Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika

Dalibor Klucký

Two theorems on homomorphism of projective nets

Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 16 (1977), No. 1, 149--153

Persistent URL: http://dml.cz/dmlcz/120045

1977 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 53

Katedra algebry a geometrie přírodovědecké fakulty
Vedoucí katedry: prof.RNDr. Ladislav Sedláček, CSc

TWO THEOREMS ON HOMOMORPHISM OF PROJECTIVE NETS

DALIBOR KLUCKÝ

(*Received on January 21st, 1976*)

In this paper, it will be shown how the well—known theorem on homomorphisms of groups, rings and in general universal algebras can be modified for special case of relation structures—projective nets. The idea contained in [3] will be developed in a natural way.

1. Preliminaries

We are dealing only with nets having the same degree whose singular points lie on the same line.

Definition 1. An ordered tripple $N = (\mathbf{P}, \mathbf{L}, \mathbf{s})$, where
(a) $\mathbf{P}$ is a (non-empty) set, whose elements are called points
(b) $\mathbf{L}$ is a system of subsets of $\mathbf{P}$ – so called lines
(c) $\mathbf{s} \in \mathbf{L}$; $\mathbf{s}$ is called the singular line
will be termed *an* (*projective*) *net*, if the following axioms are fulfilled:
A1. $\forall l \in \mathbf{L} : \text{card } l \geqq 3$;
A2. $\exists P \in \mathbf{P} \setminus \mathbf{s}$
A3. $\forall A, B \in \mathbf{P}$, $A \neq B$ there exists at most one line $l \in \mathbf{L}$ such that $A \in l \wedge B \in l$
A4. $\forall l_1, l_2 \in \mathbf{L}$, $l_1 \neq l_2$, $\exists! P \in \mathbf{P} : P \in l_1 \wedge P \in l_2$
A5. $\forall S \in \mathbf{s}$, $\forall A \in \mathbf{P} \setminus \mathbf{s}$, $\exists! l \in \mathbf{L} : S \in l \wedge A \in l$
The points of $\mathbf{P} \setminus \mathbf{s}$ as well as the lines different from $\mathbf{s}$ will be called regular. It is known that all regular lines of the net N have the common cardinality $m + 1$; m is called the order of N, the card $\mathbf{s}$ is the degree of N.

Definition 2. Let $N = (\mathbf{P}, \mathbf{L}, \mathbf{s})$, $N' = (\mathbf{P}', \mathbf{L}', \mathbf{s}')$ be two nets with the same degree. A mapping

$$\varkappa : \mathbf{P} \to \mathbf{P}'$$

will be called *a homomorphism* (of the net N into the net N′), if it has the following properties:

(1) $\varkappa(\mathbf{s}) = \mathbf{s}'$ and $\forall S' \in \mathbf{s}' \, \exists ! \, S \in \mathbf{s} : S' = \varkappa(S)$, i.e. $\varkappa \mid \mathbf{s}$ is a bijection $\mathbf{s} \to \mathbf{s}'$.
(2) $\forall l \in \mathbf{L}, \exists l' \in \mathbf{L}' : \varkappa(l) \subset l'$
(3) $\exists P \in \mathbf{P} \setminus \mathbf{s} : \varkappa(P) \in \mathbf{P}' \setminus \mathbf{s}'$.

The homomorphism $\varkappa$ of the net $\mathbf{N}$ into the net $\mathbf{N}'$ will be denoted by $\varkappa : \mathbf{N} \to \mathbf{N}'$.

If $\varkappa$ is surjective (bijective), we will say that $\varkappa$ is an epimorphism (isomorphism).

If each of homomorphisms $\varkappa_1 : \mathbf{N} \to \mathbf{N}', \varkappa_2 : \mathbf{N}' \to \mathbf{N}''$ is an epimorphism, then, obviously, $\varkappa_2 \circ \varkappa_1$ is an epimorphism.

Definition 3. Let $\mathbf{N} = (\mathbf{P}, \mathbf{L}, \mathbf{s})$ be a net. A net $\mathbf{N}^* = (\mathbf{P}^*, \mathbf{L}^*, \mathbf{s}^*)$ will be called *the subnet* of $\mathbf{N}$, if

(a) $\mathbf{P}^* \subset \mathbf{P}$

(b) $\mathbf{s}^* = \mathbf{s}$

(c) $\forall l^* \in \mathbf{L}^* \, \exists l \in \mathbf{L} : l^* \subset l$ (l is uniquely determined by l^* according to axioms A1., A3.).

It is easy to prove that for any subset l^* of $\mathbf{P}^*$ the condition

$$l^* \in \mathbf{L}^* \Leftrightarrow \exists l \in \mathbf{L} : l^* = l \cap \mathbf{P}^* \wedge \operatorname{card} l^* \geqq 3$$

is fulfilled. It follows from this that two subnets of $\mathbf{N}$ having the same set of (regular) points are equal.

Let $\mathbf{N} = (\mathbf{P}, \mathbf{L}, \mathbf{s})$, $\mathbf{N}' = (\mathbf{P}', \mathbf{L}', \mathbf{s}')$ be two nets. The following statements are proved in [3]:

(i) If $\varkappa : \mathbf{N} \to \mathbf{N}'$ is a homomorphism such that for arbitrary line $l \in \mathbf{L} : \operatorname{card} \varkappa(l) \geqq 3$ is true then $(\varkappa(\mathbf{P}), \varkappa(\mathbf{L}), \mathbf{s}')$ is a subnet of $\mathbf{N}'$. (Here $\varkappa(\mathbf{L})$ denotes the set $\{\varkappa(l) \mid l \in \mathbf{L}\}$.)

(ii) If $\varkappa : \mathbf{N} \to \mathbf{N}'$ is an epimorphism then besides $\varkappa(\mathbf{P}) = \mathbf{P}'$ also $\varkappa(\mathbf{L}) = \mathbf{L}'$ is true.

2. The first theorem on homomorphism

Let us consider an epimorphism

$$\varkappa : \mathbf{N} \to \mathbf{N}'$$

where $\mathbf{N} = (\mathbf{P}, \mathbf{L}, \mathbf{s})$, $\mathbf{N}' = (\mathbf{P}', \mathbf{L}', \mathbf{s}')$ are nets. Let $d_\varkappa$ be the equivalence relation on $\mathbf{P}$ induced by $\varkappa$, i.e.

$$d_\varkappa = \{(x, y) \in \mathbf{P} \times \mathbf{P} \mid \varkappa(x) = \varkappa(y)\}$$

and let $\mathbf{D}_\varkappa$ be the decomposition of $\mathbf{P}$ associated to $d_\varkappa \Rightarrow \mathbf{D}_\varkappa = \mathbf{P}/d_\varkappa$. We can in a very natural way establish the structure of a net onto $\mathbf{D}_\varkappa$ by asking, the cannonical mapping $\mathbf{P} \to \mathbf{D}_\varkappa$ to be an epimorphism of nets. We obtain a net $\bar{\mathbf{N}} = (\bar{\mathbf{P}}, \bar{\mathbf{L}}, \mathbf{s})$ where $\bar{\mathbf{P}} = \mathbf{D}_\varkappa$, the subset l of $\bar{\mathbf{P}}$ belongs to $\bar{\mathbf{L}}$ if and only if there exists a line $l \in \mathbf{L}$ such that

$$\bar{l} = \{\bar{X} \in \bar{\mathbf{P}} \mid \bar{X} \cap l \neq \emptyset\}$$

and finally

$$\bar{\mathbf{s}} = \{\bar{X} \in \bar{\mathbf{P}} \mid \bar{X} \cap \mathbf{s} \neq \emptyset\}.$$

We denote the net $\bar{\mathbf{N}}$ by $\mathbf{N}/\boldsymbol{d}_{\varkappa}$.

Theorem 1. Let $\varkappa_1 : \mathbf{N} \to \mathbf{N}_1$, $\varkappa_2 : \mathbf{N} \to \mathbf{N}_2$ be two epimorphisms of nets and let $\boldsymbol{d}_{\varkappa_1} \subset \boldsymbol{d}_{\varkappa_2}$ ($\Rightarrow \mathbf{D}_{\varkappa_1}$ is a refinement of $\mathbf{D}_{\varkappa_2}$). Then there exists an unique epimorphism $\varkappa : \mathbf{N}_1 \to \mathbf{N}_2$ such that the diagram

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{\varkappa_2} & \mathbf{N}_2 \\ {\scriptstyle \varkappa_1}\downarrow & \nearrow {\scriptstyle \varkappa} & \\ \mathbf{N}_1 & & \end{array}$$

is commutative. Moreover, if $\boldsymbol{d}_{\varkappa_1} = \boldsymbol{d}_{\varkappa_2}$, then $\varkappa$ is an isomorphism.

Proof: Let $\mathbf{N} = (\mathbf{P}, \mathbf{L}, \mathbf{s})$, $\mathbf{N}_i = (\mathbf{P}_i, \mathbf{L}_i, \mathbf{s}_i)$, $i = 1, 2$. The existence and the uniqueness of the mapping $\varkappa$ of $\mathbf{P}_1$ onto $\mathbf{P}_2$ with $\varkappa_2 = \varkappa \circ \varkappa_1$ follow from the elementary set theory. It remains to prove that $\varkappa$ is a homomorphism. We have to verify the conditions (1)–(3) from definition 2.

(1) Let $S_2 \in \mathbf{s}_2$, then there exists the unique point $S \in \mathbf{s}$ with $\varkappa_2(S) = S_2$. Hence:

$$S_1 \in \mathbf{s}_1 \wedge \varkappa(S_1) = S_2 \Leftrightarrow S_1 = \varkappa_1(S)$$

Such a point $S_1 \in \mathbf{s}_1$ is uniquely determined.

(2) Let $l_1 \in \mathbf{L}_1$, then there exists a line $l \in \mathbf{L}$ such that $l_1 = \varkappa_1(l)$; $\varkappa_2(l)$ is a line of $\mathbf{L}_2$ and clearly $\varkappa(l_1) = (\varkappa_0\, \varkappa_1)(l) = \varkappa_2(l)$.

(3) Let $P \in \mathbf{P}$ be the point whose image $\varkappa_2(P)$ is regular. Putting $P_1 = \varkappa_1(P)$, we obtain $\varkappa(P_1) = (\varkappa \circ \varkappa_1)(P) = \varkappa_2(P) \notin \mathbf{s}_2$.

Corolary: If $\varkappa : \mathbf{N} \to \mathbf{N}'$ is an epimorphism of nets, then there exists a cannonicaly determined isomorphism of $\mathbf{N}/\boldsymbol{d}_{\varkappa}$ onto $\mathbf{N}'$.

3. Normal decompositions and their normal coverings (The second theorem on homomorphism)

It is shown in [3] that the decomposition $\mathbf{D}$ belonging to an epimorphism $\varkappa : \mathbf{N} \to \mathbf{N}'$ of nets ($\Rightarrow \mathbf{D} = \mathbf{D}_{\varkappa}$) can be described by inner properties with respect to net $\mathbf{N}$, only:

A decomposition $\mathbf{D}$ of $\mathbf{P}$ belongs to an epimorphism of $\varkappa$ if and only if it is fulfilled:

(a) There exists at most one singular point in any class of $\mathbf{D}$.

(b) If two lines l_1, l_2 meet two different classes of $\mathbf{D}$, then each class of $\mathbf{D}$ is meeted by both or by none of them.

(c) There exists at least one class containing no singular point.

(d) Every line $l \in \mathbf{L}$ meets at least three different classes of $\mathbf{D}$.

Definition 4. The decomposition **D** of **P** having the properties (a)–(d) will be called *the normal decomposition* of the net **N** = (**P**, **L**, **s**). The equivalence relation $\boldsymbol{d}$ on **P** associated to **D** will be called *a normal equivalence relation on* **N**.

Let us consider a normal decomposition **D** of the net **N**. Let **D**′ be the covering of **D** ($\Rightarrow \boldsymbol{d} \subset \boldsymbol{d}'$, where $\boldsymbol{d}'$ is the equivalence relation associated to **D**′). The **D**′ generates a decomposition of $\mathbf{P}/\boldsymbol{d}$ denoted by $\mathbf{D}/\boldsymbol{d}'$ in the following way: Two classes $\bar{A}$, $\bar{B}$ of **D** (A, B $\in$ **P**) belong to the same class of $\mathbf{D}/\boldsymbol{d}'$ if and only if A, B belong to the same class of **D**′.

Definition 5. The covering **D**′ of the normal decomposition **D** of the net **N** will be said *the normal covering of* **D**, if **D**′ is a normal decomposition of **N**.

Theorem 2. Let **D** be a normal decomposition of the net **N** = (**P**, **L**, **s**). Then the covering **D**′ of **D** is normal covering of **D** if and only if $\mathbf{D}/\boldsymbol{d}'$ is the normal decomposition of $\mathbf{N}/\boldsymbol{d}$.

Proof: The theorem 2 follows from the elementary set considerations: If **D**′ is a normal covering of **N**, then according to theorem 1, there exists the epimorphism $\varkappa : \mathbf{N}/\boldsymbol{d} \to \mathbf{N}/\boldsymbol{d}'$ such that the diagram

$$\begin{array}{ccc} \mathbf{N} & \longrightarrow & \mathbf{N}/\boldsymbol{d}' \\ \downarrow & \nearrow_{\varkappa} & \\ \mathbf{N}/\boldsymbol{d} & & \end{array}$$

where $\mathbf{N} \to \mathbf{N}/\boldsymbol{d}$, $\mathbf{N} \to \mathbf{N}/\boldsymbol{d}'$ are the cannonical epimorphisms, is commutative. The decomposition of $\mathbf{N}/\boldsymbol{d}$ associated to $\varkappa$ is just $\mathbf{D}/\boldsymbol{d}'$. Conversely, if $\mathbf{D}/\boldsymbol{d}'$ is normal, then the decomposition associated to

$$\mathbf{N} \to \mathbf{N}/\boldsymbol{d} \to (\mathbf{N}/\boldsymbol{d}')/(\mathbf{D}/\boldsymbol{d}')$$

is just **D**′.

Corolary: Let **D** be the normal decomposition of the net **N** and **D**′ its normal covering. Then the nets

$$\mathbf{N}/\boldsymbol{d}', \qquad (\mathbf{N}/\boldsymbol{d})/(\mathbf{D}/\boldsymbol{d}')$$

are (cannonically) isomorphic.

Proof: The cannonical epimorphism $\mathbf{N} \to \mathbf{N}/\boldsymbol{d}'$ and the epimorphism $\mathbf{N} \to \to \mathbf{N}/\boldsymbol{d} \to (\mathbf{N}/\boldsymbol{d})/(\mathbf{D}/\boldsymbol{d}')$ ($\mathbf{N} \to \mathbf{N}/\boldsymbol{d}$ and $\mathbf{N}/\boldsymbol{d} \to (\mathbf{N}/\boldsymbol{d})/(\mathbf{D}/\boldsymbol{d}')$ cannonical) have the same associated decomposition $\boldsymbol{d}'$.

References

[1] *Бурбаки Н.*: Теория множеств, изд. «мир», Москва 1965 (russian translation from french original N. Bourbaki: Théorie des ensembles).
[2] *Havel*, *Václav*: Homomorphisms of nets of a fixed degree with singular points on the same line, Czech. Math. J. 25 (100) (1975).
[3] *Jachanová*, *Jaroslava*: Characterization of epimorphisms of nets by normal decompositions, AUPO 1976.

Shrnutí

DVĚ VĚTY O HOMOMORFISMU PROJEKTIVNÍCH SÍTÍ

Dalibor Klucký

V článku jsou modifikovány věty o homomorfismu grup pro homomorfismy projektivních sítí. Uvažují se jen sítě téhož stupně, jejichž singulární body leží na přímce.

Резюме

ДВЕ ТЕОРЕМЫ ОБ ГОМОМОРФИЗМЕ ПРОЕКТИВНЫХ СЕТЕЙ

Далибор Клуцки

В статье доказаны теоремы являющиеся модификациями известных теорем о гомоморфизмах групп для случая проективных сетей. Рассматриваются только сети одинаковой степени, особые точки которых росположены на одной прямой.