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NOTE ON THE PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

by

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§1. Preliminaries

In this paper we consider the system of differential equations

\[ \dot{x} = A(t)x + f(t), \]

where $A(t)$ is an $n \times n$ continuous matrix of period $p$ and $f(t)$ is a continuous vector of $n$ components of the same period $p$. It is well known [1] that a fundamental system of solutions $Y(t)$ of the corresponding homogeneous system of differential equations

\[ \dot{y} = A(t)y \]

can be obtained such that the constant matrix

\[ P = Y'(t)Y(t+p) \]

has the form $P = e^{Kp}$. Here the constant matrix $K$ has the Jordan canonical normal form

\[
\begin{pmatrix}
K_1 & & \\
& \ddots & \\
& & K_m
\end{pmatrix}
\quad \text{with} \quad
\begin{pmatrix}
z_1 & 1 \\
& \ddots & 1 \\
& & z_m
\end{pmatrix},
\]

where every submatrix $K_m$ is of order $m_v$ as well as the submatrix

\[ P_v = e^{K_v} \]

of $P$. The fundamental system of solutions $Y(t)$ of (b) takes the form

\[ Y(t) = \Phi(t)e^{Kt} \]

(see [2]), where the matrix $\Phi(t)$ has the period $p$. The corresponding adjoint system of differential equations is

\[ \dot{z} = -A^\prime(t)z, \]

where $A^\prime$ denotes the transposed of the matrix $A$. 
We write \( Y(t) \) in the form
\[
(5) \quad Y(t) = (Y_1(t), Y_2(t), \ldots, Y_s(t))
\]
with \( Y_i(t) = (y_{i,1}(t), \ldots, y_{i,m_i}(t)) \),
where \( Y_i(t) \) represents a rectangular matrix of type \( n \times m_i \). The symbols \( (\ast) \) and \( [\ast] \) are defined by
\[
(6) \quad (\ast) = \sum_{j=1}^{s} m_j / 1, \quad [\ast] = \sum_{j=1}^{s} m_j.
\]

We subdivide the fundamental system of solutions \( Z(t) \) of (c) and also \( \Phi(t) \) in a similar way. From (1), (3) and (4), we get
\[
(7) \quad Y_i(t + p) = Y_i(t) P_{ii}, \quad Y_i(t) = \Phi_i(t) e^{K_i t}.
\]

By means of the method of variation of parameters and referring to (5) and (7), the vector solution of (a) can be written as the sum of \( s \) vector components
\[
(8) \quad x(t) = \sum_{j=1}^{s} x_j(t)
\]
with
\[
(9) \quad x(t) = \sum_{j=1}^{s} x_j(t) c_j(t) = Y_i(t) c_i(t) = \Phi_i(t) e^{K_i t} c(t),
\]
where \( c_i(t) \) is the \( m_i \)-dimensional vector
\[
(10) \quad c_i = \begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{in} \end{bmatrix}.
\]

It is interesting to notice that the vector components \( x_j(t) \) of the solution \( x(t) \) of (a) satisfy the system of differential equations
\[
(11) \quad \dot{x} = A(t) x + f(t) \quad \text{with} \quad f(t) = \Phi_i e^{K_i t} Z_i f,
\]
which can be derived from the system (a) (see [3]).

In a previous paper [1] it was proved that to each submatrix \( K_i \) of \( K \) vector solutions \( x_j(t) \) exist such that
\[
(12) \quad x(t) = \sum_{j=1}^{s} x_j(t)
\]
has the period \( p \) in either one of the following two cases:
(i) If the adjoint system (c) possesses the periodic solution \( x_{a_j}(t) \), i.e., if the eigenvalue \( s_j \) of the corresponding submatrix \( K_j \) equals zero and simultaneously
\[
(13) \quad \int_0^p z_{a_j}(\tau) f(x) \, d\tau = 0.
\]
(This is the so-called exceptional case).
(ii) If the eigenvalue \( s_j \) is not equal zero (principal case).

In this paper we investigate for both the preceding cases whether the sum (12), which satisfies the system of differential equations derived from (a), can be subdivided into periodic partial sums of the period \( p \).
§2 Fundamental theorems

Let us subdivide the indices \((v), (v) + 1, \ldots, (v) + m - 1 = [v]\) into two subsets, and let us differ between them by underlining the indices in the first subset or second subset once or twice respectively. Here it will be assumed that the first index \((v)\) belongs to the first subset, i.e. \((v) = (v)\). Accordingly the sum

\[ \mathbf{x} = \sum_{(v)} x_v \]

is subdivided into two partial sums

\[ \mathbf{x} = \mathbf{x} + \mathbf{x} \]

with

\[ \mathbf{x} = \sum_{(v)} x_v \] \[ \mathbf{x} = \sum_{(v)} x_v \]

Analogously the matrix \(Y\) (see (5)) is subdivided into two matrices \(Y_1 + Y_2\), and the vector \(c\) (see (10)) is subdivided into two vectors \(c_1 + c_2\). Referring to (9), it follows that

\[ \mathbf{x} = Y_1 \cdot c_1(t) \text{ and } \mathbf{x} = Y_2 \cdot c_2 \]

We now assume that each of \(\mathbf{x}(t)\) and \(\mathbf{x}(t)\) has the period \(p\). This means the same as the assumption that the first partial sum \(\mathbf{x}(t)\) as well as the total sum \(\mathbf{x}(t) = \mathbf{x} + \mathbf{x}\) has the period \(p\). Then holds the following theorem:

**Theorem 1.** Let the vector \(\mathbf{x} = \sum_{(v)} x_v\) be subdivided into two partial sums (15), where the indices in each partial sum run over a subset of the indices \((v), (v) + 1, \ldots, [v]\) as it is described above, such that each of the partial sums has the period \(p\). Then all components \(x_v(t)\) are identically zero for \(\mu \geq \operatorname{Min}(\mu)\). Consequently the sum \(\mathbf{x}(t)\) is subdivided into two partial sums

\[ \mathbf{x}(t) = \sum_{(v)} x_v(t) \text{ and } \mathbf{x}(t) = \sum_{(v)} x_v(t) \]

with \(\gamma = \operatorname{Min}(\mu)\), in which the first sum has the period \(p\) and the second is identically zero.

**Proof.** We have only to prove that all the components \(x_v(t)\) for \(\mu \geq \operatorname{Min}(\nu)\) vanish identically. Referring to the assumption, the sum

\[ \mathbf{x}(t) = \sum_{(v)} x_v(t) \]

has the period \(p\). Instead of (18), we can also write

\[ \mathbf{x}(t) = \sum_{(v)} y_v \cdot c_v(t) \]

where all functions \(c_v(t)\) for \(\mu \geq \mu\) are identically zero. In particular

\[ c_v(t) = 0 \text{ for } \gamma \geq \operatorname{Min}(\mu) \].
By virtue of the periodicity of \( \dot{x}(t) \) and referring to (9), (17), (3), we get
\[
\dot{x}(t + p) - \dot{x}(t) = \Phi_p(t) e^{Kt} (P, \dot{c}(t + p) - \dot{c}(t))
\]
Consequently, it follows that
\[
(21) \quad P, \dot{c}(t + p) - \dot{c}(t)
\]
with \( P \), from (3). Denoting \( c(t + p) \) by \( \hat{c}(t) \), we get at once by virtue of the assumption (20)
\[
c(t) = 0 = \hat{c}(t)
\]
The system (21) takes then the scalar form
\[
(22) \quad c_p(t) = e^{n,p} \sum_{j=0}^{p-1} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \ scheduled on next page.
of order \([\nu] - \gamma\) by \(P_*\). Hence the system of equations (22) and (22) take the forms (see (20))

\[
(24) \quad e^{\lambda P_*} \dot{c}(t) = c(t)
\]

and

\[
(25) \quad q^T \dot{c}(t) = 0 = q^T c(t)
\]

respectively. By virtue of (24) and (25), we obtain successively the following identities

\[
(26) \quad q^T P_* \dot{c}(t) = 0 \quad \text{for} \quad \mu = 0, 1, \ldots, \nu - \gamma - 1.
\]

If namely (26) is valid for \(\mu\), then it will be also valid for \(\mu + 1\) because

\[
q^T P_* \dot{c}(t) = 0 = q^T P_* \dot{c}(t - p) - q^T P_* \dot{c}(t) = e^{\lambda P_*} q^T P_* \dot{c}(t).
\]

By means of linear combinations of (26), we obtain

\[
(27) \quad q^T (P_* - I) \dot{c}(t) = 0 \quad \text{for} \quad \mu = 0, 1, \ldots, \nu - \gamma - 1,
\]

where \(I\) is a unit matrix of the same order as \(P_*\). Using further linear combinations of the equations (27), we get the system of equations

\[
(28) \quad q^T K^\gamma \dot{c}(t) = 0 \quad \text{for} \quad \mu = 0, 1, \ldots, \nu - \gamma - 1,
\]

where

\[
K^\gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \frac{1}{\rho} \log P_*
\]

(see (2) with \(x = 0\)). W. r. t. convergence see e. g. [4] §8. Here we only need to expand \(K^\gamma\) in the power series

\[
K^\gamma = \frac{1}{\rho} \left[ (P_* - I) + \frac{1}{2} (P_* - I)^2 + \frac{1}{3} (P_* - I)^3 + \cdots \right] + (-1)^{\nu - \gamma} \left( \nu - 1 \right) (P_* - I)^{\nu - \gamma + 1}
\]

(Notice that \((P_* - I)^\mu = 0\) for \(\mu \geq \nu - \gamma\)). From this we get \(K^\gamma\) for \(\mu = 1, 2, \ldots, \nu - \gamma - 1\) by means of the power series. Clearly \(K^0 = I\). Setting \(q^T\)

in (28) and observing that

\[
K^\gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}
\]

we obtain the system of \((\nu - \gamma)\) equations

\[
\begin{pmatrix} p \quad p^2 \quad \cdots \quad p^{\nu - \gamma} \\ p^2 \quad \cdots \quad (p - \gamma)! \\ \vdots \quad \cdots \quad \vdots \\ p \quad \cdots \quad (p - \gamma - 1)! \end{pmatrix} c^{(\nu - \gamma)} = 0,
\]
whose coefficient matrix is a regular triangular matrix. Referring to (20) and (23) it follows that $c_{\gamma}(t) = 0$ for every index $\gamma \leq \mu \leq [\nu]$. Thus the theorem is proved.

As an organization of theorem 1, we state the following

Theorem 2. Let the vector $\mathbf{x} = \sum_{\gamma} x_{\gamma}(t)$ be subdivided into two partial sums

$$\sum_{\gamma} x_{\gamma}(t) \quad \text{and} \quad \sum_{\gamma} x_{\gamma}(t)$$

such that each of the two partial sums has the period $p$. Then it follows in the principal case that:

i) The sum $\mathbf{x}(t) = \sum_{\gamma} x_{\gamma}(t)$ is identically zero.

ii) The vector $f(t)$ satisfies the condition

$$x_{\gamma}(t)f(t) = 0 \quad \text{for} \quad \mu = \gamma, \gamma + 1, \ldots, [\nu] .$$

iii) $\mathbf{x}(t)$ is uniquely determined by means of the vectors $f(t)$. In the exceptional case, it follows that:

i) The sum $\mathbf{x}$ is again identically zero.

ii) The vector $f(t)$ satisfies the condition

$$x_{\gamma}(t)f(t) = 0 \quad \text{for} \quad \mu = \gamma, \gamma + 1, \ldots, [\nu]$$

and the condition

$$\int_{0}^{p} \int_{0}^{p} x_{\gamma}(t)f(t) d\tau = 0 .$$

iii) The sum $\mathbf{x}(t)$ is uniquely determined up to an arbitrary additive multiple of $c_{\gamma}(0)$, $y_{\gamma}(t)$.

Proof. The statement i) in both cases is a direct consequence of theorem 1, because all the functions $c_{\gamma}(t)$ for $\mu = \gamma, \gamma + 1, \ldots, [\nu]$ are identically zero. Moreover, since

$$(30) \quad c_{\gamma}(t) = \int_{0}^{1} x_{\gamma}(\tau) f(\tau) d\tau = c_{\gamma}(0)$$

(see [2], §3) then the integral on the R. S. of (30) must vanish identically in $t$. And consequently the integrand $x_{\gamma}(t)f(t)$ must be identically zero for $\mu = \gamma, \gamma + 1, \ldots, [\nu]$.

In the exceptional case we have to prove also the validity of the statement ii').

Since $c_{\gamma}(t) (\mu = \gamma), \ldots, [\nu]$ vanish identically in $t$ and $e^{(\gamma \cdot t)}$ in the exceptional case, then the $(\gamma \cdot t)$-th equation in (22) becomes

$$c_{\gamma}(t) i(t : p) = c_{\gamma + 1}(t) .$$

And by virtue of (30), we immediately obtain ii').

Whereas the statements i), ii) and ii') necessarily follow from the assumption in the theorem, the statement iii) expresses that the conditions ii) and ii') are sufficient for the existence of a periodic solution of the form $\mathbf{x} = \sum_{\gamma} x_{\gamma}(t)$.
Even the condition \( \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt = 0 \) is sufficient instead of the condition ii').

The necessary and sufficient condition for the existence of a periodic solution \( \mathbf{x}(t) \) for given \( \mathbf{f}(t) \) can be easily obtained from (3), (7) and (3) (see also [1]) as:

\[
\mathbf{x}(t + p) - \mathbf{x}(t) = Y(t) \left[ P_p \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt - (I - P_p) \mathbf{c}(0) \right] = 0
\]

i. e.

\[
(P_p^2 - I) \mathbf{c}(0) = \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt
\]

where \( I_p \) is the unit matrix of order \( m_p \). Or in components

\[
\begin{bmatrix}
    e^{-\alpha_1 t} - 1, & p_1, & \ldots, & (-p_1)^{m_p-1} \\
    e^{-\alpha_2 t} - 1, & p_2, & \ldots, & (-p_2)^{m_p-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    e^{-\alpha_p t} - 1
\end{bmatrix}
\begin{bmatrix}
    c_0(0) \\
    c_1(0) \\
    \vdots \\
    c_{(r)}(0)
\end{bmatrix} =
\begin{bmatrix}
    \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt \\
    \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt \\
    \vdots \\
    \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt
\end{bmatrix}
\]

If the condition ii) is satisfied, then the integrals \( \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt \) on the R. S. of (32) for \( \mu = \gamma, \gamma + 1, \ldots, [r] \) must vanish. Consequently, it follows from (32) in the principal case, that the constants \( c_0(0), c_1(0), \ldots, c_{(r)}(0) \) are equal zero, while the other constants \( c_{(r)}(0), \ldots, c_{(r-1)}(0) \) are uniquely determined because of the regularity of the coefficient matrix.

In the exceptional case, we consider the system (32) with \( e^{-\alpha} = 1 \) (exceptional condition). Referring to the condition ii), the integrals \( \int_{0}^{p} \mathbf{z}^T(t) \mathbf{f}(t) \, dt \) for \( \mu = \gamma, \gamma + 1, \ldots, [r] \) must vanish and hence the integral \( \int_{0}^{p} \mathbf{z}_{(r)}(t) \mathbf{f}(t) \, dt \) must vanish also. Eliminating the first column and the last row of the coefficient matrix in (32), we obtain a system of equations in the constant \( c_{(r-1)}(0), \ldots, c_{(r)}(0) \) with a regular coefficient matrix. Therefore the constants \( c_0(0), \ldots, c_{(r-1)}(0) \) are zeros and the other constants \( c_{(r)}(0), \ldots, c_{(r-1)}(0) \) are uniquely determined. Clearly the constant \( c_{(r)}(0) \) remains arbitrary. Since the functions \( c_{(r)}(t) \) vanish in both cases, (see 30), then the statement i) follows from the formula \( \mathbf{x}(t) = \mathbf{y}(t) \cdot \mathbf{c}_{(r)}(t) \).
NOTE ON THE PERIODICAL SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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In this paper we consider the system of equations \( (*) \), \( \dot{x} = A(t)x + f(t) \) with \( A(t), f(t) \) continuous and of period \( p \). Let \( Y(t) \) be a fundamental system of solutions of the homogeneous system corresponding to \( (*) \) such that the constant matrix \( P = Y^{-1}(t)Y(t + p) \) has the form \( P = e^{Kp} \) and \( K \) has the Jordan canonical normal form with the submatrices \( K^v \) (for \( v = 1, \ldots, s \)) of order \( m_r \). In a previous paper it is proved that to each submatrix \( K_r \) of \( K \), there corresponds in certain cases vector solutions \( x(t) \) such that \( \sum_{v=1}^{s} x_v(t) \) has the period \( p \). Here \( s = \sum_{r=1}^{s} m_r + 1 \), \( [e] = \sum_{r=1}^{s} m_r \) and \( x(t) = \sum_{v=1}^{s} x_v(t) \). In this paper it will be mainly investigated whether the sum \( \sum_{v=1}^{s} x_v(t) \), which satisfies the system of differential equations derived from \( (*) \), can be subdivided into periodic partial sums of the period \( p \).