establish the main result. We omit the proof as the method of proof is similar to the one employed in \([2]\).

**Theorem 2.1.** Let \( \alpha \) and \( \beta \) be lower and upper solutions of the boundary value problem \((2.1)\) respectively. Let \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be such that \( f_u(t, u) < 0 \) and \( q_i \) are continuous functions on \((0, \gamma)\) and \((\gamma, 1)\) satisfying one sided Lipschitz condition: \( q_i(u) - q_i(v) \leq L_i(u - v), \) \( 0 \leq L_i < 1, \) \( i = 1, 2, \) and \( g_i : \mathbb{R} \to \mathbb{R} \) are continuous functions satisfying one sided Lipschitz condition: \( g_i(u) - g_i(v) \leq L_i^*(u - v), \) \( 0 \leq L_i^* < 1, \) \( i = 1, 2. \) Then \( \alpha(t) \leq \beta(t). \)

**Theorem 2.2.** Assume that \( \alpha \) and \( \beta \) are lower and upper solutions of the boundary value problem \((2.1)\) respectively such that \( \alpha(t) \leq \beta(t). \) If \( f : [0, 1] \times \mathbb{R} \to \mathbb{R}, \) \( g_i : \mathbb{R} \to \mathbb{R} \) are continuous functions and \( q_i \) are continuous functions on \((0, \gamma)\) and \((\gamma, 1)\) with \( g_i \) and \( q_i \) satisfying one sided Lipschitz condition, then there exists a solution \( u(t) \) of \((2.1)\) such that \( \alpha(t) \leq u(t) \leq \beta(t), \) \( t \in [0, 1]. \)

3. MAIN RESULT

**Theorem 3.1.** Assume that

(A1) \( \alpha \) and \( \beta \in C^2[0, 1] \) are respectively lower and upper solutions of \((2.1)\) such that \( \alpha(t) \leq \beta(t); \)

(A2) \( f(t, u) \in C^2([0, 1] \times \mathbb{R}) \) be such that \( f_u(t, u) < 0 \) and \( f_{uu}(t, u) + \phi_{uu}(t, u) \geq 0, \) where \( \phi_{uu}(t, u) \geq 0 \) for some continuous function \( \phi(t, u) \) on \([0, 1] \times \mathbb{R}; \)

(A3) \( q_i \) are continuous functions on \((0, \gamma)\) and \((\gamma, 1)\) satisfying \( 0 \leq q_i''(u) < 1, \)

and \( q_i''(u) + \chi_i''(u) \geq 0 \) with \( \chi_i'' \geq 0, \) \( i = 1, 2; \)

(A4) \( g_i \in C^2(\mathbb{R}) \) be such that \( 0 \leq g_i'(u) < 1 \) and \( g_i''(u) + \psi_i''(u) \leq 0 \) with \( \psi_i'' \leq 0, \) \( i = 1, 2. \)

Then, there exists a sequence \( \{\alpha_n\} \) of approximate solutions converging monotonically and quadratically to the unique solution of the problem \((2.1).\)

**Proof.** Let \( F : [0, 1] \times \mathbb{R} \to \mathbb{R} \) and \( G_i, K_i : \mathbb{R} \to \mathbb{R} \) be defined by \( F(t, u) = f(t, u) + \phi(t, u), \)

\( G_i(u) = g_i(u) + \psi_i(u), \)

\( K_i(u) = q_i(u) + \chi_i(u) \) so that \( F_{uu}(t, u) \geq 0, \)

\( G_i''(u) \leq 0, \) \( K_i''(u) \geq 0. \) Using the generalized mean value theorem together with (A2), (A3) and (A4), we obtain

(3.1) \( f(t, u) \geq f(t, v) + F_u(t, v)(u - v) + \phi(t, v) - \phi(t, u), \)

(3.2) \( g_i(u) \leq g_i(v) + G_i'(v)(u - v) + \psi_i(v) - \psi_i(u), \) \( u, v \in \mathbb{R}, \)

(3.3) \( q_i(u) \geq q_i(v) + K_i'(v)(u - v) + \chi_i(v) - \chi_i(u), \) \( u, v \in \mathbb{R}. \)
We set
\[\tilde{F}(t, u, \alpha) = f(t, \alpha) + F_u(t\alpha)(u - \alpha) + \phi(t, \alpha) - \phi(t, u),\]
\[Q_i(u, \alpha) = q_i(\alpha) + K'_i(\alpha)(u - \alpha) + \chi_i(\alpha) - \chi_i(u),\]
\[\bar{g}_i(u(\gamma), \alpha, \beta) = g_i(\alpha(\gamma)) + G'_i(\beta(\gamma))(u(\gamma) - \alpha(\gamma)) + \psi_i(\alpha) - \psi_i(u),\]
and note that
\[\tilde{F}_u(t, u, \alpha) < 0, \quad 0 \leq (\partial/\partial u)Q_i(u, \alpha) < 1, \quad 0 \leq (\partial/\partial u)\bar{g}_i(u(\gamma), \alpha, \beta) < 1.\]
Now, we fix \(\alpha = \alpha_0\) and consider the problem
\[u''(t) + \sigma u'(t) + \tilde{F}(t, u, \alpha_0) = 0, \quad t \in [0, 1],\]
\[u(0) - \mu_1 u'(0) = \bar{g}_1(u(\gamma), \alpha_0, \beta) + \int_{\gamma^-}^{\gamma^+} Q_1(u(s), \alpha_0(s)) \, ds\]
\[+ \int_{\gamma^+}^{1} Q_1(u(s), \alpha_0(s)) \, ds,\]
\[u(1) + \mu_2 u'(1) = \bar{g}_2(u(\gamma), \alpha_0, \beta) + \int_{\gamma^-}^{\gamma^+} Q_2(u(s), \alpha_0(s)) \, ds\]
\[+ \int_{\gamma^+}^{1} Q_2(u(s), \alpha_0(s)) \, ds.\]
As a first step, it will be shown that \(\alpha_0, \beta\) are respectively lower and upper solutions of (3.4). Using (A1) together with the fact that \(\tilde{F}(t, \alpha_0, \alpha_0) = f(t, \alpha_0),\)
\[\bar{g}_1(\alpha(\gamma), \alpha_0, \beta) = g_i(\alpha_0(\gamma))\]
and \(Q_i(\alpha_0, \alpha_0) = q_i(\alpha_0),\) we have
\[\alpha_0''(t) + \sigma \alpha_0'(t) + \tilde{F}(t, \alpha_0, \alpha_0) = \alpha_0''(t) + \sigma \alpha_0'(t) + f(t, \alpha_0) \geq 0, \quad t \in [0, 1],\]
\[\alpha_0(0) - \mu_1 \alpha_0'(0) \leq g_1(\alpha_0(\gamma)) + \int_{\gamma^-}^{\gamma^+} q_1(\alpha_0(s)) \, ds + \int_{\gamma^+}^{1} q_1(\alpha_0(s)) \, ds\]
\[= \bar{g}_1(\alpha(\gamma), \alpha_0, \beta) + \int_{\gamma^-}^{\gamma^+} Q_1(\alpha_0(s), \alpha_0(s)) \, ds\]
\[+ \int_{\gamma^+}^{1} Q_1(\alpha_0(s), \alpha_0(s)) \, ds,\]
\[\alpha_0(1) + \mu_2 \alpha_0'(1) \leq g_2(\alpha_0(\gamma)) + \int_{\gamma^-}^{\gamma^+} q_2(\alpha_0(s)) \, ds + \int_{\gamma^+}^{1} q_2(\alpha_0(s)) \, ds\]
\[= \bar{g}_2(\alpha(\gamma), \alpha_0, \beta) + \int_{\gamma^-}^{\gamma^+} Q_2(\alpha_0(s), \alpha_0(s)) \, ds\]
\[+ \int_{\gamma^+}^{1} Q_2(\alpha_0(s), \alpha_0(s)) \, ds\]
and
\[\beta''(t) + \sigma \beta'(t) + \tilde{F}(t, \beta, \alpha_0) \leq \beta''(t) + \sigma \beta'(t) + f(t, \beta) \leq 0, \quad t \in [0, 1].\]
Moreover, there exists \(c_0, c_1 \in (\alpha_0(\gamma), \beta(\gamma))\) and \(c_2, c_3 \in (\alpha_0, \beta)\) so that

\[
g_1(\beta(\gamma)) - \bar{g}_1(\beta(\gamma), \alpha_0, \beta) = g_1(\beta(\gamma)) - g_1(\alpha_0(\gamma)) - G_1'(\beta(\gamma)) (\beta(\gamma) - \alpha_0(\gamma)) - \psi_1(\alpha_0(\gamma)) + \psi_1(\beta(\gamma)) = [g_1'(c_0) - g_1'(\beta(\gamma))] (\beta(\gamma) - \alpha_0(\gamma)) + [\psi_1'(c_1) - \psi_1'(\beta(\gamma))] (\beta(\gamma) - \alpha_0(\gamma)) \geq 0,
\]

\[
q_1(\beta(s)) - Q_1(\beta(s), \alpha_0(s)) = q_1(\beta(s)) - q_1(\alpha_0(s)) - K_1'(\alpha_0(s)) (\beta(s) - \alpha_0(s)) - \chi_1(\alpha_0(s)) + \chi_1(\beta(s)) = [q_1'(c_2) - q_1'(\alpha_0(s))] (\beta(s) - \alpha_0(s)) + [\chi_1'(c_3) - \chi_1'(\alpha_0(s))] (\beta(s) - \alpha_0(s)) \geq 0
\]

and consequently, we obtain

\[
\beta(0) - \mu_1 \beta'(0) \geq g_1(\beta(\gamma)) + \int_0^{\gamma_-} q_1(\beta(s)) \, ds + \int_{\gamma_+}^1 q_1(\beta(s)) \, ds
\]

\[
\geq \bar{g}_1(\beta(\gamma), \alpha_0, \beta) + \int_0^{\gamma_-} Q_1(\beta(s), \alpha_0(s)) \, ds + \int_{\gamma_+}^1 Q_1(\beta(s), \alpha_0(s)) \, ds.
\]

Similarly, it can be shown that

\[
\beta(1) + \mu_2 \beta'(1) \geq \bar{g}_2(\beta(\gamma), \alpha_0, \beta) + \int_0^{\gamma_-} Q_2(\beta(s), \alpha_0(s)) \, ds + \int_{\gamma_+}^1 Q_2(\beta(s), \alpha_0(s)) \, ds.
\]

Thus we conclude that \(\alpha_0\) and \(\beta\) are respectively lower and upper solutions of (3.4). Hence, by Theorems 2.1 and 2.2 there exists the unique solution \(\alpha_1\) of (3.4) such that

\[
\alpha_0(t) \leq \alpha_1(t) \leq \beta(t), \quad t \in [0, 1].
\]

Next, we consider

\[
u''(t) + \sigma u'(t) + \bar{F}(t, u, \alpha_1) = 0, \quad t \in [0, 1],
\]

\[
u(0) - \mu_1 \nu'(0) = \bar{g}_1(u(\gamma), \alpha_1, \beta) + \int_0^{\gamma_-} Q_1(u(s), \alpha_1(s)) \, ds
\]

\[
+ \int_{\gamma_+}^1 Q_1(u(s), \alpha_1(s)) \, ds,
\]

\[
u(1) + \mu_2 \nu'(1) = \bar{g}_2(u(\gamma), \alpha_1, \beta) + \int_0^{\gamma_-} Q_2(u(s), \alpha_1(s)) \, ds
\]

\[
+ \int_{\gamma_+}^1 Q_2(u(s), \alpha_1(s)) \, ds.
\]

Using the earlier arguments, it can be shown that \(\alpha_1\) and \(\beta\) are lower and upper solutions of (3.5) respectively and by Theorems 2.1 and 2.2, there exists the unique solution \(\alpha_2\) of (3.5) such that

\[
\alpha_1(t) \leq \alpha_2(t) \leq \beta(t), \quad t \in [0, 1].
\]

Continuing this process successively yields a sequence \(\{\alpha_n\}\) of solutions satisfying

\[
\alpha_0(t) \leq \alpha_1(t) \leq \alpha_2(t) \leq \cdots \leq \alpha_n(t) \leq \beta(t), \quad t \in [0, 1],
\]
Thus, \( \alpha_n(t) \) is a solution of the problem

\[
u''(t) + \sigma u'(t) + \tilde{F}(t, u, \alpha_{n-1}) = 0, \quad t \in [0, 1],
\]

\[
u(0) - \mu_1 u'(0) = \tilde{g}_1(u(\gamma), \alpha_{n-1}, \beta) + \int_{0}^{\gamma^-} Q_1(u(s), \alpha_{n-1}(s)) \, ds
\]

\[
+ \int_{\gamma^+}^{1} Q_1(u(s), \alpha_{n-1}(s)) \, ds,
\]

\[
u(1) + \mu_2 u'(1) = \tilde{g}_2(u(\gamma), \alpha_{n-1}, \beta) + \int_{0}^{\gamma^-} Q_2(u(s), \alpha_{n-1}(s)) \, ds
\]

\[
+ \int_{\gamma^+}^{1} Q_2(u(s), \alpha_{n-1}(s)) \, ds
\]

and is given by

\[
\alpha_n(t) = \frac{-(1 - \sigma \mu_2) e^{-\sigma} + e^{-\sigma t}}{1 + \sigma \mu_1} - (1 - \sigma \mu_2) e^{-\sigma} \left[ \tilde{g}_1(\alpha_n(\gamma), \alpha_{n-1}, \beta) + \int_{0}^{\gamma^-} Q_1(\alpha_n(s), \alpha_{n-1}(s)) \, ds + \int_{\gamma^+}^{1} Q_1(\alpha_n(s), \alpha_{n-1}(s)) \, ds \right]
\]

\[
+ \frac{(1 + \sigma \mu_1) - e^{-\sigma t}}{1 + \sigma \mu_1} - (1 - \sigma \mu_2) e^{-\sigma} \left[ \tilde{g}_2(\alpha_n(\gamma), \alpha_{n-1}, \beta) + \int_{0}^{\gamma^-} Q_2(\alpha_n(s), \alpha_{n-1}(s)) \, ds + \int_{\gamma^+}^{1} Q_2(\alpha_n(s), \alpha_{n-1}(s)) \, ds \right]
\]

\[
(3.6) \quad + \int_{0}^{1} G(t, s) \tilde{F}(s, \alpha_n(s), \alpha_{n-1}(s)) \, ds.
\]

Using the fact that \([0, 1]\) is compact and the monotone convergence of the sequence \(\{\alpha_n\}\) is pointwise, it follows that the convergence of the sequence is uniform. If \(u(t)\) is the limit point of the sequence, taking the limit \(n \to \infty\) in \((3.6)\), we obtain

\[
u(t) = \frac{-(1 - \sigma \mu_2) e^{-\sigma} + e^{-\sigma t}}{1 + \sigma \mu_1} - (1 - \sigma \mu_2) e^{-\sigma} \left[ g_1(u(\gamma)) + \int_{0}^{\gamma^-} q_1(u(s)) \, ds + \int_{\gamma^+}^{1} q_1(u(s)) \, ds \right]
\]

\[
+ \frac{(1 + \sigma \mu_1) - e^{-\sigma t}}{1 + \sigma \mu_1} - (1 - \sigma \mu_2) e^{-\sigma} \left[ g_2(u(\gamma)) + \int_{0}^{\gamma^-} q_2(u(s)) \, ds + \int_{\gamma^+}^{1} q_2(u(s)) \, ds \right]
\]

\[
+ \int_{0}^{1} G(t, s) f(s, u(s)) \, ds.
\]

Thus, \(u(t)\) is a solution of \((2.1)\). Now, we show that the convergence of the sequence is quadratic. For that we set \(\omega_n(t) = (u(t) - \alpha_n(t)) \geq 0, t \in [0, 1]\). In view of \((A_2)\),
it follows by Taylor’s theorem that

\[
\begin{align*}
\omega_n''(t) + \sigma \omega_n'(t) &= u'' + \sigma u' - (\alpha_n'' + \sigma \alpha_n') = -f(t, u) + \tilde{F}(t, \alpha_n, \alpha_{n-1}) \\
&= -f(t, u) + f(t, \alpha_{n-1}) + F_u(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1}) + \phi(t, \alpha_{n-1}) - \phi(t, \alpha_n) \\
&= -f_u(t, c_4)(u - \alpha_{n-1}) - F_u(t, \alpha_{n-1})(u - \alpha_n) + F_u(t, \alpha_{n-1})(u - \alpha_{n-1}) - \phi_u(t, c_5)(\alpha_n - \alpha_{n-1}) \\
&= \left[-f_u(t, c_4) + F_u(t, \alpha_{n-1}) - \phi_u(t, c_5)\right] \omega_{n-1} \\
&\quad + \left[-f_u(t, \alpha_{n-1}) + \phi_u(t, c_5)\right] \omega_n \\
&\geq \left[-f_u(t, \alpha_{n-1}) + \phi_u(t, \alpha_{n-1}) + \phi_u(t, c_5)\right] \omega_{n-1} \\
&\quad + \left[-f_u(t, \alpha_{n-1}) + \phi_u(t, c_5)\right] \omega_n \\
&\geq \left[-f_{uu}(t, c_6) - \phi_{uu}(t, c_7)\right] \omega_{n-1}^2 - f_u(t, \alpha_{n-1}) \omega_n \geq -A_1 \|\omega_{n-1}\|^2, \\
\end{align*}
\]

where \(\alpha_{n-1} \leq c_4, c_6 \leq u, \alpha_{n-1} \leq c_5, c_7 \leq \alpha_n, A\) is a bound on \(\|F_{uu}\|, B\) is a bound on \(\|\phi_{uu}\|\) and \(A_1 = A + B\). Further, we have

\[
\begin{align*}
\omega_n(0) - \mu_1 \omega_n'(0) &= g_1(u(\gamma)) - \tilde{g}_1(\alpha_n(\gamma), \alpha_{n-1}, \beta) \\
&\quad + \int_0^{\gamma^-} \left[q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\
&\quad + \int_{\gamma^+}^1 \left[q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\
&= g_1(u(\gamma)) - g_1(\alpha_n(\gamma)) - G'_1(\beta(\gamma))(\alpha_n - \alpha_{n-1}) \\
&\quad + \int_0^{\gamma^-} \left[q_1(u(s)) - q_1(\alpha_n(s)) - K'_1(\alpha_n(s))(\alpha_n - \alpha_{n-1}) - \chi_1(\alpha_{n-1}) \\
&\quad + \chi_1(\alpha_n)\right] ds + \int_{\gamma^+}^1 \left[q_1(u(s)) - q_1(\alpha_{n-1}(s)) - K'_1(\alpha_{n-1}(s))(\alpha_n - \alpha_{n-1}) \\
&\quad - \chi_1(\alpha_{n-1}) + \chi_1(\alpha_n)\right] ds \\
&\leq \left[\frac{1}{2} g''_1(\xi_1) + \psi''(\eta_2)\right] \omega_{n-1}^2(\gamma) + \left[G'_1(\beta(\gamma)) - \psi'(\eta_1)\right] \omega_n(\gamma) \\
&\quad + \int_0^{\gamma^-} \left[(K'_1(\alpha_{n-1}(s)) - \chi_1'(\eta_3)) \omega_n(s) + \left(\frac{1}{2} q_1''(\xi_2) + \chi_1''(\eta_4)\right) \omega_{n-1}^2(s)\right] ds \\
&\quad + \int_{\gamma^+}^1 \left[(K'_1(\alpha_{n-1}(s)) - \chi_1'(\eta_3)) \omega_n(s) + \left(\frac{1}{2} q_1''(\xi_2) + \chi_1''(\eta_4)\right) \omega_{n-1}^2(s)\right] ds
\end{align*}
\]
and
\[
\omega_n(1) + \mu_2 \omega_n'(1) = g_2(u(\gamma)) - \bar{g}_2(\alpha_n(\gamma), \alpha_{n-1}, \beta) \\
+ \int_0^{\gamma_-} [q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))] \, ds \\
+ \int_{\gamma_+}^{1} [q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))] \, ds \\
\leq \left[ \frac{1}{2} g''_2(\xi_3) + \psi''_2(\eta_0) \right] \omega_{n-1}^2(\gamma) + \left[ G'_2(\beta(\gamma)) - \psi'_2(\eta_0) \right] \omega_n(\gamma) \\
+ \int_0^{\gamma_-} \left[ (K'_2(\alpha_{n-1}(s)) - \chi'_2(\eta_7)) \omega_n(s) + \left( \frac{1}{2} g''_2(\xi_4) + \chi''_2(\eta_8) \right) \omega_{n-1}^2(s) \right] \, ds \\
+ \int_{\gamma_+}^{1} \left[ (K'_2(\alpha_{n-1}(s)) - \chi'_2(\eta_7)) \omega_n(s) + \left( \frac{1}{2} g''_2(\xi_4) + \chi''_2(\eta_8) \right) \omega_{n-1}^2(s) \right] \, ds,
\]
where \( \alpha_{n-1} \leq \xi_j \leq u, \ j = 1, \ldots, 4, \alpha_{n-1} \leq \eta_\nu \leq \alpha_n \leq u, \nu = 1, \ldots, 8. \) In view of (A_3) and (A_4), there exists \( \lambda_i < 1, \lambda^*_i < 1, \ M_i \geq 0 \) and \( M^*_i \geq 0 \) such that \( |G'_i - \psi'_i| \leq \lambda_i, \ |K'_i - \chi'_i| \leq \lambda_i, \ |\frac{1}{2} g''_i + \chi''_i| \leq M_i \) and \( |\frac{1}{2} g''_i + \psi''_i| \leq M^*_i. \) Letting \( \lambda = \max\{\lambda_1, \lambda_2\}, \lambda^* = \max\{\lambda^*_1, \lambda^*_2\}, \ M^* = \max\{M^*_1, M^*_2\}, \) and \( M = \max\{M_1, M_2\}, \) we get
\[
\begin{align*}
\omega_n(0) - \mu_1 \omega_n'(0) & \leq M^* \omega_{n-1}^2(\gamma) + \lambda^* \omega_n(\gamma) \\
& + \lambda \left[ \int_0^{\gamma_-} \omega_n(s) \, ds + \int_{\gamma_+}^{1} \omega_n(s) \, ds \right] \\
& + M \left[ \int_0^{\gamma_-} \omega_{n-1}^2(s) \, ds + \int_{\gamma_+}^{1} \omega_{n-1}^2(s) \, ds \right],
\end{align*}
\]
(3.8)
\[
\omega_n(1) + \mu_2 \omega_n'(1) \leq M^* \omega_{n-1}^2(\gamma) + \lambda^* \omega_n(\gamma) \\
+ \lambda \left[ \int_0^{\gamma_-} \omega_n(s) \, ds + \int_{\gamma_+}^{1} \omega_n(s) \, ds \right] \\
+ M \left[ \int_0^{\gamma_-} \omega_{n-1}^2(s) \, ds + \int_{\gamma_+}^{1} \omega_{n-1}^2(s) \, ds \right].
\]
Using the estimates \( (3.7) \) and \( (3.8) \), we obtain
\[
\omega_n(t) = \frac{- (1 - \sigma \mu_2)e^{-\sigma t} + e^{-\sigma t}}{(1 + \sigma \mu_1) - (1 - \sigma \mu_2)e^{-\sigma}} \left( g_1(u(\gamma)) - \bar{g}_1(\alpha_n(\gamma), \alpha_{n-1}, \beta) \right) \\
+ \int_0^{\gamma_-} [q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))] \, ds \\
+ \int_{\gamma_+}^{1} [q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))] \, ds \\
+ \frac{(1 + \sigma \mu_1) - e^{-\sigma t}}{(1 + \sigma \mu_1) - (1 - \sigma \mu_2)e^{-\sigma}} \left( g_2(u(\gamma)) - \bar{g}_2(\alpha_n(\gamma), \alpha_{n-1}, \beta) \right)
\]
where \( A \) the sequence of iterates.

The results obtained in [2] appear as a special case of our results if we take \( \gamma = 1/2 \) in (2.1) and \( \psi_i \equiv 0 \equiv \chi_i, i = 1, 2 \) in the assumptions (A3) and (A4) of Theorem 3.1.

\[
\begin{align*}
&+ \int_0^{\gamma^-} [q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))] ds \\
&+ \int_0^{\gamma^+} [q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))] ds \\
&+ \int_0^1 G(t, s) [f(s, u(s)) - F(t, \alpha_n, \alpha_{n-1})] ds \\
&\leq -\frac{(1 - \sigma \mu_2) e^{-\sigma} + e^{-\sigma t}}{(1 + \sigma \mu_1) - (1 - \sigma \mu_2)e^{-\sigma}} [M^* \omega_{n-1}^2(\gamma) + \lambda^* \omega_n(\gamma) \\
&+ \lambda \left( \int_0^{\gamma^-} \omega_n(s) ds + \int_0^{\gamma^+} \omega_n(s) ds \right) + M \left( \int_0^{\gamma^-} \omega_{n-1}^2(s) ds + \int_0^{\gamma^+} \omega_{n-1}^2(s) ds \right)] \\
&+ \frac{(1 + \sigma \mu_1) - e^{-\sigma t}}{(1 + \sigma \mu_1) - (1 - \sigma \mu_2)e^{-\sigma}} [M^* \omega_{n-1}^2(\gamma) + \lambda^* \omega_n(\gamma) \\
&+ \lambda \left( \int_0^{\gamma^-} \omega_n(s) ds + \int_0^{\gamma^+} \omega_n(s) ds \right) + M \left( \int_0^{\gamma^-} \omega_{n-1}^2(s) ds + \int_0^{\gamma^+} \omega_{n-1}^2(s) ds \right)] \\
&- \int_0^1 G(t, s) [\omega''_n(s) + \sigma \omega'_n(s)] ds \\
&\leq M^* \omega_{n-1}^2(\gamma) + \lambda^* \omega_n(\gamma) + \lambda \left( \int_0^{\gamma^-} \omega_n(s) ds + \int_0^{\gamma^+} \omega_n(s) ds \right) \\
&+ M \left( \int_0^{\gamma^-} \omega_{n-1}^2(s) ds + \int_0^{\gamma^+} \omega_{n-1}^2(s) ds \right) + A_1 \|\omega_{n-1}\|^2 + \int_0^1 G(t, s) ds \\
&\leq M^* \|\omega_{n-1}\|^2 + \lambda^* \|\omega_n\| + \lambda \|\omega_n\| + M \|\omega_{n-1}\|^2 + A_2 \|\omega_{n-1}\|^2 \\
&= \lambda^{**} \|\omega_n\| + M^{**} \|\omega_{n-1}\|^2
\end{align*}
\]

where \( A_2 \) provides a bound on \( A_1 \int_0^1 G(t, s) \). We choose \( \lambda^* \) and \( \lambda \) so that \( \lambda^{**} = \lambda^* + \lambda < 1 \) and \( M^{**} = M^* + M + A_2 \). Taking the maximum over \([0, 1]\), we get

\[
\|\omega_n\| \leq \frac{M^{**}}{1 - \lambda^{**}} \|\omega_{n-1}\|^2,
\]

where \( \|u\| = \max \{ |u(t)| : t \in [0, 1] \} \). This establishes the quadratic convergence of the sequence of iterates.

\[ \square \]

**Remark.** The results obtained in [2] appear as a special case of our results if we take \( \gamma = 1/2 \) in (2.1) and \( \psi_i \equiv 0 \equiv \chi_i, i = 1, 2 \) in the assumptions (A3) and (A4) of Theorem 3.1.

\[ \text{References} \]


**Department of Mathematics, Faculty of Science**

**King Abdulaziz University**

**P.O. Box. 80257, Jeddah 21589, Saudi Arabia**

**E-mail:** aalsaedi@hotmail.com