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LATTICE-VALUED BOREL MEASURES III

Surjit Singh Khurana

Abstract. Let \( X \) be a completely regular \( T_1 \) space, \( E \) a boundedly complete vector lattice, \( C(X) \) the space of all (all, bounded), real-valued continuous functions on \( X \). In order convergence, we consider \( E \)-valued, order-bounded, \( \sigma \)-additive, \( \tau \)-additive, and tight measures on \( X \) and prove some order-theoretic and topological properties of these measures. Also for an order-bounded, \( E \)-valued (for some special \( E \)) linear map on \( C(X) \), a measure representation result is proved. In case \( E^*_n \) separates the points of \( E \), an Alexanderov’s type theorem is proved for a sequence of \( \sigma \)-additive measures.

1. Introduction and notation

All vector spaces are taken over reals. \( E \), in this paper, is always assumed to be a Dedekind complete Riesz space (and so, necessarily Archimedean) ([1], [15], [14]). For a completely regular \( T_1 \) space \( X \), \( vX \) is the real-compactification, \( \tilde{X} \) is the Stone-Čech compactification of \( X \), \( B(X) \) is the space of all real-valued bounded functions on \( X \), \( C(X) \) (resp. \( C_b(X) \)) is the space of all real-valued, (resp. real-valued and bounded) continuous functions on \( X \); sets of the form \( \{ f^{-1}(0); f \in C_b(X) \} \) are called zero-sets of \( X \) and their complements positive subsets of \( X \), and the elements of the \( \sigma \)-algebra generated by zero-sets are called Baire sets ([20], [19]); \( B(X) \) and \( B_1(X) \) will denote the classes of Borel and Baire subsets of \( X \) and \( F(X) \) will be the algebra generated by the zero-sets of \( X \). \( \beta_1(X)(\beta(X)) \) are, respectively the spaces of bounded Baire (Borel) measurable functions on \( X \). It is easily verified that the order \( \sigma \)-closure of \( C_b(X) \) in \( \beta_1(X) \), in the topology of pointwise convergence, is \( \beta_1(X) \) and the order \( \sigma \)-closure, in \( \beta(X) \), of the vector space generated by bounded lower semi-continuous functions on \( X \), is \( \beta(X) \) ([3], [4]).

In ([21], [23]), the author discussed the positive measures taking values in Dedekind complete Riesz spaces and proved some basic results about the integration relative to these measures; he also proves some Riesz representation type theorems; it was proved there that when \( X \) is a compact Hausdorff space and \( \mu: C(X) \to E \) is a positive linear mapping then \( \mu \) arises from a unique quasi-regular Borel measure \( \mu: B(X) \to E \) which is countably additive in order convergence (quasi-regular means that the measure of any open set is inner regular by the compact subsets

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of $X$). In [7, 8] new proofs were given for these Riesz representation theorems for positive measures and then the study was extended to completely regular $T_1$ spaces and $\sigma$-additive, $\tau$-additive and tight positive measures were studied on these spaces. In [17, 18], some decomposition theorems for measures, which take values in Dedekind complete Riesz spaces and are not necessarily positive, were proved. In [16], the authors proved some results about the countable additivity of the order-theoretic modulus of a countable additive measures taking values in a Banach lattice.

In the present paper, we consider measures, not necessarily positive, on completely regular $T_1$ spaces, taking values in Dedekind complete Riesz spaces. In Section 2 some order-theoretic and topological properties of $\sigma$-additive, $\tau$-additive and tight measures are proved. In Section 3 a well-known result about the measure representation of real-valued, order-bounded linear map on $C(X)$ is extended to the case when the order-bounded linear map on $C(X)$ takes values in $C(S)$, $S$ being a Stone space. In Section 4 assuming that the continuous order dual $E'_\tau$ separates the points of $E$, an Alexanderov’s type theorem is proved about a sequence of $\sigma$-additive measures.

For locally convex spaces and vector lattices, we will be using notations and results for (15, 1, 13). For a locally convex space $E$ with $E'$ its dual, with an $x \in E$ and $f \in E'$, $\langle f, x \rangle$ will stand for $f(x)$. For measures, results and notations from (21, 10, 2) will be used, and for lattice-valued measures, results of (17, 18) will be used.

2. ORDER-BOUNDED MEASURES ON COMPLETELY REGULAR $T_1$ SPACE IN ORDER CONVERGENCE

We start with a compact Hausdorff space $X$ and an order-bounded, countably additive (countable additivity in the order convergence of $E$) Borel measure $\mu: \mathcal{B}(X) \to E$ be an order-bounded, countably additive (countable additivity in the order convergence of $E$) Borel measure on $X$, having the property that for any decreasing net $\{C_\alpha\}$ of closed subsets of $X$, $\mu(\cap C_\alpha) = \omega - \lim \mu(C_\alpha)$ (if $\mu$ has this property then we say $\mu$ is $\tau$-smooth). We first prove the following theorem.

**Theorem 1.** Suppose $X$ is a compact Hausdorff space and $\mu: \mathcal{B}(X) \to E$ be an order-bounded, countably additive (countable additivity in the order convergence of $E$) Borel measure on $X$, having the property that for any decreasing net $\{C_\alpha\}$ of closed subsets of $X$, $\mu(\cap C_\alpha) = \omega - \lim \mu(C_\alpha)$. Let $\{f_\alpha\}$ be a net of $[0, 1]$-valued, upper semi-continuous functions on $X$, decreasing pointwise to a function $f$ on $X$. Then $\omega - \lim \mu(f_\alpha) = \mu(f)$.

**Proof.** Since $\mu$ is order-bounded, we can take $E = C(S)$, $S$ being a compact Stone space and $|\mu(\mathcal{B}(X))| \leq 1 \in C(S)$; this implies, that for any Borel function $h: X \to [-1, 1]$, $|\mu(h)| \leq 1$. Fix a $k \in N$ and let $Z_\alpha^i = f_\alpha^{-1}[\frac{i}{k}, 1]$ and $Z^i = f^{-1}[\frac{i}{k}, 1]$, for $i = 1, 2, \ldots, (k - 1)$. By hypothesis, $\omega - \lim_\alpha \mu(Z_\alpha^i) = \mu(Z^i)$, $\forall i$. We have $\frac{1}{k} \sum_{i=1}^{k-1} Z_\alpha^i \leq f_\alpha \leq \frac{1}{k} \sum_{i=1}^{k-1} Z^i$ and $\frac{1}{k} \sum_{i=1}^{k-1} Z^i \leq f \leq \frac{1}{k} \sum_{i=1}^{k-1} Z^i$. This implies $|f_\alpha - \frac{1}{k} \sum_{i=1}^{k-1} Z_\alpha^i| \leq \frac{1}{k}$ and $|f - \frac{1}{k} \sum_{i=1}^{k-1} Z^i| \leq \frac{1}{k}$. This gives $|\mu(f_\alpha) - \frac{1}{k} \sum_{i=1}^{k-1} \mu(Z_\alpha^i)| \leq \frac{1}{k}$ and $|\mu(f) - \frac{1}{k} \sum_{i=1}^{k-1} \mu(Z^i)| \leq \frac{1}{k}$. So $-\frac{1}{k} \sum_{i=1}^{k-1} \mu(Z_\alpha^i) \leq \mu(f_\alpha) \leq \frac{1}{k} \sum_{i=1}^{k-1} \mu(Z^i) \leq \frac{1}{k} \sum_{i=1}^{k-1} \mu(Z_\alpha^i)$.
Theorem 2. Suppose we get \( E \) \( \mu \) is easily verified. \( \mu \) \( \mu \) \( \leq 0 \) \( \beta \) lower semi-continuous functions is \( \beta \) \( \mu \) \( \mu \) \( \leq 2 \frac{k}{k} \). Combining these two, we get \( o \) \( \lim \sup \alpha \mu(f_\alpha) - o \) \( \lim \inf \alpha \mu(f_\alpha) \leq 2 \frac{k}{k} \). Letting \( k \to \infty \), \( o \) \( \lim \mu(f_\alpha) \) exists. Using the fact that \( \mu(f) - p \leq \frac{1}{k} \), we get \( o \) \( \lim \mu(f_\alpha) - \mu(f) \leq \frac{2}{k} \). Letting \( k \to \infty \), we get the result. □

We denote by \( M_{(\alpha)}(X, E) \) the set of all order-bounded linear mappings \( \mu: C(X) \to E \). Now we come to the next theorem.

**Theorem 2.** Suppose \( X \) is a compact Hausdorff space and \( \mu: C(X) \to E \) be an order-bounded, linear mapping.

(i) Then there is a unique countably additive Baire measure, which again we denote by \( \mu \), on \( X \), such that the corresponding linear mapping \( \mu: \beta_1(X) \to E \) extends the given mapping. Further \( \mu \) can also be uniquely extended to a countably additive \( \tau \)-smooth Baire measure.

(ii) The modulus of the Baire measure \( \mu \), determined from \( \mu: C(X) \to E \) and \( \mu: \beta_1(X) \to E \) are equal and also modulus of the Baire measure \( \mu \), determined from \( \mu: C(X) \to E \) and \( \mu: \beta(X) \to E \) are equal. Thus \( \mu \) can be written as \( \mu = \mu^+ - \mu^- \). For every \( \tau \)-smooth Baire measure \( \mu \) on \( X \), there is the largest open set \( V \subset X \) such that \( |\mu|(V) = 0 \); \( C = X \setminus V \) is called the support of \( \mu \) and has the property that any open \( U \subset X \) such that \( U \cap C \neq \emptyset \), we have \( |\mu|(U) > 0 \).

(iii) \( M_{(\alpha)}(X, E) \) is a Dedekind-complete vector lattice.

**Proof.** (i) Since \( \mu \) is order-bounded and \( E \) is a boundedly order-complete, we can write \( \mu = \mu^+ - \mu^- \) ([13] Theorem 1.3.2, p. 24]). Now \( \mu^+ \) and \( \mu^- \) can be uniquely extended to \( E^+ \)-valued, countably additive Baire measures and also to \( E^+ \)-valued, countably additive \( \tau \)-smooth Baire measures ([7], [21], [24]). Thus we get a countably additive Baire measure \( \mu: \beta_1(X) \to E \) and a countably additive \( \tau \)-smooth Baire measure \( \mu: \beta(X) \to E \). Since the order \( \sigma \)-closure, in \( \beta_1(X) \), of \( C(X) \) is \( \beta_1(X) \), for Baire measure, the uniqueness follows. Now we consider the case of Baire measure. Suppose two \( \tau \)-smooth Baire measures \( \mu_1, \mu_2 \) are equal on \( C(X) \). By Theorem [1] they are equal on bounded lower semi-continuous functions and so they are equal on the vector space generated by lower semi-continuous functions. Since the order \( \sigma \)-closure, in \( \beta(X) \), of the vector space generated by lower semi-continuous functions is \( \beta(X) \), by countable additivity they are equal on \( \beta(X) \).

(ii) Let \( \mu_1, \mu_2 \) be the \( \mu^+ \)'s coming from \( \mu: C(X) \to E \) and \( \mu: \beta_1(X) \to E \) respectively. Evidently \( \mu_2 \geq \mu_1 \). Fix a \( g \in C(X) \), \( g \geq 0 \) and take an \( h \in \beta_1(X) \), \( 0 \leq h \leq g \). Since \( \mu(h) \leq \mu_1(g) \), taking \( \sup_{0 \leq h \leq g} \), we get \( \mu_2(g) \leq \mu_1(g) \). By ([18], Theorem 2.3, p.25), \( \mu_2 \) is countably additive. Since \( \mu_1 = \mu_2 \) on \( C(X) \), we get \( \mu_1 = \mu_2 \) on \( \beta_1(X) \). The result follows now. The other result about the support of \( \mu \) is easily verified. □

(iii) It is a simple verification.
Now we consider the case when \( X \) is a completely regular \( T_1 \) space and \( \mu: \mathcal{F}(X) \rightarrow E \) a finitely additive, order-bounded measure. Because of order-boundedness, order modulus \(|\mu|\) exists. \( \mu \) will be called regular if for any \( A \in \mathcal{F}(X) \), there exists an increasing net \( \{Z_\alpha\} \) of zero-sets in \( X \), \( Z_\alpha \subset A \), \( \forall \alpha \), and a deceasing net \( \{\eta_\alpha\} \) in \( E \) such that \( \eta_\alpha \downarrow 0 \) and \(|\mu|(A \setminus Z_\alpha) < \eta_\alpha \), \( \forall \alpha \).

**Theorem 3.** Suppose \( X \) be a completely regular \( T_1 \) space and \( \mu: C_b(X) \rightarrow E \) be an order-bounded, linear mapping. Then there is unique, finitely additive, order-bounded measure, regular measure \( \nu: \mathcal{F}(X) \rightarrow E \) such that \( \mu(f) = \int f \, d\nu, \forall f \in C_b(X) \).

\( M_{(o)}(X, E) \) is a Dedekind-complete vector lattice.

**Proof.** When \( \mu \) is positive, then result is proved in ([12], p. 353). Since \( \mu = \mu^+ - \mu^- \), using the result ([12], p. 353), we get a \( \nu \) with the required properties. We denote \( \nu \) by \( \mu \) also.

Uniqueness: Let \( \mu: \mathcal{F}(X) \rightarrow E \) be an order-bounded, finitely additive, order-bounded measure such that \( \mu = 0 \) on \( C_b(X) \). Denoting by \( S(X) \) the norm closure of \( \mathcal{F}(X) \)-simple real valued functions on \( X \), we have \( S(X) \supset C_b(X) \). Thus \( \mu \) extends to \( \mu: S(X) \rightarrow E \), is linear and order-bounded. Split \( \mu = \mu^+ - \mu^- \).

By the definition of regularity, \(|\mu|\) is regular and so \( \mu^+, \mu^- \) are regular and \( \mu^+ = \mu^- \) on \( C_b(X) \). Since both are regular, there is unique extension to \( \mathcal{F}(X) \). This means \( \mu^+ = \mu^- \) on \( \mathcal{F}(X) \) and consequently \( \mu^+ = \mu^- \) on \( S(X) \). This proves uniqueness. It is easy to verify that \( M_{(o)}(X, E) \) is a Dedekind-complete vector lattice. \( \square \)

We come to countably additive (in order convergence), of order-bounded Baire measures on a completely regular \( T_1 \) space \( X \). A countably additive, order-bounded \( \mu: \mathcal{B}_1(X) \rightarrow E \) is called an order-bounded Baire measure on \( X \). The collection of all such measures will be denoted by \( M_{(o, \sigma)}(X, E) \).

**Theorem 4.** For a be a completely regular \( T_1 \) space \( X \), \( M_{(o, \sigma)}(X, E) \) is a band in \( M_{(o)}(X, E) \).

**Proof.** Take a \( \mu \in M_{(o, \sigma)}(X, E) \). By ([18], Theorem 2.3, p.25 ), \(|\mu|, \mu^+, \mu^- \) are also in \( M_{(o, \sigma)}(X, E) \). so \( M_{(o, \sigma)}(X, E) \) is a vector sublattice of \( M_{(o)}(X, E) \). Let \( \{\mu_\alpha\} \) be positive, bounded, increasing net in \( M_{(o, \sigma)}(X, E) \) and \( \mu = \sup \mu_\alpha \) in \( M_{(o)}(X, E) \). Then \( \mu \), defined for every \( A \in \mathcal{B}_1(X) \), \( \mu(A) = \sup \mu_\alpha(A) \), is finitely additive. Take an increasing sequence \( \{A_n\} \subset \mathcal{B}_1(X) \) and let \( A = \bigcup A_n \). Now \( \mu(A) = o - \lim_n \mu_\alpha(A) = o - \lim_n (o - \lim_n \mu_\alpha(A_n)) \leq o - \lim_n \mu(A_n) \leq \mu(A) \). This proves \( \mu \) is countably additive. This proves the result. \( \square \)

We denote by \( M_{(o, \tau)}(X, E) \) those \( \mu \in M_{(o, \sigma)}(X, E) \) which can be extended to \( \mu: \mathcal{B}(X) \rightarrow E \) and are \( \tau \)-smooth, in the sense, that for any increasing net \( \{V_\alpha\} \) of open subsets of \( X \), \( \mu(\bigcup V_\alpha) = o - \lim \mu(V_\alpha) \) (extension will obviously be unique if it exists).

**Theorem 5.** For a completely regular \( T_1 \) space \( X \), \( M_{(o, \tau)}(X, E) \) is a band in \( M_{(o, \sigma)}(X, E) \).

**Proof.** Take a \( \mu \in M_{(o, \tau)}(X, E) \). This gives a \( \bar{\mu} \in M_{(o)}(\bar{X}, E) \), \( \bar{\mu}(B) = \mu(B \cap X) \) with the property that \( \bar{\mu}(B) = 0 \) if \( B \cap X = \emptyset \). It is a routine verification that \( \bar{\mu}^+, (\bar{\mu})^- \), \(|\bar{\mu}| \) all are \( = 0 \) on those Borel sets \( B \) for which \( B \cap X = \emptyset \). For this it easily
follows that, for any Borel set $B \subset X$, $\mu^+(B) = (\bar{\mu})^+(B_0)$, where $B_0$ is any Borel subset of $\tilde{X}$ with $B_0 \cap X = B$; similar result for $\mu^-$ and $|\mu|$. To prove $\tau$-smoothness of $|\mu|$, take a collection $\{V_\gamma; \gamma \in I\}$ of open subsets of $X$ and select open subsets $\{U_\gamma; \gamma \in I\}$ in $\tilde{X}$ such that $U_\gamma \cap X = V_\gamma$. Let $J$ be the collection of all finite subsets of $I$ and order them by inclusion; also denote by $\alpha$ a general element of $J$. By the $\tau$-smooth property of $[\bar{\mu}]$ (Theorem 2), we have, $|\bar{\mu}|(\cup U_\gamma) = o - \lim_\alpha |\bar{\mu}|(\cup_{\gamma \in \alpha} U_\gamma)$. This means $|\mu|(\cup V_\gamma) = o - \lim_\alpha |\mu|(\cup_{\gamma \in \alpha} V_\gamma)$. This proves $|\mu|$ in $\tau$-smooth. In a similar way $\mu^+$ and $\mu^-$ are also $\tau$-smooth.

Now the proof that it is a band in $M_{(\alpha,\sigma)}(X, E)$ is very similar to what is done in Theorem 4. □

We denote by $M_{(\alpha,t)}(X, E)$ those $\mu \in M_{(\alpha,\tau)}(X, E)$ which have the property that, for the measure $|\mu|$, open sets are inner regular by the compact subsets of $X$. From this definition it follows that if $\mu \in M_{(\alpha,t)}(X, E)$ then $\mu^+, \mu^-, |\mu|$ are also in $M_{(\alpha,t)}(X, E)$.

**Theorem 6.** For a completely regular $T_1$ space $X$, $M_{(\alpha,t)}(X, E)$ is a band in $M_{(\alpha,\tau)}(X, E)$.

**Proof.** $M_{(\alpha,t)}(X, E)$ is already seen to be a vector sub-lattice of $M_{(\alpha,\tau)}(X, E)$. Let $\{\mu_\alpha\}$ be positive, bounded, increasing net in $M_{(\alpha,t)}(X, E)$ and $\mu = \sup \mu_\alpha$ in $M_{(\alpha,\tau)}(X, E)$. Let $V$ be an open subset of $X$. Let $\{C_\beta\}$ be the family of all compact subsets of $V$; this is filtering upwards. $\mu(V) = o - \lim_\alpha \mu_\alpha(V) = o - \lim_\alpha (o - \lim_\beta \mu_\alpha(C_\beta)) \leq o - \lim_\beta \mu(C_\beta) \leq \mu(V)$. This proves $\mu \in M_{(\alpha,t)}(X, E)$. This proves the result. □

If $\mu \in M_{(\alpha,\tau)}(X, E)$, then it is easily seen that there is a smallest closed subset $Y \subset X$ such that $|\mu|(Y) = |\mu|(X)$. This $Y$ is called the support of $\mu$.

The following two theorems are well-known for scalar-valued measures ([20, 19]). We prove some extensions.

**Theorem 7.** Let $(X, d)$ be a metric space and $E$ super Dekekind complete ([14, p.78]) and $\mu \in M_{(\alpha,\tau)}(X, E^+)$. Then the support of $\mu$ is a separable subset of $X$.

**Proof.** Let the support of $\mu$ be $Y$. Fix an $n \in N$ and let $A = \{A \subset Y : d(x, y) \geq \frac{1}{n}, \forall x \in A, \forall y \in A, x \neq y\}$. By Zorn’s Lemma, $A$ has a maximal element, say $A_n$. It is easily verified that that for any $x \in (Y \setminus A_n)$, there is a $y \in A_n$ such that $d(x, y) < \frac{1}{n}$. We claim that $A_n$ is countable. Suppose not. Thus there is an uncountable collection $\{B(x, \frac{1}{2n}) : x \in A_n\}$ of mutually disjoint open subsets of $Y$ and $\mu(B(x, \frac{1}{2n})) > 0, \forall x \in A_n$. Using $\tau$-additivity of $\mu$ and the hypothesis that $E$ is super Dekekind complete, we get, that except for countable $x \in A_n$, $\mu(B(x, \frac{1}{2n})) = 0$. Since $Y$ is the support of $\mu$, this is a contradiction. Thus $A_n$ is countable and so $\cup A_n$ is dense in $Y$. This proves the result. □

**Theorem 8.** Let $(X, d)$ be a complete metric space and $E$ super Dekekind complete and also weakly $\sigma$-distributive ([25]). Then $M_{(\alpha,\tau)}(X, E) = M_{(\alpha,t)}(X, E)$.

**Proof.** Take a $\mu \in M_{(\alpha,\tau)}(X, E^+)$. By Theorem 7, we can assume $X$ to be separable. Let $Z$ be a compact metric space which is a compactification of $X$. It is well-known
that $X$ is a $G_\delta$ set in $Z$. Define $\tilde{\mu}: \mathcal{B}(Z) \to E^+$, $\tilde{\mu}(B) = \mu(B \cap X)$. It is obvious that $\tilde{\mu} \in M_{(0)}(Z, E^+)$. It is Baire measure. Since $E$ is weakly $\sigma$-distributive, $\tilde{\mu}$ is inner regular by compact subset of $Z$. This means, since $X$ is a Baire subset of $Z$, $\mu(X) = \sup \{\mu(C) : C \text{ compact and } C \subset X\}$. From this, it is a routine verification that $\mu \in M_{(0,1)}(X, E)$ (cf. [5]).

3. Representation theorem for $C(X)$, $X$ completely regular

It is well-known that a linear map $\mu: C(X) \to R$, which maps order-bounded sets into bounded sets, gives a unique $\nu \in M_\sigma(X)$ such that $C(X) \subset L^1(\nu)$, $\mu(f) = \int fd\nu$, $\forall f \in C(X)$ and $\text{supp}(\tilde{\nu}) \subset vX$ (the real-compactification of $X$) ([9, Theorem 23]). We will extend it to the vector case.

In this section $E = (C(S), \|\cdot\|)$, $S$ being a Stone space and $X$ completely regular $T_1$ space. We will prove a representation theorem for a positive linear map $\mu: C(X) \to E$. $B(X)$ denotes the space of all bounded real-valued functions. We will use the following results.

(A) Suppose $F$ is a locally convex space whose topology is generated by the family $\{\|\cdot\|_p : p \in P\}$ of semi-norms, $M_\sigma(X, F)$ the space of all $F$-valued Baire measures on $X$, and $\mu: C(X) \to F$ a linear map such that order-bounded subsets are mapped into relatively weakly compact subsets of $F$. Then:

(i) There is a unique $\nu \in M_\sigma(X, F)$ such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \int fd\nu$, $\forall f \in C(X)$;

(ii) for every $p \in P$, there is compact $C \subset vX$ (the real-compactification of $X$), depending on $p$, such that $\tilde{\nu}_p(\hat{X} \setminus C) = 0$ ([9, Theorem 7]), $\tilde{\nu}_p$ being the semi-variation of $\nu$.

(B) There is an order $\sigma$-continuous positive linear map $\psi_1: \beta_1(S) \to C(S)$ such that for every $f \in \beta_1(S)$, we get $f = \psi_1(f) = 0$ except on a meager set ([7, Lemma 2, p. 379]).

In the following theorem countable additivity is taken in the context of order convergence and integration and integrability in the sense of [21].

Theorem 9. Suppose $\mu: C(X) \to E$ be a positive linear map. Then there is a unique $E$-valued positive Baire measure $\nu$ on $X$ such that $\forall f \in C(X)$ is $\nu$-integrable and $\mu(f) = \int fd\nu$, $\forall f \in C(X)$. Also the supp$(\tilde{\nu}) \subset vX$.

Proof. By taking the pointwise topology $pt$ on $B(S)$ and noting that $C(S) \subset B(S)$, we have a positive linear map $\mu: C(X) \to (B(S), pt)$ with the property that order-bounded subsets of $C(X)$ are mapped into relatively weakly compact subsets of $(B(S), pt)$. By (A) there is a Baire measure $\lambda: \mathcal{B}_1(X) \to (B(S), pt)$ such that $C(X) \subset L_1(\lambda)$ ([10]) and $\mu(f) = \int fd\lambda$, $\forall f \in C(X)$. This measure is easily seen to be positive. Fix an $f \in C(X)$, $f \geq 0$ and let $f_n = f \wedge n$ ($n \in N$). Put $h = \mu(f)$, $h_n = \mu(f_n)$. Since $f \in L_1(\lambda)$, $\lambda(f_n) \to \lambda(f)$ ([10]). From $\lambda^{-1}(\beta_1(S)) \supset C_0(X)$, we get $\lambda^{-1}(f_1(S)) \supset h_1(X)$. Thus $\lambda: \mathcal{B}_1(X) \to \beta_1(S)$. Using (B) and defining $\nu = \psi_1 \circ \lambda$, we see that $\nu: \mathcal{B}_1(X) \to C(S)$ is countably additive in order convergence and $h_n = \mu(f_n) = \lambda(f_n) = \nu(f_n)$, $\forall n$. This means $h_n \uparrow h$ pointwise in $C(S)$ and so $o - \lim h_n = h$ in $C(S)$. By ([21, Prop. 3.3, p. 113]) $f$ is $\nu$-integrable.
and \( \int fd\nu = o - \lim \int f_n d\nu = o - \lim h_n = h = \lim h_n \) pointwise. This proves \( \mu(f) = \int fd\nu \). This proves the result.

Uniqueness: If there is another \( E \)-valued positive Baire measure \( \nu_0 \) on \( X \) having the above properties then \( \mu(f) = \int fd\nu_0, \forall f \in C(X) \). Thus \( \nu_0(f) = \nu(f), \forall f \in C_b(X) \). Because of order countable additivity of \( \nu_0 \) and \( \nu \), we get \( \nu_0 = \nu \) on Baire subsets of \( X \). This proves uniqueness.

Now we prove that \( \text{supp}(\tilde{\nu}) \subset \nu X \). Suppose \( z \in \tilde{X} \setminus \nu X \) and \( z \in (\text{supp})(\tilde{\mu}) \). Take an \( f \geq 0, f \in C(X) \) with \( \tilde{f}(z) = \infty \). Thus, for every \( n, \tilde{\mu}(A_n) = 0 \) where \( A_n = \{ x : \tilde{f}(x) > n \} \).

Suppose first that \( \bigwedge_{n=1}^{\infty}(\tilde{\mu}(A_n)) = h > 0 \) and put \( f_n = f \wedge n \). Then \( \tilde{f}_n = \tilde{f} \wedge n \). Now \( \mu(f) \geq \mu(f_n) = \tilde{\mu}(f \wedge n) = \int(\tilde{f} \wedge n)d\tilde{\mu} \geq n\tilde{\mu}(A_n) \geq nh \). Since \( E \) is Archimedean, we get \( h = 0 \) which is a contradiction. Thus \( h = 0 \).

Since \( \tilde{\mu}(A_n) > 0 \) for every \( n \) and \( h = 0 \), select a strictly increasing sequence \( \{a_k\} \) of positive integers such that \( a_{k+1} - a_k > 4 \forall k \) and \( h_k = \tilde{\mu}(\{ x : a_{k+1} < \tilde{f}(x) < a_{k+2} \}) > 0, \forall k \). Let \( p_k = \| h_k \| > 0 \). Putting \( B_k = f^{-1}([a_{k+1},a_{k+2}]) \), \( C_k = f^{-1}((a_{k+1} - 1,a_{k+2} + 1)) \), we see that \( B_k \) and \( C_k \) are two disjoint zero subsets of \( X \). Define a \( g_k \in C_b(X), g_k \geq 0, g_k \equiv 0 \) on \( C_k \) and \( g_k \equiv k \frac{1}{p_k} \) on \( B_k \). It is a routine verification that \( g = \sum_{k=1}^{\infty} g_k \in C(X) \).

For \( A \subset \tilde{X}, \overline{A} \) will denote its closure in \( \tilde{X} \). Now \( B_k \supset V \cap X \), where \( V = \{ x : a_{k+1} < \tilde{f}(x) < a_{k+2} \} \) is an open non-void subset of \( \tilde{X} \). Since \( X \) is dense in \( \tilde{X}, V \cap \overline{X} \supset V \) and so \( \overline{B_k} \supset V \). Also \( g_k \equiv k \frac{1}{p_k} \) on \( B_k \) implies \( \tilde{g}_k \equiv k \frac{1}{p_k} \) on \( \overline{B_k} \). So we get
\[
\tilde{\mu}(\tilde{g}_k) \geq \int_{B_k} \tilde{g}_k \, d\tilde{\mu} \geq k \frac{1}{p_k} \tilde{\mu}(V) = kh_k \frac{1}{p_k}.
\]
We have, for every \( n \in N, \mu(g) \geq \sum_{k=1}^{n} \mu(g_k) = \sum_{k=1}^{n} \tilde{\mu}(\tilde{g}_k) \geq \sum_{k=1}^{n} k h_k \frac{1}{p_k} \).

Now \( \| k h_k \frac{1}{p_k} \| = k \) and so \( \| \mu(g) \| = \infty \) (note \( E \) is an AM space) which is a contradiction. This proves that \( \text{supp}(\tilde{\nu}) \subset \nu X \). □

**Corollary 10.** Suppose \( \mu: C(X) \to E \) be an order-bounded linear map ([13], p.24]). Then there is a unique \( E \)-valued Baire measure \( \nu \) on \( X \) such that every \( f \in C(X) \) is \( \nu \)-integrable and \( \mu(f) = \int fd\nu, \forall f \in C(X) \) and \( \text{supp}(\tilde{\mu}) \subset \nu X \).

**Proof.** By [13] Theorem 1.3.2, p.24], \( \mu = \mu^+ - \mu^- \). Now \( \mu^+ \) and \( \mu^- \) are positive linear maps. Applying Theorem 9 to \( \mu^+ \) and \( \mu^- \) we get an \( E \)-valued Baire measure \( \nu \) on \( X \) such that every \( f \in C(X) \) is \( \nu \)-integrable and \( \mu(f) = \int fd\nu, \forall f \in C(X) \). As in Theorem 9, the uniqueness of \( \nu \) and \( \text{supp}(\tilde{\mu}) \subset \nu X \) can be proved.

### 4. The Case of \( E \) with Points Separated by \( E_n^* \)

For the order complete vector lattice \( E \), let \( E^* \) be its order dual and \( E_n^* \) its continuous order dual. In this section we assume that \( E_n^* \) separates the points of \( E \)\). It is known that \( E_n^* \) is a band in \( E^* \) and order intervals in \( E_n^* \) are \( \sigma(E_n^*,E) \)-compact and convex ([13], [13]). \( \sigma(E, E_n) \) will denote the locally convex topology on \( E \), of uniform convergence on the order intervals of \( E_n^* \); in this topology the lattice
operations are continuous and so the positive cone is closed and convex. Since this topology is compatible with the duality $< E, E_n^* >$, $E_+$ is also closed in $\sigma(E, E_n^*)$. 

The following theorem is well-known. We include a new proof.

**Theorem 11** ([10 Theorem 3]). Suppose $\mathcal{A}$ be a $\sigma$-algebra of subsets of a set $X$ and $\mu: \mathcal{A} \to E$ a finitely additive measure. Then $\mu$ is countably additive in order convergence iff $\mu$ is countably additive in the locally convex topology $\sigma(E, E_n^*)$.

**Proof.** Obviously countably additivity in order convergence implies countably additivity in $\sigma(E, E_n^*)$. Assume that $\mu$ is countably additive in $\sigma(E, E_n^*)$; this means $\mu$ is countably additive in $o(E, E_n)$. We first prove that $\mu^+$ countably additive in order convergence.

Fix a sequence $B_n \downarrow \emptyset$ in $\mathcal{A}$. Take a $C \subset X, C \in \mathcal{A}$. From $\mu(C - C \cap B_n) = \mu(B_n \cup C - B_n)$, we get $\mu(C) - \mu(C \cap B_n) \leq \mu^+(X) - \mu^+(B_n)$. Let $0 \leq z = \inf_n(\mu^+(B_n))$. Thus $z \leq \mu(C \cap B_n) + \mu^+(X) - \mu(C)$. Since $\mu(C \cap B_n) \to 0$ in $\sigma(E, E_n^*)$, we get, for every $f \in (E_n^*)_+$, $\langle f, z \rangle \leq \langle f, \mu(C \cap B_n) \rangle + \langle f, \mu^+(X) - \mu(C) \rangle$; using the fact $\mu(C \cap B_n) \to 0$ in $\sigma(E, E_n^*)$, this gives $\langle f, z \rangle \leq \langle f, \mu^+(X) - \mu(C) \rangle$ for every $f \in (E_n^*)_+$. Thus $z \leq \mu^+(X) - \mu(C)$ for every $C \in \mathcal{A}$. Taking inf of the right hand side as $C$ varies in $\mathcal{A}$, we get $z = 0$. This proves $\mu^+$ is countably additive in order convergence. Similarly $\mu^-$ is countably additive in order convergence and so $\mu$ is countably additive in order convergence. This proves the theorem. 

The next theorem extends the well-known Alexanderov’s theorem ([19], p. 195) about the convergent sequence of real-valued measures to our setting.

**Theorem 12.** Suppose $X$ is a completely regular $T_1$ space, $E$ is a boundedly order-complete vector-lattice, $E^+$ its order dual and $E_n^*$ its continuos order dual. Assume that $E_n^*$ separates the points of $E$. Let $\{\mu_n\} \subset M(o, \sigma)(X, E)$ be a uniformly order-bounded sequence such that, in order convergence, $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. Then the order-bounded $\mu: C_b(X) \to E$ is generated by $E$-valued order-bounded Baire measure on $X$.

**Proof.** Since the $\{\mu_n\}$ is uniformly order-bounded, we can assume that $E$ has an order unit. By taking the order unit norm ([13, p.8]), we assume $E = C(S)$ for some hyperstonian space $S$. Thus $F = E_n^*$ is a band in $E'$ and $E = F'$. Note the locally convex space $(E, \tau(E, E_n^*)) = (F', \tau(F', F'))$ is complete (Grothendieck completeness theorem ([15, Theorem 6.2, p.148])).

For every $g \in E_n^*$, $g \circ \mu_n \to g \circ \mu$, pointwise on $C_b(X)$ and $g \circ \mu_n \in M_\sigma(X)$, $\forall n$. Fix a $g \in E_n^*$ and take a sequence $\{f_m\} \subset C_b(X)$, $f_m \downarrow 0$. By ([19, p.195]), $g \circ \mu_n(f_m) \to g \circ \mu(f_m)$ as $n \to \infty$, uniformly in $m$. Thus $g \circ \mu(f_m) \to 0$. By ([20 Corollary 11.16]), $g \circ \mu: (C_b(X), \beta_\sigma) \to R$ is continuous, $\beta_\sigma$ being the strict topology ([20]). Thus the weakly compact map $\mu: (C_b(X), \beta_\sigma) \to (E, \tau(E, E_n^*))$ is continuous in the weak topology $\sigma(E, E_n^*)$ on $E$ ($\tau(E, E_n^*)$ is the Mackey topology in the duality $< E, E_n^*>$; since the topology $\beta_\sigma$ is Mackey ([20]), it is continuous. Since $(E, \tau(E, E_n^*))$ is complete, by ([19 Theorem 2]), $\mu$ can be extended to an $E$-valued Baire measure which is countably additive in $\tau(E, E_n^*)$. This implies that
\( \mu \) is countably additive in \( \sigma(E, E_n) \). By Theorem 11, \( \mu \) is countably additive in order convergence. \( \square \)

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**References**


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