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GENERALIZATION OF AMPLITUDE PHASE AND ACCOMPANYING DIFFERENTIAL EQUATION

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Introduction. In paper [1] O. Borůvka introduced a notion of the first and the second amplitude and a notion of the first and the second phase of basis \((u, v)\) of the differential equation

\[ y'' = q(t)y. \]

where the function \(q(t)\) — which is the carrier of this equation—belongs to the class \(C_0\) in the interval \(I\).

These notions were generalized by M. Laitoch in paper [3] under assumption that the carrier \(q(t)\) is negative in the interval \(I\).

In paper [1] pg. 6 a notion of an accompanying differential equation \(q_0\) towards an equation \(q\) is introduced whereby the carrier \(q(t)\) belongs to the class \(C_0\) in the interval \(I\).

This notion is generalized in paper [3] under assumption that the carrier \(q(t)\) is negative for every \(t \in I\).

In this paper we are going to introduce the preceding notions more generally than in paper [3].

1. In this section we’ll investigate the properties of integrals and their derivatives of the differential equation \(q\), where \(q(t) \in C_0(I)\) and \(q(t) < 0\) for every \(t \in I\).

We shan’t take into consideration such an integral of \(q\) which is identically equal to zero. The fact that the function \(u(t)\) is an integral of \(q\) we’ll denote by \(u \in (q)\).

We know from the classical theory that the exactly one integral of \(q\) is determined by the Cauchy initial conditions, i.e. if \(\tau_0 \in I\), \(u_0, u_0'\) are arbitrary numbers, there exists exactly one integral \(u \in (q)\) defined in the interval \(I\) that fulfills the initial conditions

\[ u(\tau_0) = u_0, \quad u'(\tau_0) = u_0'. \]

For simplicity let us have the following registrations:

\[ f(t, u) = \alpha(t)u(t) + \beta(t)u'(t) \]
\[ f(t, v) = \alpha(t)v(t) + \beta(t)v'(t) \]
\[ f(t, u/v) = f(t, u) / f(t, v) \]
We easily find out that there always exists an integral \( v \in (q) \) defined in the interval \( j \) with such a property that the functions \( f(t, v) \) is equal to zero at \( t_0 \in j \), where the functions \( \alpha(t), \beta(t) \) belong to the class \( C_0 \) in the interval \( j \), \( \alpha(t), \beta(t) \) don’t change their signs in \( j \) and at least one of these functions hasn’t zero values in the interval \( j \). Let us choose such an integral \( v(t) \) that fulfills the initial conditions

\[
U(T_0) = \alpha(T_0), \quad v'(T_0) = -x\alpha(T_0),
\]

where \( x \) is a constant value different from zero.

**Definition:** Let’s denote by \( z_n(T_0), n = 1, 2, 3, \ldots \), the \( n \)-th root of the function \( f(t, v) \) which lies after (before) the root \( T_0 \), so far there such a case exists. Number \( z_n(T_0) \) is called the \( n \)-th conjugate number towards \( T_0 \) lying on the right (on the left) from \( T_0 \) with respect to the weighing functions \( \alpha(t), \beta(t) \).

**Lemma:** Let \((u, v)\) be an ordered pair of independent integrals of equation \((q)\) defined in the interval \( j \); \( w = uv' - u'v \) is the Wronskian belonging to it. Let the functions \( \alpha(t), \beta(t) \), both of the class \( C^1 \), be given in the interval \( j \) not changing their signs there and at least one of them having no zero values in the interval \( j \). If \( \beta(t) \geq 0 \) for every \( t \in j \), let the function \( \frac{\beta(t)}{\alpha(t)} \) be nonincreasing in the interval \( j \). If \( \alpha(t) \geq 0 \) for every \( t \in j \), let the function \( \frac{\alpha(t)}{\beta(t)} \) be nondecreasing in the interval \( j \). Then the function

\[
F(t, u/v)
\]

continually increases or continually decreases for every \( t \in j \) satisfying inequality \( f(t, v) \neq 0 \), according to whether \( w < 0 \) or \( w > 0 \).

**Proof:** We easily derive the following formula

\[
\frac{d}{dt} F(t, u/v) = -w(z^2 + z\beta' - z'\beta - \beta^2q)/f^2(t, v), \quad (1)
\]

which holds for every \( t \in j \) where \( f(t, v) \neq 0 \). If \( \beta(t) \geq 0 \) for every \( t \in j \) and \( \frac{\alpha(t)}{\beta(t)} \) is nonincreasing in \( j \), then it follows that

\[
\left( \frac{\alpha}{\beta} \right)' = \frac{z\beta' - z\beta}{\beta^2} \leq 0 \Rightarrow z\beta' - z\beta \geq 0
\]

and therefore

\[
z^2 + z\beta' - z\beta - \beta^2q > 0, \quad (i)
\]
If \( a(t) \neq 0 \) for every \( t \in j \), we similarly find out that (1) is fulfilled. The assertion of the lemma follows now directly from the formula (1).

Note: If \( \eta < \xi \) are arbitrary numbers in \( j \) having the property that \( f(t, v) \) is in \( \langle \eta, \xi \rangle \) different from zero, then we obtain from (1)

\[
F(\xi, u|v) - F(\eta, u|v) = -\int_\eta^\xi \frac{w(\beta^2 + \alpha^2 - \alpha' \beta - \beta' \eta)}{f^2(t, v)} \, dt.
\]

2. In this section we'll introduce several theorems concerning the zero points of the function \( f(t, u) \), if \( u \in (q) \). Let \( q(t) \) be continually negative in the interval \( j \).

**Theorem 1:** Let \( u \in (q) \). Let functions \( a(t), \beta(t) \) fulfil the assumptions of the lemma in the interval \( j \). Then if at \( t_0 \in j \) holds \( f(t_0, u) = 0 \) then the first derivative of \( f(t, u) \) at \( t_0 \) is different from zero.

**Proof:** Let

\[
f(t_0, u) = 0
\]

and simultaneously

\[
\left[ \frac{d}{dt} f(t, u) \right]_{t=t_0} = 0.
\]

This system has non-trivial solution if and only if the determinant of this system is equal to zero, i.e.

\[
\alpha^2(t_0) + \alpha(t_0) \beta(t_0) - \alpha'(t_0) \beta(t_0) - \beta'(t_0) q(t_0) = 0.
\]

This result is in contradiction to (1). Thus Theorem 1 is proved.

**Note:** It is evident that the function \( f(t, u) \) changes its sign at \( t_0 \).

**Theorem 2:** Let \( u \in (q) \). Let functions \( a(t), \beta(t) \) fulfil the assumptions of the lemma in the interval \( j \). Then the function \( f(t, u) \) cannot have an infinite number of zero points in the interval \( \langle a, b \rangle \subset j \).

**Proof:** Let the function \( f(t, u) \) have an infinite number of zero points in the interval \( j \) and let \( t_0 \) be their limit point. We'll consider a sequence \( \{t_n\} \), \( t_n \neq t_0 \), \( n = 1, 2, 3, \ldots \) of zero points of the function \( f(t, u) \) so that these zero points converge to \( t_0 \). It holds that

\[
\frac{f(t_n, u) - f(t_0, u)}{t_n - t_0} = 0.
\]

As the function \( f(t, u) \) has the first derivative in the interval \( j \), we get

\[
\lim_{n \to \infty} \frac{f(t_n, u) - f(t_0, u)}{t_n - t_0} = \left[ \frac{d}{dt} f(t, u) \right]_{t=t_0} = 0;
\]

which is in contradiction to the assertion of Theorem 1.
Theorem 3: Let $u, v$ be linearly independent integrals of $(q)$. Let functions $a(t), \beta(t)$ fulfill the assumptions of the lemma in the interval $j$. Then, if $\tau_0 < \tau_1$ are two neighbouring zero points of $f(t, u)$ in the interval $j$, so the function $f(t, v)$ has exactly one zero point between $\tau_0$ and $\tau_1$.

Proof: It is evident that $f(t, v) = 0$ at $\tau_0$ and $\tau_1$. If, namely, there were 

$$f(\tau_k, t) = 0 \quad k = 0, 1$$

and simultaneously 

$$f(\tau_k, u) = 0 \quad k = 0, 1$$

the determinant $u(\tau_k) v(\tau) - u(\tau) v(\tau_k) = -w(\tau_k)$ would have to be equal to zero. Hence it would follow that $w = 0$ and $u, v$ would be dependent integrals.

Suppose that there exists no zero point of $f(t, v)$ in the interval $[\tau_0, \tau_1]$. Evidently it holds that $w > 0$ or $w < 0$ for every $t \in j$. Now we use the relation (1), which is positive for $w < 0$ and negative for $w > 0$. On integrating this relation from $\tau_0$ to $\tau_1$ we have the following equality:

$$[F(t, u/v)]^\tau_1_{\tau_0} = \int_{\tau_0}^{\tau_1} \frac{w(x^2 + \alpha \beta - x^2 \beta^2 - \beta^2 q)}{f^2(t, v)} \, dt.$$ 

The term on the left-hand side is equal to zero and that one on the right is positive for $w < 0$ and negative for $w > 0$ which is a contradiction. Thus we get that at least one zero point of $f(t, v)$ lies between $\tau_0$ and $\tau_1$.

If there were two zero points $\tau_0; \tau_1$ between $\tau_0$ and $\tau_1$, we could easily prove in the preceding way that at least one zero point $\tau$ of $f(t, v)$ lies between $\tau_0$ and $\tau_1$. Herefrom we have 

$$\tau_0 < \tau_0 < \tau < \tau_1$$

which is impossible, because $\tau_0$ and $\tau_1$ are two neighbouring zero points of $f(t, u)$.

Now we'll introduce the polar coordinates of independent integrals $u, v$ with the weighing functions $[a(t), \beta(t)]$.

Let $(u, v)$ be an ordered pair of independent integrals of $(q)$ and let $w$ be its Wronskian. Let $a(t), \beta(t)$ be the functions of the class $C_2$ fulfilling the assumptions of the lemma in the interval $j$. Let $q(t)$ belong to the class $C_2$ continually negative in the interval $j$. Now we define the following function in the interval $j$:

$$\delta = \sqrt{f^2(t, u) + f^2(t, v)}. \quad (2)$$

This function will be called the generalized amplitude of the ordered pair $(u, v)$ with the weighing functions $[a(t), \beta(t)]$.

Note: If $\beta(t) \equiv 0$, we get the first generalized amplitude. If $a(t) \equiv 0$, we get the second generalized amplitude. If $a, \beta$ are constants and $a^2 + \beta^2 > 0$, we have the amplitude with respect to basis $[a, \beta]$ (see pg. 48). If $a \equiv 1, \beta \equiv 0$, we get the first amplitude, if $a \equiv 0, \beta \equiv 1$, we have the second amplitude. (see [1] pg. 32).
The function $\delta(t)$ satisfies the following differential equation of the second order:

$$\delta'' = q\delta + \frac{w^2(x' + z'b - \beta')^2}{\beta^3} +$$

$$+ \left(2x'' + x'b' + 2a'b'q + a'q' + \beta'b'q' - a'b' - 2x'p' + \frac{a''p'q - \beta'b'q}{\beta^2 q} \right) \delta +$$

$$+ \left(2x'' - \beta'b'q + 2\beta'' - \beta^2 q \right) \delta' \quad (3)$$

which can be verified by direct calculation.

**Theorem 4:** Let $t_0 \in I$, $\delta_0 \neq 0$, be arbitrary real numbers. Then the solutions $\delta(t)$ of the differential equation (3), where $\delta(t_0) = \delta_0$, $\delta'(t_0) = \delta'_0$, satisfy the following relation:

$$\delta(t) = \text{sgn} \delta_0 \sqrt{\int^t_{t_0} f^2(u, v) \, du}, \quad (4)$$

where $(u, v)$ is a fundamental system of solutions of (4) which satisfies the initial conditions as follows:

$$u(t_0) = \left[ p(t_0) + v(t_0) q_0 \right] \delta_0 - \beta(t_0) \delta'_0,$$

$$u'(t_0) = -\left[ q(t_0) + u(t_0) \beta_0 + u(t_0) \delta_0 \right],$$

$$v(t_0) = -\beta k,$$

$$v'(t_0) = a k,$$

where $q_0 = q(t_0)$ and $k \neq 0$ is constant.

**Proof:** It is evident that the function (4) determines the solution of (3). For every function (4) there are fulfilled the initial conditions $\delta(t_0) = \delta_0$, $\delta'(t_0) = \delta'_0$; therefore it is necessary that

$$\delta_0 = \text{sgn} \delta_0 \cdot \left[ f^2(u_0, v_0) + f^2(t_0, v_0) \right]^{\frac{1}{2}},$$

$$\delta_0 = \text{sgn} \delta_0 \cdot \left[ f^2(u_0, v_0) + f^2(t_0, v_0) \right]^{-1},$$

where $u(t) = u(t_0)$, $v(t) = v(t_0)$, $u'_0 = u'(t_0)$, $v'_0 = v'(t_0)$, $q_0 = q(t_0)$. Hence we obtain

$$\delta_0, \delta'_0 = f(t_0, u_0) \left[ \frac{d}{dt} f(t, u) \right]_{t=t_0} + f(t_0, v_0) \left[ \frac{d}{dt} f(t, v) \right]_{t=t_0},$$

$$\delta_0 = f^2(t_0, u_0) + f^2(t_0, v_0). \quad (5)$$

which is a system of two algebraic equations with four unknown values $u_0, v_0, u'_0, v'_0$. 11
Let's take two conditions: 
\[ f(t_0, v_0) = \delta_0 \]
\[ \left[ \frac{d}{dt} f(t, v) \right]_{t=t_0} = \delta_0, \]
whence we obtain that
\[ u_0 = \frac{[\alpha(t_0) + \beta'(t_0) \cdot q_0] \cdot \delta_0 - \beta(t_0) \cdot \delta_0}{\alpha'(t_0) + \alpha(t_0) \beta'(t_0) - \alpha'(t_0) \beta(t_0) - \beta'(t_0) \cdot q_0} \]
and
\[ u_0 = \frac{[\alpha(t_0) + \beta(t_0) \cdot q_0] \cdot \delta_0 + \alpha(t_0) \cdot \delta_0}{\alpha'(t_0) + \alpha(t_0) \beta'(t_0) - \alpha'(t_0) \beta(t_0) - \beta'(t_0) \cdot q_0}. \]
Now equations (5) assume the form
\[ f(t_0, v_0) \left[ \frac{d}{dt} f(t, v) \right]_{t=t_0} = 0, \]
\[ f'(t_0, v_0) = 0, \]
wherefrom we have the condition
\[ f(t_0, v_0) = 0 \]
with one solution
\[ v_0 = -\beta k, \quad v_0' = \alpha k \]
where \( k + 0 \) is a constant value. The relations (6) and (7) prove thus the assertion of this theorem.

Let \( t_0 \) be any root of \( f(t, v) \) in the interval \( f \) and \( t_0(\tau_{-1}) \) be the \( n \)-th zero point on the right (on the left) from \( t_0 \). It is evident from the preceding results that the function \( f(t, u/v) \) increases from \( -\infty \) to \( +\infty \) in every interval \( (\tau_{-1}, \tau_{-2}) \) where \( w < 0 \) and decreases from \( +\infty \) to \( -\infty \), if \( w > 0 \). In this case there exists for every \( t \in (\tau_{-1}, \tau_{-2}), \)
\[ \nu = 0, \pm 1, \pm 2, \ldots, \] exactly one number
\[ \varphi(t) = \arctg F(t, u/v) \]
in the interval \( -\frac{\pi}{2}, \frac{\pi}{2} \) and we can define in the interval \( f \) the following function:
\[ \varphi(t) = \begin{cases} \frac{\pi}{2} - \nu \cdot \text{sgn} w & \text{for } t = \tau_{-1} \\ \arctg F(t, u/v) - \nu \cdot \text{sgn} w & \text{for } t \in (\tau_{-1}, \tau_{-2}) \end{cases} \]
This function will be called the phase of an ordered pair \( (u, v) \) of independent integrals of \( (q) \) with the weighing functions \( [\alpha(t), \beta(t)] \).

Note: If \( \alpha, \beta \) are constants and \( \alpha^2 + \beta^2 > 0 \) then we get the phase of an ordered pair \( (u, v) \) with respect to the basis \( [x, \beta] \), (see [3] pg. 49). If \( x = 1, \beta = 0 \), we get the first phase; if \( x = 0, \beta = 1 \), we have the second phase of an ordered pair \( (u, v) \) (see [1] pp. 31 or. 36).
The function \( \varphi(t) \) has the following properties:

a. it is continuous for every \( t \in I \) and derivable as well. For its first derivative we obtain a formula

\[
\varphi'(t) = -\frac{w[x^2(t) + \alpha(t) \beta'(t) - \beta(t)]}{\varphi'(t)}.
\]

b. with respect to the formula (3) the function \( \varphi(t) \) satisfies the following relation:

\[
\left(\frac{\sqrt{-w(x^2 + \alpha\beta' - \alpha'\beta - \beta^2q)}}{\varphi'}\right)^2 = \left(\frac{1}{\sqrt{x^2 + \alpha\beta' - \alpha'\beta - \beta^2q}}\right).
\]

Note: The relation (8) can be written by the help of the Schwarz derivative in another form. With respect to it we can say, that the function \( \varphi(t) \) satisfies the following differential equation of the 3-rd order:

\[
-\{\varphi; t\} - \varphi^3 = q + \frac{\alpha x^2 + 2\beta q + 2\beta'q - \alpha'\beta - 2\alpha\beta - 2\beta^2q + \beta^2q - \alpha'\beta - 2\alpha\beta^2 - 2\beta^2q}{\sqrt{x^2 + \alpha\beta' - \alpha'\beta - \beta^2q}}.
\]

If \( \alpha, \beta \) are constants, \( x^2 + \beta^2 > 0 \), then the equation (9) has the form

\[
-\{\varphi; t\} - \varphi^3 = q + \frac{\beta^2q}{\sqrt{x^2 + \beta^2q}}.
\]

If \( \alpha = 0, \beta = 1 \), then we obtain the form

\[
-\{\varphi; t\} - \varphi^2 = q + \sqrt{-q}
\]

which is the well known equation satisfied by the second phases of \( \varphi(t) \). If \( \alpha = 1, \beta = 0 \), then we obtain

\[
-\{\varphi; t\} - \varphi^2 = q
\]

i.e. the Kummer's equation satisfied by the first phases of \( \varphi(t) \). Concluding this note we can say that the equation (9) is a certain generalization of the Kummer's equation (12).
Theorem 5: Let \( u, v \) be linearly independent integrals of \( q \). Let the weighing functions \( \alpha(t), \beta(t) \) belong to the class \( C_3 \) and fulfil the assumption of the lemma in the interval \( j \). Let \( \tau_0 \) be a zero point of \( f(t, v) \). Then for every \( t \in j \) there holds:

\[
\begin{align*}
f(t, u) &= \text{sgn} \left[ \frac{d}{dt} f(t, v) \right]_{\tau_0} \delta(t) \sin \varphi(t), \\
f(t, v) &= \text{sgn} \left[ \frac{d}{dt} f(t, v) \right]_{\tau_0} \delta(t) \cos \varphi(t).
\end{align*}
\]

Proof: If \( t \in (\tau_0; \tau_1) \), there holds

\[
\tan \varphi = f(t, u/v)
\]
and \(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\). Then evidently

\[
\begin{align*}
\sin \varphi &= k f(t, u) \\
\cos \varphi &= k f(t, v),
\end{align*}
\]

where \( k > 0 \). On squaring and adding we have

\[
1 = k^2 \delta^2 \Rightarrow |k| = \frac{1}{\delta}.
\]  (14)

As the function \( \cos \varphi \) is positive for \( t \in (\tau_0, \tau_1) \), we can take the sign of (14) so that the second equation of (13) is fulfilled. But it holds that the function \( f(t, v) \) is positive (negative) in \((\tau_0, \tau_1)\) if and only if \( \frac{d}{dt} f(t, v) \left|_{\tau_0} \right. \) is positive (negative). Now we get the assertion of the theorem from the relations (13) and (14). Thus the theorem is proved.

4. Now we are going to introduce a notion of the accompanying differential equation towards \( q \) with the weighing functions \( \alpha(t), \beta(t) \).

Theorem 6: Let \( u \in \langle q \rangle \). Let \( \alpha(t), \beta(t) \) be of the class \( C_3 \) and fulfil the assumptions of the lemma in the interval \( j \). Let \( q(t) \) belong to the class \( C_2 \) and be continually negative in the interval \( j \). Then the function

\[
U(t) = \frac{f(t, u)}{\sqrt{a^2 + 2a'q + a''q'} + a'q + a''q'' + a'q'}
\]  (15)

is a solution of the differential equation

\[
y'' = q_1(t) y, \quad (q_1)
\]

where

\[
q_1 = q + \frac{2\beta'q' + \beta''q'' - 2\beta q' - 2\beta'q - \beta''q}{a^2 + 2a'q + a''q'} + \frac{1}{\sqrt{a^2 + 2a'q + a''q'}}
\]  (16)

\[
+ \sqrt{a^2 + 2a'q + a''q'} - a'q' - a''q - a'q - a''q' + \frac{1}{\sqrt{a^2 + 2a'q + a''q'}}.
\]
The proof will be easily verified by direct calculation.

**Definition**: Differential equation \((qx)\) is called the first accompanying equation towards \((q)\) with the weighing functions \([a(t), \beta(t)]\). The first accompanying equation towards \((q_1)\) is called the second accompanying equation towards \((q)\) with the weighing functions \([a(t), \beta(t)]\), etc.

**Note**: If \(a, \beta\) are constants, then

\[
q_1 = q + \frac{\alpha \beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\frac{1}{\alpha^2 - \beta^2 q}}
\]

is the carrier of the first accompanying equation towards \((q)\) with respect to the basis \([a, \beta]\) (see [3] pg. 50).

If \(a = 0, \beta = 1\), then

\[
q_1 = q + \sqrt{-q} \left( \frac{1}{\sqrt{-q}} \right)^*
\]

is the carrier of the first accompanying equation towards \((q)\), if \(q < 0\) (see [1] pg. 7).

**Note**: With respect to the preceding definition we can write the relation (9) in the form

\[
-\{q; t\} - \varphi^2 = q_1(t).
\]

where \(q_1(t)\) is the carrier of the first accompanying equation towards \((q)\) with the weighing functions \([a(t), \beta(t)]\).

**Example**: Consider that for \(q_1(t)\) in relation (17) there holds:

\[
\left( \frac{1}{\sqrt{a^2 - \beta^2 q}} \right)^* = 0.
\]

Hence it directly follows that

\[
a^2 + \beta^2 q = \frac{1}{C^2(t + d)^2},
\]

wherefrom

\[
q(t) = \frac{-1}{\beta^2 C^2(t + d)^2} + \frac{a^2}{\beta^2}
\]

and the differential equation \((q)\) has the form

\[
y'' = \left( \frac{-1}{\beta^2 C^2(t + d)^2} + \frac{a^2}{\beta^2} \right)y.
\]

(19)

We find out by calculation that the carrier of the first accompanying equation \((q_1)\) towards (19) with respect to the basis \([a, \beta]\) has the form:

\[
q_1 = \frac{-1}{\beta^2 C^2(t + d)^2} + \frac{a^2}{\beta^2} + \frac{2x}{\beta(t + d)}
\]
and prove by complete induction that the n-th accompanying equation towards (19) has the form
\[ y'' = \left\{ \frac{-1}{\beta^2 C^2 (t + d)^2} + \frac{x^2}{\beta^2} + \frac{2n\pi}{\beta(t + d)} - (n - 1) n C^2 x^2 \right\} y. \] (20)

**Note:** If we put \( \alpha = 0, \beta = 1 \) in relation (20), then the differential equation (20) is identical with (19) and we get the case solved in [2] for \( q < 0 \).

**Theorem 7:** If \( U \in (q_j) \) is an arbitrary integral, then there exists the integral \( u \in (q) \) such that
\[ f(t, u) \sqrt{\alpha^2 + \alpha\beta - \alpha' - \beta' q} = U(t). \]

**Proof:** \( U(t) \) is defined by the following initial conditions for \( \tau_0 \in j \):
\[ U(\tau_0) = U_0, \quad U'(\tau_0) = U'_0. \]

It is necessary to choose for \( u \in (q) \) from relations
\[ f(t_0; u_0) \sqrt{\alpha^2(\tau_0) + \alpha(\tau_0) \beta(\tau_0) - \alpha'(\tau_0) \beta(\tau_0) - \beta'(\tau_0) q(\tau_0)} = U_0, \]
\[ \sqrt{\alpha^2(\tau_0) + \alpha(\tau_0) \beta(\tau_0) - \alpha'(\tau_0) \beta(\tau_0) - \beta'(\tau_0) q(\tau_0)} = U_0, \]
\[ f(t_0; u_0) \left( \sqrt{\alpha^2 + \alpha\beta - \alpha' - \beta' q} \right)_{\tau_0} = U'_0 \]
an integral \( u \in (q) \) satisfying the initial conditions
\[ u(\tau_0) = u_0, \quad u'(\tau_0) = u'_0. \]

It is however sufficient to choose
\[ u_0 = \frac{1}{\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0) \beta(\tau_0) - \alpha'(\tau_0) \beta(\tau_0) - \beta'(\tau_0) q(\tau_0)}} \cdot \left\{ U_0 \left[ \alpha(\tau_0) + \beta(\tau_0) \sqrt{\alpha^2(\tau_0) + \alpha(\tau_0) \beta(\tau_0) - \alpha'(\tau_0) \beta(\tau_0) - \beta'(\tau_0) q(\tau_0)} \right] \right\}, \]
\[ u'_0 = \frac{1}{\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0) \beta(\tau_0) - \alpha'(\tau_0) \beta(\tau_0) - \beta'(\tau_0) q(\tau_0)}} \cdot \left\{ \alpha(\tau_0) U'_0 - U_0 \left[ \alpha(\tau_0) \sqrt{\alpha^2(\tau_0) + \alpha(\tau_0) \beta(\tau_0) - \alpha'(\tau_0) \beta(\tau_0) - \beta'(\tau_0) q(\tau_0)} \right] \right\}. \]

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The proof will be easily verified by direct calculation.

Concluding this paper I should like to express my gratitude to Prof. RNDr. M. Laitoch CSc., for suggesting the idea to study this problem, and for his valuable advice.

REFERENCES


Resume

ZOVŠEOBECNENIE AMPLITÚDY, FÁZY A SPRIEVODNEJ DIFERENCIÁLNEJ ROVNICE

MILOŠ HÁČIK

V tomto článku je skúmaná diferenciálna rovnica

$$y'' = q(t)y,$$

(kde $q(t) \in C^1(j)$ a $q(t) < 0$ pre všetky $t \in j$, z hľadiska vlastností lineárnych kombinácií jej integrálov a ich prvých derivácií vzhľadom na váhové funkcie $[\alpha(t), \beta(t)]$.

Funkciu

$$\delta(t) = \sqrt{[\alpha(t) u(t) + \beta(t) u'(t)]^2 + [\alpha(t) v(t) + \beta(t) v'(t)]^2}$$

je zavedená zovšeobecná amplitúda usporiadané dvojice $(u, v)$ nezávislých integrálov rovnice $(q)$ s váhovými funkciami $[\alpha(t), \beta(t)]$. Ďalej sa z tohto hľadiska pri-chádza k pojmu fáza usporiadané dvojice riešení $(u, v)$ rovnice $(q)$ a tiež k pojiu sprievodnej rovnici k rovnici $(q)$ vzhľadom na váhové funkcie $[\alpha(t), \beta(t)]$. 
