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NOTE ON THE OSCILLATORY BEHAVIOUR OF BOUNDED SOLUTIONS OF A HIGHER ORDER DIFFERENTIAL EQUATION WITH RETARDED ARGUMENT

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This paper contains two theorems giving sufficient conditions for bounded solutions of the nth order differential equation with retarded argument to be oscillatory. The assertions of those theorems are not true for the corresponding ordinary differential equation.

Some theorems which have specific character for the first, second and the nth order differential equations with retarded argument are given in works [3] – [5]. In [1] D. L. Lovelady had been studying the asymptotic behavior of bounded solutions of differential equations

$$(p_{n-1}(\dots p_2(p_1u')'\dots)')' + (-1)^{n+1}qu = 0,$$
$$(p_{n-1}(t)(\dots p_2(t)(p_1(t)\,u'(t))'\dots)')' + (-1)^{n+1}F(t, u) = 0$$

and the oscillatory behaviour of bounded solutions of differential equations

$$(p_{n-1}(\dots p_2(p_1u')'\dots)')' + (-1)^{n}qu = 0,$$
$$(p_{n-1}(t)(\dots p_2(t)(p_1(t)\,u'(t))'\dots)')' + (-1)^{n}F(t, u) = 0.$$

We consider the nth order differential equation with retarded argument

$$(p_{n-1}(t)(\dots p_2(t)(p_1(t)\,y'(t))'\dots)')' + (-1)^{n+1}q(t)\,y(g(t)) = 0, \tag{1}$$

where

$$p_1, \dots, p_{n-1} \in C^1[[0, \infty), (0, \infty)], \tag{2}$$
$$q \in C[[0, \infty), [0, \infty)], \tag{3}$$
$$g \in C[[0, \infty), R], \qquad g(t) \leqq t, \qquad \lim_{t\to\infty} g(t) = \infty. \tag{4}$$

A function $y \in C[[0, \infty), R]$ which satisfies the initial conditions $y(t) = \Phi(t)$, $t \leqq 0$, $\Phi \in C[E_0, R]$, (E_0 is the initial set) $y^{(k)}(0) = y_0^{(k)}$, $k = 1, 2, \dots, n-1$, is called a solution of (1) if and only if y is differentiable, $p_1 y'$ is differentiable, $p_2(p_1 y')'$ is differentiable, ..., $p_{n-1}(\dots p_2(p_1 y')'\dots)'$ is differentiable, and (1) is true.

A solution $y(t)$ of the equation (1) is called oscillatory if the set of zeros of $y(t)$ is not bounded from the right. A solution $y(t)$ of the equation (1) is called nonoscillatory if it is eventually of constant sign. We consider only such solutions that are not trivial for all sufficiently large t.

Theorem 1. *Assume that*

$$p_1, \dots, p_{n-1} \quad \text{are nonincreasing functions,} \tag{5}$$

$$t - g(t) \geqq h_0 > 0, \tag{6}$$

$$\limsup_{t\to\infty} \left[p_1(t) \dots p_{n-1}(t) - \int_t^{t+h_0} \frac{(s-t)^{n-1}}{(n-1)!} q(s)\,ds\right] < 0. \tag{7}$$

Then every bounded solution of (1) *is oscillatory.*

Proof. We shall use the methods from [1] and [2]. Let $y(t)$ be a bounded nonoscillatory solution of (1). We may suppose without any loss of generality that $y(t) > 0$ for $t \geqq t_0$, $t_0 \in [0, \infty)$ (the case $y(t) < 0$ is treated similarly). By (4) there exists $t_1 \geqq t_0$ such that $g(t) \geqq t_0$ for $t \geqq t_1$. Thus, $y(g(t)) > 0$ for $t \geqq t_1$. Let $v_1 = y(t)$, $v_2 = p_1 v_1'$, ..., $v_n = p_{n-1} v_{n-1}'$ on $[t_1, \infty)$. Now the system

$$\begin{aligned} v_1' &= \frac{v_2}{p_1} \\ v_2' &= \frac{v_3}{p_2} \\ &\vdots \\ v_{n-1}' &= \frac{v_n}{p_{n-1}} \\ v_n' &= -(-1)^{n+1} q y(g) \end{aligned} \tag{8}$$

is satisfied.

By (8), v_n' is one-signed on $[t_1, \infty)$, so v_n is eventually one-signed. Thus v_{n-1}' is eventually one-signed, so v_{n-1} is eventually one-signed. Continuing this, we see that there is t_2 in (t_1, ∞) such that each v_k, $1 \leqq k \leqq n$, is one-signed on $[t_2, \infty)$. Now we shall prove that if $k \geqq 2$ then $v_k v_k' \leqq 0$ in $[t_2, \infty)$. If $k \geqq 2$ and $t \geqq t_2$ then

$$v_{k-1}(t) = v_{k-1}(t_2) + \int_{t_2}^{t} \frac{v_k(s)}{p_{k-1}(s)}\,ds. \tag{9}$$

Suppose that $k \geqq 2$ and $v_k v_k' \leqq 0$ fails on $[t_2, \infty)$. Since v_k and v_k' are both one-signed on $[t_2, \infty)$, we see that $v_k v_k' > 0$ on $[t_2, \infty)$ for some $k \geqq 2$. Thus v_k is either eventually positive and nondecreasing or eventually negative and nonincreasing. In either case, (9) and (5) say that v_{k-1} is unbounded and has the same eventual sign as v_k. Repeating this procedure $k - 1$ times, we see that $y(t)$ is unbounded, a contradiction, so we conclude that $v_k v_k' \leqq 0$ on $[t_2, \infty)$ whenever $k \geqq 2$. By (8) and $v_k v_k' \leqq 0$ for $k \geqq 2$ is $v_k \leqq 0$ on $[t_2, \infty)$ if k is even and $v_k \geqq 0$ on $[t_2, \infty)$ if k is odd. Thus each v_k, $k \geqq 1$, is either nonnegative and nonincreasing or nonpositive and nondecreasing

Integrating the last equation of (8) from $t > t_2$ to ∞, we have

$$(-1)^{n+1}[v_n(\infty) - v_n(t)] = -\int_t^\infty q(s)\, y(g(s))\, ds,$$

$$(-1)^{n+1} v_n(t) \geqq \int_t^\infty q(s)\, y(g(s))\, ds,$$

$$(-1)^{n+1} p_{n-1}(t)\, v_{n-1}'(t) \geqq \int_t^\infty q(s)\, y(g(s))\, ds.$$

Integrating the above inequality from t to $u > t > t_2$,

$$(-1)^{n+1}[p_{n-1}(u)\, v_{n-1}(u) - p_{n-1}(t)\, v_{n-1}(t) - \int_t^u p_{n-1}'(s)\, v_{n-1}(s)\, ds] \geqq$$
$$\geqq \int_t^u (s - t)\, q(s)\, y(g(s))\, ds + \int_u^\infty (u - t)\, q(s)\, y(g(s))\, ds.$$

In view of (5) and letting $u \to \infty$, we have

$$(-1)^{n+1}[-p_{n-1}(t)\, v_{n-1}(t)] \geqq \int_t^\infty (s - t)\, q(s)\, y(g(s))\, ds,$$

$$(-1)^{n+1}[-p_{n-1}(t)\, p_{n-2}(t)\, v_{n-2}'(t)] \geqq \int_t^\infty (s - t)\, q(s)\, y(g(s))\, ds.$$

Proceeding in this fashion we see that for $t > t_2$

$$p_{n-1}(t) \dots p_2(t)\, v_2(t) \leqq -\int_t^\infty \frac{(s - t)^{n-2}}{(n-2)!}\, q(s)\, y(g(s))\, ds,$$

$$p_1(t) \dots p_{n-1}(t)\, y'(t) \leqq -\int_t^\infty \frac{(s - t)^{n-2}}{(n-2)!}\, q(s)\, y(g(s))\, ds.$$

Integrating the above inequality from $t_3 > t_2$ to $t > t_3$, we obtain

$$p_1(t) \dots p_{n-1}(t)\, y(t) - p_1(t_3) \dots p_{n-1}(t_3)\, y(t_3) - \int_{t_3}^t [p_1(s) \dots p_{n-1}(s)]'\, y(s)\, ds \leqq$$
$$\leqq -\int_{t_3}^t \frac{(s - t_3)^{n-1}}{(n-1)!}\, q(s)\, y(g(s))\, ds - \int_t^\infty \frac{(t - t_3)^{n-1}}{(n-1)!}\, q(s)\, y(g(s))\, ds.$$

With regard to (5) we get

$$p_1(t) \dots p_{n-1}(t)\, y(t) \leqq p_1(t_3) \dots p_{n-1}(t_3)\, y(t_3) -$$
$$-\int_{t_3}^t \frac{(s - t_3)^{n-1}}{(n-1)!}\, q(s)\, y(g(s))\, ds.$$

For $t = t_3 + h_0$ we get

$$p_1(t_3 + h_0) \dots p_{n-1}(t_3 + h_0)\, y(t_3 + h_0) \leqq p_1(t_3) \dots p_{n-1}(t_3)\, y(t_3) - \\ - \int_{t_3}^{t_3+h_0} \frac{(s - t_3)^{n-1}}{(n-1)!} q(s)\, y(g(s))\, \mathrm{d}s.$$

In view of (7) we can choose t_3 so large that

$$p_1(t_3) \dots p_{n-1}(t_3) < \int_{t_3}^{t_3+h_0} \frac{(s - t_3)^{n-1}}{(n-1)!} q(s)\, \mathrm{d}s.$$

Then for $t \in [t_3, t_3 + h_0]$ we have

$$p_1(t_3 + h_0) \dots p_{n-1}(t_3 + h_0)\, y(t_3 + h_0) \leqq \\ \leqq y(t_3) \left[p_1(t_3) \dots p_{n-1}(t_3) - \int_{t_3}^{t_3+h_0} \frac{(s - t_3)^{n-1}}{(n-1)!} q(s)\, \mathrm{d}s \right],$$

which is the contradiction with $y(t) > 0$ for $t \geqq t_0$.

Corollary. *Assume that the conditions* (2)–(6) *are satisfied and, in addition,*

$$\limsup_{t \to \infty} \left[p_1(t) - \int_t^{t+h_0} q(s)\, \mathrm{d}s \right] < 0.$$

Then each solution of the differential equation with retarded argument

$$p_1(t)\, y'(t) + q(t)\, y(g(t)) = 0$$

is oscillatory.

Proof. This follows from Theorem 1 with $n = 1$ and the observation that each nonoscillatory solution of (10) is bounded.

Theorem 2. *Assume that the condition* (6) *is satisfied and, in addition,*

$$r \in C^1[[0, \infty), (0, \infty)] \qquad \text{is nonincreasing function,} \tag{11}$$

$$p_i(t) \leqq r(t), \qquad i = 1, \dots, n - 1, \tag{12}$$

$$\limsup_{t \to \infty} \left[(r(t))^{n-1} - \int_t^{t+h_0} \frac{(s - t)^{n-1}}{(n-1)!} q(s)\, \mathrm{d}s \right] < 0. \tag{13}$$

Then every bounded solution of (1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory bounded solution of (1). We shall assume that $y(t)$ is eventually positive. Now we are proceeding similarly as in the proof of Theorem 1 and we shall prove that if $k \geqq 2$ then $v_k v_k' \leqq 0$ on some interval $[t_2, \infty)$. If we suppose that $v_k v_k' > 0$ for some $k \geqq 2$, then v_k is either eventually positive and increasing or eventually negative and decreasing. In either case with regard to the inequalities

$$v_{k-1}(t) \geqq v_{k-1}(t_2) + \int_{t_2}^{t} \frac{v_k(s)}{r(s)}\, \mathrm{d}s, \qquad \text{for } v_k > 0,$$

$$v_{k-1}(t) \leqq v_{k-1}(t_2) + \int_{t_2}^{t} \frac{v_k(s)}{r(s)}\, ds, \qquad \text{for } v_k < 0,$$

and (11) we see that v_{k-1} is unbounded and has the same eventual sign as v_k. Repeating this procedure $k - 1$ times, we see that $y(t)$ is unbounded, which is a contradiction. Thus $v_k \leqq 0$ on $[t_2, \infty)$ if k is even and $v_k \geqq 0$ on $[t_2, \infty)$ if k is odd.

Integrating the last equation of (8) from $t > t_2$ to ∞ and with regard to (12), we get

$$(-1)^{n+1} r(t)\, v'_{n-1}(t) \geqq \int_t^{\infty} q(s)\, y(g(s))\, ds.$$

Integrating the last inequality from t to $u > t > t_2$ and then letting $u \to \infty$, we have

$$(-1)^{n+1} [-r^2(t)\, v'_{n-2}(t)] \geqq \int_t^{\infty} (s - t)\, q(s)\, y(g(s))\, ds.$$

If we proceed similarly as in the proof of Theorem 1, for $t \in [t_3, t_3 + h_0]$ we get

$$(r(t_3 + h_0))^{n-1}\, y(t_3 + h_0) \leqq y(t_3)\left[(r(t_3))^{n-1} - \int_{t_3}^{t_3+h_0} \frac{(s - t_3)^{n-1}}{(n-1)!}\, q(s)\, ds\right].$$

If we choose t_3 so large that

$$(r(t_3))^{n-1} < \int_{t_3}^{t_3+h_0} \frac{(s - t_3)^{n-1}}{(n-1)!}\, q(s)\, ds,$$

then we get a contradiction with $y(t) > 0$ for $t \geqq t_0$.

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Shrnutí

POZNÁMKA K OSCILATORICKÝM VLASTNOSTIAM OHRANIČENÝCH RIEŠENÍ DIFERENCIÁLNEJ ROVNICE VYŠŠIEHO RÁDU S ONESKORENÝM ARGUMENTOM

Rudolf Oláh

Článok obsahuje dve vety, ktoré dávajú postačujúce podmienky pre oscilatoričnosť ohraničených riešení diferenciálnej rovnice n-tého rádu

$$(p_{n-1} \ldots p_2(p_1 y')' \ldots)')' + (-1)^{n+1} q y(g) = 0.$$

Резюме

ЗАМЕТКА О СВОЙСТВАХ КОЛЕБЛЕМОСТИ ОГРАНИЧЕННЫХ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ—ВЫСШЕГО ПОРЯДКА С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

Рудолф Олах

Работа содержает две теоремы девающие достаточные условия колеблемости ограниченных решений дифференциального уравнения n-го порядка

$$(p_{n-1} \ldots p_2(p_1 y')' \ldots)')' + (-1)^{n+1} q y(g) = 0.$$