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GENERALIZATION OF AMPLITUDE PHASE AND ACCOMPANYING DIFFERENTIAL EQUATION

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Introduction. In paper [1] O. Borůvka introduced a notion of the first and the second amplitude and a notion of the first and the second phase of basis (u, v) of the differential equation

$$y'' = q(t)y, \tag{q}$$

where the function $q(t)$ — which is the carrier of this equation—belongs to the class C_0 in the interval j.

These notions were generalized by M. Laitoch in paper [3] under assumption that the carrier $q(t)$ is negative in the interval j.

In paper [1] pg. 6 a notion of an accompanying differential equation (q_1) towards an equation (q) is introduced whereby the carrier $q(t)$ belongs to the class C_3 in the interval j.

This notion is generalized in paper [3] under assumption that the carrier $q(t)$ is negative for every $t \in j$.

In this paper we are going to introduce the preceding notions more generally than in paper [3].

1. In this section we'll investigate the properties of integrals and their derivatives of the differential equation (q), where $q(t) \in C_0(j)$ and $q(t) < 0$ for every $t \in j$.

We shan't take into consideration such an integral of (q) which is identically equal to zero. The fact that the function $u(t)$ is an integral of (q) we'll denote by $u \in (q)$.

We know from the classical theory that the exactly one integral of (q) is determined by the Cauchy initial conditions, i.e. if $\tau_0 \in j$, u_0, u'_0 are arbitrary numbers, there exists exactly one integral $u \in (q)$ defined in the interval j that fulfils the initial conditions

$$u(\tau_0) = u_0, \qquad u'(\tau_0) = u'_0.$$

For simplicity let us have the following registrations:

$$f(t, u) = \alpha(t)\, u(t) + \beta(t)\, u'(t)$$
$$f(t, v) = \alpha(t)\, v(t) + \beta(t)\, v'(t)$$
$$F(t, u/v) = \frac{f(t, u)}{f(t, v)}$$

$$\frac{d}{dt} f(t, u) = [\alpha'(t) + \beta(t) q(t)] u(t) + [\alpha(t) + \beta'(t)] u'(t),$$

$$\frac{d}{dt} f(t, v) = [\alpha'(t) + \beta(t) q(t)] v(t) + [\alpha(t) + \beta'(t)] v'(t).$$

We easily find out that there always exists an integral $v \in (q)$ defined in the interval j with such a property that the functions $f(t, v)$ is equal to zero at $\tau_0 \in j$, where the functions $\alpha(t)$, $\beta(t)$ belong to the class C_0 in the interval j, $\alpha(t)$, $\beta(t)$ don't change their signs in j and at least one of these functions hasn't zero values in the interval j. Let us choose such an integral $v(t)$ that fulfils the initial conditions

$$v(\tau_0) = \varkappa\beta(\tau_0), \qquad v'(\tau_0) = -\varkappa\alpha(\tau_0),$$

where $\varkappa$ is a constant value different from zero.

Definition: Let's denote by $\tau_n(\tau_{-n})$, $n = 1, 2, 3, \ldots$, the n-th root of the function $f(t, v)$ which lies after (before) the root τ_0, so far there such a case exists. Number $\tau_n(\tau_{-n})$ is called *the n-th conjugate number* towards τ_0 lying on the right (on the left) from τ_0 with respect to the weighing functions $[\alpha(t), \beta(t)]$.

Lemma: Let (u, v) be an ordered pair of independent integrals of equation (q) defined in the interval j; $w = uv' - u'v$ is the Wronskian belonging to it. Let the functions $\alpha(t)$, $\beta(t)$, both of the class C_1, be given in the interval j not changing their signs there and at least one of them having no zero values in the interval j. If $\beta(t) \neq 0$ for every $t \in j$, let the function $\dfrac{\alpha(t)}{\beta(t)}$ be nonincreasing in the interval j. If $\alpha(t) \neq 0$ for every $t \in j$, let the function $\dfrac{\beta(t)}{\alpha(t)}$ be nondecreasing in the interval j. Then the function

$$F(t, u/v)$$

continually increases or continually decreases for every $t \in j$ satisfying inequality $f(t, v) \neq 0$, according to whether $w < 0$ or $w > 0$.

Proof: We easily derive the following formula

$$\frac{d}{dt} F(t, u/v) = \frac{-w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{f^2(t, v)}, \tag{1}$$

which holds for every $t \in j$ where $f(t, v) \neq 0$. If $\beta(t) \neq 0$ for every $t \in j$ and $\dfrac{\alpha(t)}{\beta(t)}$ is nonincreasing in j, then it follows that

$$\left(\frac{\alpha}{\beta}\right)' = \frac{\alpha'\beta - \alpha\beta'}{\beta^2} \leqq 0 \Rightarrow \alpha\beta' - \alpha'\beta \geqq 0$$

and therefore

$$\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q > 0. \tag{$\bar{1}$}$$

If $\alpha(t) \neq 0$ for every $t \in j$, we similarly find out that $(\bar{1})$ is fulfilled. The assertion of the lemma follows now directly from the formula (1).

Note: If $\eta < \xi$ are arbitrary numbers in j having the property that $f(t, v)$ is in $\langle \eta, \xi \rangle$ different from zero, then we obtain from (1)

$$F(\xi, u/v) - F(\eta, u/v) = -\int_{\eta}^{\xi} \frac{w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{f^2(t, v)}\, \mathrm{d}t.$$

2. In this section we'll introduce several theorems concerning the zero points of the function $f(t, u)$, if $u \in (q)$. Let $q(t)$ be continually negative in the interval j.

Theorem 1: Let $u \in (q)$. Let functions $\alpha(t)$, $\beta(t)$ fulfil the assumptions of the lemma in the interval j. Then if at $\tau_0 \in j$ holds $f(\tau_0, u) = 0$ then the first derivative of $f(t, u)$ at τ_0 is different from zero.

Proof: Let

$$f(\tau_0, u) = 0$$

and simultaneously

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t, u)\right]_{t=\tau_0} = 0.$$

This system has non-trivial solution if and only if the determinant of this system is equal to zero, i.e.

$$\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha'(\tau_0)\,\beta(\tau_0) - \beta^2(\tau_0)\,q(\tau_0) = 0.$$

This result is in contradiction to $(\bar{1})$. Thus Theorem 1 is proved.

Note: It is evident that the function $f(t, u)$ changes its sign at τ_0.

Theorem 2: Let $u \in (q)$. Let functions $\alpha(t)$, $\beta(t)$ fulfil the assumptions of the lemma in the interval j. Then the function $f(t, u)$ cannot have an infinite number of zero points in the interval $\langle a, b \rangle \subset j$.

Proof: Let the function $f(t, u)$ have an infinite number of zero points in the interval j and let τ_0 be their limit point. We'll consider a sequence $\{\tau_n\}$, $\tau_n \neq \tau_0$; $n = 1, 2, 3, \ldots$ of zero points of the function $f(t, u)$ so that these zero points converge to τ_0. It holds that

$$\frac{f(\tau_n, u) - f(\tau_0, u)}{\tau_n - \tau_0} = 0.$$

As the function $f(t, u)$ has the first derivative in the interval j, we get

$$\lim_{n \to \infty} \frac{f(\tau_n, u) - f(\tau_0, u)}{\tau_n - \tau_0} = \left[\frac{\mathrm{d}}{\mathrm{d}t} f(t, u)\right]_{t=\tau_0} = 0;$$

which is in contradiction to the assertion of Theorem 1.

Theorem 3: Let u, v be linearly independent integrals of (q). Let functions $\alpha(t)$, $\beta(t)$ fulfil the assumptions of the lemma in the interval j. Then, if $\tau_0 < \tau_1$ are two neighbouring zero points of $f(t, u)$ in the interval j, so the function $f(t, v)$ has exactly one zero point between τ_0 and τ_1.

Proof: It is evident that $f(t, v) \neq 0$ at τ_0 and τ_1. If, namely, there were

$$f(\tau_k, v) = 0 \qquad k = 0,1$$

and simultaneously

$$f(\tau_k, u) = 0 \qquad k = 0,1$$

the determinant $u'(\tau_k)\, v(\tau_k) - u(\tau_k)\, v'(\tau_k) = -w(\tau_k)$ would have to be equal to zero. Hence it would follow that $w = 0$ and u, v would be dependent integrals.

Suppose that there exists no zero point of $f(t, v)$ in the interval (τ_0, τ_1). Evidently it holds that $w > 0$ or $w < 0$ for every $t \in j$. Now we use the relation (1), which is positive for $w < 0$ and negative for $w > 0$. On integrating this relation from τ_0 to τ_1 we have the following equality:

$$[F(t, u/v)]_{\tau_0}^{\tau_1} = -\int_{\tau_0}^{\tau_1} \frac{w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{f^2(t, v)}\, \mathrm{d}t.$$

The term on the left-hand side is equal to zero and that one on the right is positive for $w < 0$ and negative for $w > 0$ which is a contradiction. Thus we get that at least one zero point of $f(t, v)$ lies between τ_0 and τ_1.

If there were two zero points $\bar{\tau}_0; \bar{\tau}_1$ between τ_0 and τ_1, we could easily prove in the preceding way that at least one zero point τ of $f(t, v)$ lies between $\bar{\tau}_0$ and $\bar{\tau}_1$. Herefrom we have

$$\tau_0 < \bar{\tau}_0 < \tau < \bar{\tau}_1 < \tau_1$$

which is impossible, because τ_0 and τ_1 are two neighbouring zero points of $f(t, u)$.

3. Now we'll introduce the polar coordinates of independent integrals u, v with the weighing functions $[\alpha(t), \beta(t)]$.

Let (u, v) be an ordered pair of independent integrals of (q) and let w be its Wronskian. Let $\alpha(t)$, $\beta(t)$ be the functions of the class C_3 fulfilling the assumptions of the lemma in the interval j. Let $q(t)$ belong to the class C_2 continually negative in the interval j. Now we define the following function in the interval j:

$$\delta = \sqrt{f^2(t, u) + f^2(t, v)}. \qquad (2)$$

This function will be called the *generalized amplitude* of the ordered pair (u, v) with the weighing functions $[\alpha(t), \beta(t)]$.

Note: If $\beta(t) \equiv 0$, we get the first generalized amplitude. If $\alpha(t) \equiv 0$, we get the second generalized amplitude. If α, β are constants and $\alpha^2 + \beta^2 > 0$, we have the amplitude with respect to basis $[\alpha, \beta]$ (see 3 pg. 48). If $\alpha \equiv 1$, $\beta \equiv 0$, we get the first amplitude, if $\alpha \equiv 0$, $\beta \equiv 1$, we have the second amplitude.. (see [1] pg. 32).

The function $\delta(t)$ satisfies the following differential equation of the second order:

$$\delta'' = q\delta + \frac{w^2(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)^2}{\delta^3} +$$
$$+ \frac{(\alpha\alpha'' + \alpha''\beta' + 2\alpha\beta' q + 2\beta'^2 q + \alpha\beta q' + \beta\beta' q' - \alpha'\beta'' - 2\alpha'^2 - 2\alpha'\beta q - \beta\beta'' q)\,\delta}{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q} +$$
$$+ \frac{(2\alpha\alpha' - 2\beta\beta' q - \alpha''\beta + \alpha\beta'' - \beta^2 q')\,\delta'}{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}. \tag{3}$$

which can be verified by direct calculation.

Theorem 4: Let $t_0 \in j$, $\delta_0 \neq 0$, be arbitrary real numbers. Then the solutions $\delta(t)$ of the differential equation (3), where $\delta(t_0) = \delta_0$, $\delta'(t_0) = \delta'_0$, satisfy the following relation:

$$\delta(t) = \operatorname{sgn} \delta_0 \sqrt{f^2(t, u) + f^2(t, v)}, \tag{4}$$

where (u, v) is a fundamental system of solutions of (q) which satisfies the initial conditions as follows:

$$u(t_0) = \frac{[\alpha(t_0) + \beta'(t_0)\, q_0]\, \delta_0 - \beta(t_0)\, \delta'_0}{\alpha^2(t_0) + \alpha(t_0)\, \beta'(t_0) - \alpha'(t_0)\, \beta(t_0) - \beta^2(t_0)\, q_0},$$
$$u'(t_0) = \frac{-[\alpha'(t_0) + \beta(t_0)\, q_0]\, \delta_0 + \alpha(t_0)\, \delta'_0}{\alpha^2(t_0) + \alpha(t_0)\, \beta'(t_0) - \alpha'(t_0)\, \beta(t_0) - \beta^2(t_0)\, q_0},$$
$$v(t_0) = -\beta k$$
$$v' t_0 = \alpha k,$$

where $q_0 = q(t_0)$ and $k \neq 0$ is constant.

Proof: It is evident that the function (4) determines the solution of (3). For every function (4) there are fulfilled the initial conditions $\delta(t_0) = \delta_0$, $\delta'(t_0) = \delta'_0$; therefore it is necessary that

$$\delta_0 = \operatorname{sgn} \delta_0 \,.\, [f^2(t_0, u_0) + f^2(t_0, v_0)]$$

and

$$\delta'_0 = \operatorname{sgn} \delta_0 \,.\, [f^2(t_0, u_0) + f^2(t_0, v_0)]^{-\frac{1}{2}} \,.$$
$$.\left[f(t_0, u_0) \,.\left(\frac{d}{dt} f(t, u)\right)_{t=t_0} + f(t_0, v_0) \,.\left(\frac{d}{dt} f(t, v)\right)_{t=t_0}\right],$$

where $u_0 = u(t_0)$, $v_0 = v(t_0)$, $u'_0 = u'(t_0)$, $v'_0 = v'(t_0)$, $q_0 = q(t_0)$. Hence we obtain

$$\delta_0 \,.\, \delta'_0 = f(t_0, u_0)\left[\frac{d}{dt} f(t, u)\right]_{t=t_0} + f(t_0, v_0)\left[\frac{d}{dt} f(t, v)\right]_{t=t_0},$$
$$\delta_0^2 = f^2(t_0, u_0) + f^2(t_0, v_0); \tag{5}$$

which is a system of two algebraic equations with four unknown values u_0, v_0, u'_0, v'_0.

Let's take two conditions:

$$f(t_0, u_0) = \delta_0$$
$$\left[\frac{d}{dt} f(t, u)\right]_{t=t_0} = \delta_0',$$

whence we obtain that

$$u_0 = \frac{[\alpha(t_0) + \beta'(t_0) q_0] \delta_0 - \beta(t_0) \delta_0'}{\alpha^2(t_0) + \alpha(t_0) \beta'(t_0) - \alpha'(t_0) \beta(t_0) - \beta^2(t_0) q_0}$$

and (6)

$$u_0' = \frac{-[\alpha'(t_0) + \beta(t_0) q_0] \delta_0 + \alpha(t_0) \delta_0'}{\alpha^2(t_0) + \alpha(t_0) \beta'(t_0) - \alpha'(t_0) \beta(t_0) - \beta^2(t_0) q_0}.$$

Now equations (5) assume the form

$$f(t_0, v_0) \left[\frac{d}{dt} f(t, v)\right]_{t=t_0} = 0,$$
$$f^3(t_0, v_0) = 0,$$

wherefrom we have the condition

$$f(t_0, v_0) = 0$$

with one solution

$$v_0 = -\beta k, \qquad v_0' = \alpha k \tag{7}$$

where $k \neq 0$ is a constant value. The relations (6) and (7) prove thus the assertion of this theorem.

Let τ_0 be any root of $f(t, v)$ in the interval j and $\tau_n(\tau_{-n})$ be the n-th zero point on the right (on the left) from τ_0. It is evident from the preceding results that the function $F(t, u/v)$ increases from $-\infty$ to $+\infty$ in every interval $(\tau_\nu, \tau_{\nu+1})$ where $w < 0$ and decreases from $+\infty$ to $-\infty$, if $w > 0$. In this case there exists for every $t \in (\tau_\nu, \tau_{\nu+1})$, $\nu = 0, \pm 1, \pm 2, \ldots$, exactly one number

$$\varphi(t) = \operatorname{arctg} F(t, u/v)$$

in the interval $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ and we can define in the interval j the following function:

$$\varphi(t) = \begin{cases} \frac{\pi}{2} - \nu\pi \operatorname{sgn} w & \text{for } t = \tau_\nu \\ \operatorname{arctg} F(t, u/v) - \nu\pi \operatorname{sgn} w & \text{for } t \in (\tau_\nu, \tau_{\nu+1}). \end{cases}$$

This function will be called the phase of an ordered pair (u, v) of independent integrals of (q) with the weighing functions $[\alpha(t), \beta(t)]$.

Note: If α, β are constants and $\alpha^2 + \beta^2 > 0$ then we get the phase of an ordered pair (u, v) with respect to the basis $[\alpha, \beta]$, (see [3] pg. 49). If $\alpha \equiv 1, \beta \equiv 0$, we get the first phase; if $\alpha \equiv 0$, $\beta \equiv 1$, we have the second phase of an ordered pair $(u, v.)$ (see [1] pp. 31 or. 36).

The function $\varphi(t)$ has the following properties:

a. it is continuous for every $t \in j$ and derivable as well. For its first derivative we obtain a formula

$$\varphi'(t) = \frac{-w[\alpha^2(t) + \alpha(t)\beta'(t) - \alpha'(t)\beta(t) - \beta^2(t)q(t)]}{\delta^2(t)},$$

b. with respect to the formula (3) the function $\varphi(t)$ satisfies the following relation:

$$\left(\sqrt{\frac{-w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{\varphi'}}\right)'' = (q + \varphi'^2)\sqrt{\frac{-w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{\varphi'}} +$$

$$+ \frac{\alpha\alpha'' + \alpha''\beta + 2\alpha\beta'q + 2\beta'^2 q + \alpha\beta q' + \beta\beta'q' - \alpha'\beta'' - 2\alpha'^2 - 2\alpha'\beta q - \beta\beta''q}{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q} \cdot$$

$$\cdot \sqrt{\frac{-w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{\varphi'}} +$$

$$+ \frac{2\alpha\alpha' - 2\beta\beta'q - \alpha''\beta + \alpha\beta'' - \beta^2 q'}{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}\left(\sqrt{\frac{-w(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)}{\varphi'}}\right)'. \tag{8}$$

Note: The relation (8) can be written by the help of the Schwarz derivative in another form. With respect to it we can say, that the function $\varphi(t)$ satisfies the following differential equation of the 3-rd order:

$$-\{\varphi; t\} - \varphi'^2 =$$

$$= q + \frac{\alpha\alpha'' + 2\alpha\beta'q + \alpha\beta q' + \alpha''\beta' + 2\beta'^2 q + \beta\beta'q' - \alpha'\beta'' - 2\alpha'^2 - 2\alpha'\beta q - \beta\beta''q}{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q} +$$

$$+ \sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}\left(\frac{1}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}}\right)''. \tag{9}$$

If α, β are constants, $\alpha^2 + \beta^2 > 0$, then the equation (9) has the form

$$-\{\varphi; t\} - \varphi'^2 = q + \frac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\alpha^2 - \beta^2 q}\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)''. \tag{10}$$

If $\alpha \equiv 0$, $\beta \equiv 1$, then we obtain the form

$$-\{\varphi; t\} - \varphi'^2 = q + \sqrt{-q}\left(\frac{1}{\sqrt{-q}}\right)'' \tag{11}$$

which is the well known equation satisfied by the second phases of (q).

If $\alpha \equiv 1$, $\beta \equiv 0$, then we obtain

$$-\{\varphi; t\} - \varphi'^2 = q \tag{12}$$

i.e. the Kummer's equation satisfied by the first phases of (q). Concluding this note we can say that the equation (9) is a certain generalization of the Kummer's equation (12).

Theorem 5: Let u, v be linearly independent integrals of (q). *Let the weighing functions* $\alpha(t)$, $\beta(t)$ *belong to the class* C_3 *and fulfil the assumption of the lemma in the interval j. Let* τ_0 *be a zero point of* $f(t, v)$. *Then for every* $t \in j$ *there holds:*

$$f(t, u) = \operatorname{sgn}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t, v)\right]_{t=\tau_0} \delta(t) \sin \varphi(t),$$

$$f(t, v) = \operatorname{sgn}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t, v)\right]_{t=\tau_0} \delta(t) \cos \varphi(t).$$

Proof: If $t \in (\tau_0; \tau_1)$, there holds

$$\operatorname{tg} \varphi = F(t, u/v)$$

and $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Then evidently

$$\begin{aligned} \sin \varphi &= kf(t, u) \\ \cos \varphi &= kf(t, v), \end{aligned} \tag{13}$$

where $k \neq 0$. On squaring and adding we have

$$1 = k^2\delta^2 \Rightarrow |k| = \frac{1}{\delta}. \tag{14}$$

As the function $\cos \varphi$ is positive for $t \in (\tau_0, \tau_1)$, we can take the sign of (14) so that the second equation of (13) is fulfilled. But it holds that the function $f(t, v)$ is positive (negative) in (τ_0, τ_1) if and only if $\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t, v)\right]_{t=\tau_0}$ is positive (negative). Now we get the assertion of the theorem from the relations (13) and (14). Thus the theorem is proved.

4. Now we are going to introduce a notion of the accompanying differential equation towards (q) with the weighing functions $[\alpha(t), \beta((t)]$.

Theorem 6: Let $u \in (q)$. *Let* $\alpha(t)$, $\beta(t)$ *be of the class* C_3 *and fulfil the assumptions of the lemma in the interval j. Let* $q(t)$ *belong to the class* C_2 *and be continually negative in the interval j. Then the function*

$$U(t) = \frac{f(t, u)}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}} \tag{15}$$

is a solution of the differential equation

$$y'' = q_1(t)\, y, \tag{q_1}$$

where

$$q_1 = q + \frac{\alpha\alpha'' + 2\alpha\beta' q + \alpha\beta q' + \alpha''\beta' + 2\beta'^2 q + \beta\beta' q' - \alpha'\beta'' - 2\alpha'^2 - 2\alpha'\beta q - \beta\beta'' q}{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q} + \sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}\left(\frac{1}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}}\right)''. \tag{16}$$

The proof will be easily verified by direct calculation.

Definition: Differential equation (q_1) is called the *first accompanying equation* towards (q) with the weighing functions $[\alpha(t), \beta(t)]$. The first accompanying equation towards (q_1) is called the *second accompanying equation* towards (q) with the weighing functions $[\alpha(t), \beta(t)]$, etc.

Note: If α, β are constants, then

$$q_1 = q + \frac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\alpha^2 - \beta^2 q}\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)'' \tag{17}$$

is the carrier of the first accompanying equation towards (q) with respect to the basis $[\alpha, \beta]$ (see [3] pg. 50).
If $\alpha \equiv 0$, $\beta \equiv 1$, then

$$q_1 = q + \sqrt{-q}\left(\frac{1}{\sqrt{-q}}\right)''$$

is the carrier of the first accompanying equation towards (q), if $q < 0$ (see [1] pg. 7).

Note: With respect to the preceding definition we can write the relation (9) in the form

$$-\{\varphi; t\} - \varphi'^2 = q_1(t), \tag{18}$$

where $q_1(t)$ is the carrier of the first accompanying equation towards (q) with the weighing functions $[\alpha(t), \beta(t)]$.

Example: Consider that for $q_1(t)$ in relation (17) there holds:

$$\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)'' = 0.$$

Hence it directly follows that

$$\alpha^2 + \beta^2 q = \frac{1}{C^2(t + d)^2},$$

wherefrom

$$q(t) = \frac{-1}{\beta^2 C^2(t + d)^2} + \frac{\alpha^2}{\beta^2}$$

and the differential equation (q) has the form

$$y'' = \left(\frac{-1}{\beta^2 C^2(t + d)^2} + \frac{\alpha^2}{\beta^2}\right)y. \tag{19}$$

We find out by calculation that the carrier of the first accompanying equation (q_1) towards (19) with respect to the basis $[\alpha, \beta]$ has the form:

$$q_1 = \frac{-1}{\beta^2 C^2(t + d)^2} + \frac{\alpha^2}{\beta^2} + \frac{2\alpha}{\beta(t + d)}$$

and prove by complete induction that the n-th accompynying equation towards (19) has the form

$$y'' = \left\{ \frac{-1}{\beta^2 C^2 (t+d)^2} + \frac{\alpha^2}{\beta^2} + \frac{2n\alpha}{\beta(t+d)} - (n-1)\, nC^2\alpha^2 \right\} y. \qquad (20)$$

Note: If we put $\alpha = 0$, $\beta = 1$ in relation (20), then the differential equation (20) is identical with (19) and we get the case solved in [2] for $q < 0$.

Theorem 7: If $U \in (q_1)$ is an arbitrary integral, then there exists the integral $u \in (q)$ such that

$$\frac{f(t, u)}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}} = U(t).$$

Proof: $U(t)$ is defined by the following initial conditions for $\tau_0 \in j$:

$$U(\tau_0) = U_0, \qquad U'(\tau_0) = U_0'.$$

It is necessary to choose for $u \in (q)$ from relations

$$\frac{f(\tau_0 ; u_0)}{\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha'(\tau_0)\,\beta(\tau_0) - \beta^2(\tau_0)\,q(\tau_0)}} = U_0,$$

$$\frac{\left[\frac{\mathrm{d}}{\mathrm{d}t} - f(t, u)\right]_{t=\tau_0}}{\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha(\tau_0)\,\beta'(\tau_0) - \beta^2(\tau_0)\,q(\tau_0)}} +$$

$$+ f(\tau_0 ; u_0)\left(\frac{1}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}}\right)_{t=\tau_0} = U_0'$$

an integral $u \in (q)$ satisfying the initial conditions

$$u(\tau_0) = u_0, \qquad u'(\tau_0) = u_0';$$

it is however sufficient to choose

$$u_0 = \frac{1}{\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha'(\tau_0)\,\beta(\tau_0) - \beta^2(\tau_0)\,q(\tau_0)}} \cdot$$

$$\cdot \left\{ U_0 \left[\alpha(\tau_0) + \beta'(\tau_0) + \beta(\tau_0)\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha'(\tau_0)\,\beta(\tau_0) - \beta^2(\tau_0)\,q(\tau_0)}\, \cdot \right.\right.$$

$$\left.\left. \cdot \left(\frac{1}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}}\right)'_{t=\tau_0} \right] - \beta(\tau_0)\, U_0' \right\},$$

$$u_0' = \frac{1}{\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha'(\tau_0)\,\beta(\tau_0) - \beta^2(\tau_0)\,q(\tau_0)}} \cdot$$

$$\cdot \left\{ \alpha(\tau_0)\, U_0' - U_0 \left[\alpha(\tau_0)\sqrt{\alpha^2(\tau_0) + \alpha(\tau_0)\,\beta'(\tau_0) - \alpha'(\tau_0)\,\beta(\tau_0) - \beta^2(\tau_0)\,q(\tau_0)}\, \cdot \right.\right.$$

$$\left.\left. \cdot \left(\frac{1}{\sqrt{\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q}}\right)'_{t=s_0} - \alpha'(\tau_0) - \beta(\tau_0)\, q(\tau_0) \right] \right\}.$$

The proof will be easily verified by direct calculation.

Concluding this paper I should like to express my gratitude to Prof. RNDr. M. Laitoch CSc., for suggesting the idea to study this problem, and for his valuable advice.

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Resume

ZOVŠEOBECNENIE AMPLITÚDY, FÁZY A SPRIEVODNEJ DIFERENCIÁLNEJ ROVNICE

MILOŠ HÁČIK

V tomto článku je skúmaná diferenciálna rovnica

$$y'' = q(t)\,y, \qquad \text{(q)}$$

kde $q(t) \in C_2(j)$ a $q(t) < 0$ pre všetky $t \in j$, z hľadiska vlastností lineárnych kombinácií jej integrálov a ich prvých derivácií vzhľadom na váhové funkcie $[\alpha(t), \beta(t)]$. Funkciou

$$\delta(t) = \sqrt{[\alpha(t)\,u(t) + \beta(t)\,u'(t)]^2 + [\alpha(t)\,v(t) + \beta(t)\,v'(t)]^2}$$

je zavedená zovšeobecnená amplitúda usporiadanej dvojice (u, v) nezávislých integrálov rovnice (q) s váhovými funkciami $[\alpha(t), \beta(t)]$. Ďalej sa z tohto hľadiska prichádza k pojmu fáza usporiadanej dvojice riešení (u, v) rovnice (q) a tiež k pojmu sprievodnej rovnice k rovnici (q) vzhľadom na váhové funkcie $[\alpha(t), \beta(t)]$.