Dina Štěrbová

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Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 16 (1977), No. 1, 155--160

Persistent URL: `http://dml.cz/dmlcz/120046`

1977 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 53

Katedra algebry a geometrie přírodovědecké fakulty
Vedoucí katedry: prof. RNDr. Ladislav Sedláček, CSc.

ON THE SPECTRAL RADIUS IN LOCALLY MULTIPLICATIVELY CONVEX TOPOLOGICAL ALGEBRAS

DINA ŠTĚRBOVÁ

(*Received on April 15th, 1976*)

In this paper, we will show a characterization of hermitian complete locally multiplicatively—conves topological in general noncommutative algebras with a projective involution, based on the V. Pták's fundamental inequality for a hermitian Banach algebra with involution.

1. Introduction

The notion of a seminormed algebra was introduced by R. Arens as a generalization of Banach algebras. They are called locally multiplicatively-convex algebras by E. A. Michael [3]. Several properties of Banach algebras have been generalized by A. E. Michael [3], Sa-do-šin [7] and Ch. Wenjen [8] to semi-normed algebras. A Banach algebra with involution is said to be hermitian if the spectrum of each selfadjoint element is real.

V. Pták [5] fully recognized the importance of the function $p(x) = \sqrt{| x^*x |_\sigma}$, the square root of the spectral radius of the element x^*x. The inequality $| x |_\sigma \leqq p(x)$ which V. Pták proved for hermitian Banach algebras with involution plays a fundamental rôle in the theory of hermitian algebras. The same autor [6] built a theory of hermitian Banach algebras and their connections with C^*-algebras based on this fundamental inequality.

The aim of the present paper is to generalize some of V. Pták's results to locally multiplicatively-convex algebras. Speaking more closely we are going to show the rôle of V. Pták's fundamental inequality for hermitian complete in general noncommutative locally multiplicatively-convex algebras with a projective involution.

2. Preliminaries

Let A be an algebra over the complex field. A is said to be a topological algebra if it is also a topological space. An involution on A is a map $x \to x^*$ of A onto itself such that for each $x, y \in A$ and for each complex λ

1. $x^{**} = x$
2. $(x + y)^* = x^* + y^*$
3. $(\lambda x)^* = \bar{\lambda} x^*$
4. $(x \,.\, y)^* = y^* \,.\, x^*$

A *-algebra is an algebra endowed with an involution. An element $x \in A$ is said to be regular, selfadjoint, normal respectively if it holds that there exists an inverse to x, $x = x^*$, $x^* \,.\, x = x \,.\, x^*$ respectively. The set of all regular, selfadjoint, normal elements of A will be denoted by $R(A)$, $H(A)$, $N(A)$ respectively. For any set $S \subset A$ let $S^* = \{x^* : x \in S\}$. If $S = S^*$, we say S is selfadjoint. If the elements of $S \cup S^*$ are pairwise commutative, we say S is normal. A nonnegative real valued function p defined on A is called a semi-norm if it satisfies the following conditions:

1. $p(x + y) \leqq p(x) + p(y)$
2. $p(x \,.\, y) \leqq p(x) \,.\, p(y)$
3. $p(\lambda x) = |\lambda| \, p(x)$, for each $x, y \in A$ and λ complex.

A topological algebra A is said to be locally multiplicatively convex if its topology can be given by means of a family $(p_\alpha)_{\alpha \in \Sigma}$ of semi-norms on A which separates points of A. It means there is a base of neighbourhoods of the origin in A consisting of sets $\left\{x \in A : p_\alpha(x) < \frac{1}{n}\right\}$ for $n = 1, 2, \dots$ and $\alpha \in \Sigma$. The class of locally multiplicatively-convex algebras will be designated by LMC. In this paper we are dealing with LMC *-algebras. The spectrum of an element $x \in A$ will be denoted by $\sigma(x)$. If it is necessary to specify the algebra with respect to which the spectrum is taken, we use the notation $\sigma(A, x)$. The spectral radius of the element $x \in A$ is denoted by $|x|_\sigma$ and it is defined $|x|_\sigma = \sup \{|\lambda| : \lambda \in \sigma(x)\}$. In *-algebras we can define the spectral norm of an element $x \in A$ as follows: $p(x) = \sqrt{|x^* \,.\, x|_\sigma}$. The unit element of A (if it exists) will be designated by e and will be left in expressions like $\lambda - x$. An involution * on A is called hermitian if the spectrum $\sigma(x)$ is real for each $x \in H(A)$. The algebra A with involution is called hermitian if this involution is hermitian. If we set $N_\alpha = \{x \in A : p_\alpha(x) = 0\}$ for $\alpha \in \Sigma$ where A is LMC algebra with system of seminorms $(p_\alpha)_{\alpha \in \Sigma}$, we obtain a closed ideal in A. Let us designate by A_α the Banach algebra obtained by completion of the normed algebra $(A/N_\alpha, p_\alpha)$. Designate by π_α the natural homomorfism of A into A_α. Let $\tilde{A} = \prod_{\alpha \in \Sigma} A_\alpha$ (with Cartesian product topology and coordinatewise operations). Let's designate by π the mapping $\pi : A \to \prod_{\alpha \in \Sigma} A_\alpha$, $\pi(x) = (\pi_\alpha(x))_{\alpha \in \Sigma}$. This mapping is an isomorfism. This fact yields the following theorem:

2.1. Theorem: Let A be a LMC algebra, then A is isomorfic with a subalgebra of the Cartesian product of Banach algebras. If A is complete the subalgebra is closed.

Proof: Can be found in [9, p. 89].

We recall now the concept of projective limit of topological linear spaces. Let Σ be an index set, directed by the relation $<$. Let $\{X_\alpha, \alpha \in \Sigma\}$ be a system of topological linear spaces. Suppose that for every α, $\beta \in \Sigma$ such that $\alpha < \beta$ there is a linear map $\pi_{\alpha,\beta}$, $\pi_{\alpha,\beta} : X_\beta \to X_\alpha$ and the system of maps $(\pi_{\alpha,\beta})$ satisfies $\pi_{\alpha\beta} \cdot \pi_{\beta\gamma} = \pi_{\alpha\gamma}$ if $\alpha < \beta < < \gamma$. The inverse or projective limit of $(X_\alpha)_{\alpha\in\Sigma}$ designated by $\varprojlim X_\alpha$ is the subset of the Cartesian product $\prod_{\alpha\in\Sigma} X_\alpha$ consisting of those elements $(x_\alpha)_{\alpha\in\Sigma}$, $x_\alpha \in X_\alpha$ for which it is $\pi_{\alpha\beta}(x_\beta) = x_\alpha$ whenever $\alpha < \beta$. If X_α are Banach spaces, then $\lim X_\alpha$ is a complete locally-convex space. (If p_α is the norm in X_α, then for $x = (x_\alpha)_{\alpha\in\Sigma} \in \varprojlim X_\alpha$ we put $q_\alpha(x) = p_\alpha(x_\alpha)$ and this is a system of seminorms defining the topology of $\varprojlim X_\alpha$).

Let now A be a complete LMC algebra with a system of semi-norms $(p_\alpha)_{\alpha\in\Sigma}$, satisfying the conditions above. Write $\alpha < \beta$ for α, $\beta \in \Sigma$ if p_α is continuous with respect to p_β. This relation makes of Σ a directed system, since we can assume without any loss of generality that the maximum of a finite number of members of Σ is again in Σ. Let A_α be the algebra obtained by completion of $(A/N_\alpha, p_\alpha)$ and let again π_α be the natural homomorfism of A into A_α. Let $\alpha < \beta$. We define $\pi_{\alpha\beta}$ as the mapping of normed algebras $(A/N_\beta, p_\beta)$ into $(A/N_\alpha, p_\alpha)$ given by

$$\pi_{\alpha,\beta}(\pi_\beta(x)) = \pi_\alpha(x),$$

so that $\pi_{\alpha,\beta}$ is a homomorfism of A/N_β onto A/N_α. It is evident that $\pi_{\alpha,\beta}$ is a continuous function and thus can be extended to a homomorfism of A_β into A_α. This extended homomorfism will also be designated by $\pi_{\alpha,\beta}$. Following theorem is wellknown:

2.2. Theorem: Let A be a complete LMC algebra. Then A is isomorfic with the inverse limit of the Banach algebras A_α, with mappings $\pi_{\alpha,\beta}$.

Proof: Can be found for instance in [9].

We recall now some useful corollaries of this theorem.

2.3. Corollary: If $(x_\alpha)_{\alpha\in\Sigma} \in \prod_{\alpha\in\Sigma} A_\alpha$ and $\pi_{\alpha,\beta}(x_\beta) = (x_\alpha)$ whenever $\alpha < \beta$, then there exists an element $x \in A$ such that $\pi_\alpha(x) = x_\alpha$ for all $\alpha \in \Sigma$.

Proof: It is obvious by the definition of the projective limit.

2.4. Corollary: If A is as above and $x \in A$, then $x \in R(A)$ if and only if $\pi_\alpha(x) \in R(A_\alpha)$ for all $\alpha \in \Sigma$.

Proof: Can be found in [9].

Now it is clear that in a complete LMC algebra the following equality for the spectra holds:

$$\sigma(x) = \bigcup_{\alpha\in\Sigma} \sigma(\pi_\alpha(x), A_\alpha).$$

For spectral radius follows:

$$|\, x\,|_\sigma = \sup\{|\,\lambda\,| : \lambda \in \sigma(A, x)\} = \sup_{\alpha\in\Sigma} |\,\pi_\alpha(x)\,|_\sigma .$$

2.5. Definition: Let A be a complete LMC *-algebra. Its involution * is said to be projective if the following holds:

(i) For each $\alpha \in \Sigma$ the A_α is *-algebra.

(ii) For each $x \in A$, $x \leftrightarrow (\pi_\alpha(x))_{\alpha\in\Sigma}$ where $\leftrightarrow$ denotes the identifying isomorfiic mapping π from theorem 2.2. then $x^* \leftrightarrow ((\pi_\alpha(x))^*)_{\alpha\in\Sigma} = (\pi_\alpha(x^*))_{\alpha\in\Sigma}$.

A complete LMC *-algebra is said to be projective if its involution is projective.

3. Hermitian projective complete LMC*algebras

In this part we shall give a characterization of hermitian projective complete LMC *-algebras based on V. Pták's fundamental inequality. At first some modified version of square root lemma for such algebras must be developed. We begin with the original J. W. M. Ford's lemma [2, 5].

3.1. Theorem: Let A be a Banach *-algebra. Let $h \in H(A)$ and suppose that the real part of each point $\lambda \in \sigma(h)$ is positive. Then there exists a $u \in H(A)$ such that u commutes with h and $u^2 = h$. Moreover, if $\sigma(h)$ is positive, then so is $\sigma(u)$.

Proof: Can be found in [2, 5].

3.2. Corollary: Let A be a projective complete LMC *-algebra. Let $h \in H(A)$ and suppose that the real part of each point $\lambda \in \sigma(h)$ is positive. Then for each $\alpha \in \Sigma$ there exists an element $u_\alpha \in H(A_\alpha)$ such that u_α commutes with $\pi_\alpha(h)$ and $u_\alpha^2 = \pi_\alpha(h)$. Moreover, if $\sigma(h)$ is positive, then so is $\sigma(u)$.

Proof: Since $\sigma(h) = \bigcup_{\alpha\in\Sigma} \sigma(\pi_\alpha(x), A_\alpha)$, we have the real part of each $\lambda \in \sigma(\pi_\alpha(h), A_\alpha)$ $(= \sigma(\pi_\alpha(h)))$ positive for all $\alpha \in \Sigma$. Due to the projectivity of the involution in A we have

$$(\pi_\alpha(h))^* = \pi_\alpha(h^*) = \pi_\alpha(h)$$

and so $\pi_\alpha(h) \in H(A_\alpha)$ for all $\alpha \in \Sigma$. Thus we can use theorem 3.1. to each $\pi_\alpha(h)$ and A_α separately and we immediately obtain the desired result. Q.E.D.

Now we are able to state the main result.

3.3. Theorem: Let A be a complete projective LMC *-algebra. Then the following conditions are equivalent:

(i) A is Hermitian,

(ii) $|\pi_\alpha(x)|_\sigma \leqq p(\pi_\alpha(x))$ for each $x \in A$ and all $\alpha \in \Sigma$,

(iii) $|\pi_\alpha(x)|_\sigma \leqq p(\pi_\alpha(x))$ for each $x \in N(A)$ and all $\alpha \in \Sigma$.

Proof: (i) $\rightarrow$ (ii).

This implication is equivalent to the following one: Given $x \in A$, $\alpha \in \Sigma$ and complex number λ with $|\lambda| > p(\pi_\alpha(x))$ implies the existence of inverse element to the element $(\lambda - \pi_\alpha(x))$ in A_α. We shall prove that in this case there exists the right and also left inverse in A_α. At first we note that $\sigma(x^*x, A) = \sigma(xx^*, A)$. See e.g. [1]. Since xx^*, $x^*x \in H(A)$ it follows from the fact that A is hermitian that both spectra $\sigma(xx^*, A)$, $\sigma(x^*x, A)$ are real. It is clear that $p(x) = p(x^*)$ and $p(\pi_\alpha(x)) = p(\pi_\alpha(x^*))$ for all $\alpha \in \Sigma$. Setting $y = \lambda^{-1} . x$ it suffices to prove that $p(\pi_\alpha(y)) = p(\pi_\alpha(y^*)) < 1$ implies the left and right regularity of $\pi_\alpha(y)$. Hence assume $p(\pi_\alpha(y)) < 1$. Then it follows that $\sigma(\pi_\alpha(1 - yy^*)) = \sigma(\pi_\alpha(1 - y^*y))$ consists of only positive real numbers and so by the square root theorem there exists suitable $w_1, w_2 \in H(A_\alpha)$ such that the following holds: $w_1^2 = \pi_\alpha(1 - y^*y)$, $w_2^2 = \pi_\alpha(1 - yy^*)$.

Now, we have:

$$\pi_\alpha((1 + y^*)(1 - y)) = \pi_\alpha(1 + y^* - y - y^*y) = w_1^2 + \pi_\alpha(y^* - y) =$$
$$= w_1(\pi_\alpha(1 + w_1^{-1}\pi_\alpha(y^* - y)\, w_1^{-1})\, w_1$$

and

$$\pi_\alpha((1 - y)(1 + y^*)) = w_2(\pi_2(1 + w_2^{-1}\pi_\alpha(y^* - y)\, w^{-1}))\, w_2 .$$

Here w_1, w_2 are regular and so are the expressions in the last two brackets since $w_1^{-1}(i\pi_\alpha(y^* - y))\, w_1^{-1}$ and $w_2^{-2}(i\pi_\alpha(y^* - y))\, w_2$ are selfadjoint.

So, the first line of equalities implies the left regularity of $\pi_\alpha(1 - y)$ while the second line of enequalities implies the right regularity.

(ii) → (iii) is trivial,

(iii) → (i).

Suppose there exists an $h \in H(A)$ such that $(\gamma + i\delta) \in \sigma(h)$, γ, δ real, $\delta \neq 0$. Then $a = \delta^{-1}(h - \gamma)$ is selfadjoint and $i \in \sigma(a)$. Using the fact $\sigma(a, A) = \bigcup_{\alpha \in \Sigma} \sigma(\pi_\alpha(a))$ there exists $\alpha_0 \in \Sigma$ such that $i \in \sigma(\pi_{\alpha_0}(a))$. For each $\tau > 0$ the element $a + i\tau$ is normal and $(\tau + 1)\, i \in \sigma(\pi_{\alpha_0}(a + \tau i))$. Now, from the Gelfand representation theory and by (iii) it follows:

$$|(\tau + 1)\, i|^2 \leqq |\pi_{\alpha_0}(a + \tau i)|^2 \leqq |\pi_{\alpha_0}(a + \tau i)\, \pi_{\alpha_0}(a + \tau i)^*|_\sigma =$$
$$= |\pi_{\alpha_0}((a + \tau i)(a + \tau i)^*)|_\sigma = |\pi_{\alpha_0}(a^2 + \tau^2)|_\sigma \leqq \tau^2 + |\pi_{\alpha_0}(a^2)|_\sigma .$$

Whence

$$2\tau + 1 \leqq |\pi_{\alpha_0}(a^2)|_\sigma$$

for all $\tau > 0$ which is a contradiction with the compactness of spectra in Banach algebras Q.E.D.

References

[1] *Bonsall, F. F., Duncan, J.*: Complete Normed Algebras. Springer-Verlag 1973.

[2] *Ford, J. W. M.*: A square root lemma for Banach star algebras. J. Lond. Math. Soc. 42, p. 521—522 (1967).

[3] *Michael, E. A.*: Locally Multiplicatively-convex Topological Algebras, Memoirs of AMS nb. 11, 1952.

[4] *Najmark, M. A.*: Normirovannyje kolca, Moskva 1968.
1952.
[5] *Pták, V.*: Banach Algebras with involution, manuscripta math. 6, p. 245—290 (1972), Springer Verlag.
[6] *Pták, V.*: On the spectral radius in Banach algebras with involution, Bull. Lond. Math. Soc. 2, p. 327—334 (1970).
[7] *Sa-do-šin*: O polunormirovannych kolcach s involuciej Izv. Ak. Nauk SSSR 23 (1959) 509—528.
[8] *Wenjen, Ch.*: Os semi-normed * -algebras, Proc. J. Math. 8 (1958), p. 177—186.
[9] *Zełazko, W.*: Selected Topics in Topological Algebras Lecture Notes Series N. 31 (1971), Aarhus University.
[10] *Zełazko, W.*: Algebry Banacha, Warszava 1966.

Shrnutí

O SPEKTRÁLNÍM POLOMĚRU V LOKÁLNĚ MULTIPLIKATIVNĚ KONVEXNÍCH TOPOLOGICKÝCH ALGEBRÁCH

Dina Štěrbová

V práci jsou charakterizovány hermiteovské úplné lokálně multiplikativně-konvexní topologické algebry s projektivní involucí. Charakterizace je založena na fundamentální nerovnosti V. Ptáka pro Banachovy algebry s hermiteovskou involucí.

Резюме

О СПЕКТРАЛЬНОМ РАДИУСЕ В ПОЛУНОРМИРОВАННЫХ КОЛЬЦАХ С ИНВОЛЮЦИЕЙ

Дина Штербова

В статье охарактеризированы полные полунормированные кольца с вполне симметрической инволюцией. При этом существительным образом используется фундаментальное неравенство В. Птака для вполне симметрических полных колец с инволюцией.