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A criterion for determining the $2^n d$ order linear differential equations possessing the central dispersion with the index $n$ equal to $t + \pi$

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In this paper we shall concern ourselves with a differential equation

\[ y'' = q(t)y, \quad q \in C^0_j, \quad j = (-\infty, \infty). \quad (q) \]

Throughout, the equation (q) will be understood to be oscillatory on both sides on \( j \) (which implies every nontrivial solution of (q) with infinitely many zeros on each of the intervals \( (-\infty, a) \) and \( (b, \infty), a \in j, b \in j \)).

We now recall some definitions and results adopted from the monograph [1] that will be of need below. Trivial solutions of (q) will be always from our considerations eliminated.

Let \( n \) be a positive integer, \( x \in j \) and \( y \) be a solution of (q) such that \( y(x) = 0 \). If \( \varphi_n(x) \) denotes the \( n \)-th zero of the \( y \) lying to the right of \( x \), then \( \varphi_n \) is called the 1st kind central dispersion (from now on only the central dispersion) with the index \( n \) of (q). Instead of \( \varphi_1 \) we write \( \varphi \) which is called the basic dispersion (of the 1st kind) of (q).

Let \( (u, v) \) be a basis of (q) and \( w \) its Wronskian \( (w = uv' - u'v) \). Then \( r(t) = \sqrt{u'^2(t) + v^2(t)}, t \in j \), is called the (first) amplitude of the basis \( (u, v) \) and every function \( \alpha, \alpha \in C^0_j \), satisfying the equation \( \tan \alpha(t) = -\frac{u(t)}{v(t)} \) wherever \( v(t) \neq 0 \) is called the (first) phase of the basis \( (u, v) \). Let us say that \( \alpha \) is a phase of (q) if there is a basis \( (u, v) \) of (q) possessing the function \( \alpha \) as a phase. If \( \alpha \) is a phase and \( r \) the amplitude of the basis \( (u, v) \) with the Wronskian equal to \( w \) then \( \alpha'(t) = \frac{w}{r^2(t)}, t \in j \).
Let \( \varphi \) be the basic dispersion and \( \alpha \) a phase of \((q)\). Then

(i) \( \alpha \in C^3, \alpha'(t) \neq 0 \) on \( J \),

(ii) \[ -\frac{1}{2} \frac{\varphi''(t)}{\alpha'(t)} + \frac{3}{4} \left( \frac{\varphi'(t)}{\alpha'(t)} \right)^2 = q(t), \quad t \in J, \]

(iii) \[ \varphi_n(t) = \varphi \circ \varphi \circ \ldots \circ \varphi(t), \varphi \in C^3, \varphi'(t) \neq 0 \text{ on } J, \]

(iv) \( \alpha \circ \varphi_n(t) = \alpha(t) + n\pi \text{ sgn } \alpha', \quad t \in J, \)

(v) \( \alpha_1 \) is a phase of \((q)\) if and only if there are the numbers \( a_{11}, a_{12}, a_{21}, a_{22} \), \( \det(a_{ik}) \neq 0 \) such that

\[ \tan \alpha_1(t) = \frac{a_{11} \tan \alpha(t) + a_{12}}{a_{21} \tan \alpha(t) + a_{22}} \]

for all \( t \) for which both sides of the last formula are meaningful,

(vi) If \( \alpha \) is a phase of the basis \((u, v)\) of \((q)\), \( w = uv' - u'v \) then \( r(t) := \sqrt{-\frac{w}{\alpha'(t)}}, \quad t \in J \), is a solution of the equation

\[ r'' = q(t) r + \frac{w^2}{r^3}. \]

**Theorem.** Let \( q \in C^1, q'(t) \neq 0 \). Let next \( \alpha \) be a phase and \( \varphi \) the basic dispersion of \((q)\). Then there exists a positive integer \( n \) such that \( \varphi_n(t) = t + \pi \) if and only if it holds:

\[ q(t + \pi) = q(t), \quad t \in J, \quad (1) \]

\[ \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \, ds = 0, \quad \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \sin^2 \alpha(s) \, ds = 0, \quad t \in J. \quad (2) \]

**Proof:** a) Let for a positive integer \( n \) \( \varphi_n(t) = t + \pi, \quad t \in J \). Let \( \alpha \) be a phase of the basis \((u, v)\) of \((q)\) whose Wronskian is equal to \( w \). Following (iv) we have

\[ \alpha(t + \pi) = \alpha(t) + n\pi \text{ sgn } \alpha', \quad t \in J, \quad (3) \]

and according to (vi)

\[ r'(t) = q(t) r(t) + \frac{w^2}{r^3(t)}, \quad t \in J, \quad (4) \]

for \( r(t) := \sqrt{-\frac{w}{\alpha'(t)}}, \quad t \in J \).

From the formula \( q(t) = -\frac{1}{2} \frac{\varphi''(t)}{\alpha'(t)} + \frac{3}{4} \left( \frac{\varphi'(t)}{\alpha'(t)} \right)^2 - \alpha'^2(t) \) and \( \alpha'(t + \pi) = \alpha'(t) \) that follows from (3), we get (1).
On multiplying out both sides of (4) by $2r'$ we get after an elementary modification the equality

$$(r^2(t))' = q(t)(r^2(t))' - \left(\frac{w^2}{r^2(t)}\right)'$$

and integrating this from $t$ to $t + \pi$ we have

$$r^2(t + \pi) - r^2(t) = \int_t^{t + \pi} q(s)(r^2(s))' ds - w^2\left(\frac{1}{r^2(t + \pi)} - \frac{1}{r^2(t)}\right), \quad t \in j.$$  

The functions $qr^2$, $r$, $r'$ are periodic with the period $\pi$ which follows from (1), (3) and from the definition of the function $r$. Next we have

$$\int_{-\infty}^{\infty} q(s) r^2(s) ds = 0, \quad t \in j.$$  

Let us note that $\alpha$ in (6) is an arbitrary phase of $(q)$. We will utilize this fact to the proof of (2). Let $x \neq 0$ and $\alpha_x \in C^0$ a function such that $\tan \alpha_x(t) = x \tan \alpha(t)$ for all $t$ for which $\tan \alpha(t)$ has been defined. Then $\alpha_x$ is a phase of $(q)$ as follows from (vi)

$$\begin{pmatrix} a_{11} = \frac{1}{a_{22}} = x, a_{12} = a_{21} = 0 \end{pmatrix}.$$  

Next we have

$$\alpha_x'(t) = \frac{x^2}{\cos^2 \alpha(t) + x^4 \sin^2 \alpha(t)} \alpha'(t), \quad t \in j, x \neq 0.$$  

Since

$$0 = \int_t^{t + \pi} \frac{q'(s)}{\alpha'(s)} ds = \frac{1}{x^2} \int_t^{t + \pi} \frac{q'(s)}{\alpha'(s)} \left(\cos^2 \alpha(s) + x^4 \sin^2 \alpha(s)\right) ds$$

for every $x \neq 0$ and hence also

$$\int_t^{t + \pi} \frac{q'(s)}{\alpha'(s)} \cos^2 \alpha(s) ds = -x^4 \int_t^{t + \pi} \frac{q'(s)}{\alpha'(s)} \sin^2 \alpha(s) ds,$$

it is necessarily

$$\int_t^{t + \pi} \frac{q'(s)}{\alpha'(s)} \sin^2 \alpha(s) ds = 0, \quad \int_t^{t + \pi} \frac{q'(s)}{\alpha'(s)} \cos^2 \alpha(s) ds = 0, \quad t \in j.$$  

By this we have proved statement of the Theorem in one direction.
b) Let the phase \( \alpha \) and \( q \) satisfy the assumptions (1), (2), \( q \in C^1 \) and \( q' \neq 0 \). The function \( q \) is periodic with the period \( \pi \) and therefore exists (uniquely) a phase \( \varepsilon \) of the equation
\[
y'' = -y : \alpha(t + \pi) = \varepsilon \circ \alpha(t) \quad ([2], \S 3.8).
\]
By the assumption \( q \in C^1 \), \( q' \neq 0 \) thus there exists an interval \( (\lambda, \mu) \), where \( q'(t) \neq 0 \). By differentiating (2)
\[
q'(t + \pi) = q'(t) \quad \frac{q'(t + \pi)}{\alpha'(t + \pi)} \sin^2 \alpha(t + \pi) = \frac{q'(t)}{\alpha'(t)} \sin^2 \alpha(t) \quad (t \in j)
\]
and making use of (1) we get \( \alpha'(t + \pi) = \alpha'(t) \) and \( \sin^2 \alpha(t + \pi) = \sin^2 \alpha(t) \) for \( t \in (\lambda, \mu) \). Therefore \( \alpha(t + \pi) = \alpha(t) + c, \) where \( c \neq 0 \) is a constant, \( \text{sgn} \ c = \text{sgn} \ \alpha' \) and from \( \sin^2(\alpha(t) + c) = \sin^2 \alpha(t) \) then follows \( c = n\pi \text{sgn} \ \alpha' \) (\( n \) is a positive integer). So, we have proved \( \alpha(t + \pi) = \alpha(t) + n\pi \text{sgn} \ \alpha' \), \( t \in (\lambda, \mu) \). From the last equality and from \( \alpha(t + \pi) = \varepsilon \circ \alpha(t) \) we get \( \varepsilon(t) = t + n\pi \text{sgn} \ \alpha' \) for \( t \) from the open interval with the end points \( \alpha(\lambda), \alpha(\mu) \). By the Theorem in [1] p. 209 there is the phase \( \varepsilon \) uniquely determined by the values of \( \varepsilon, \varepsilon', e'' \) at a point \( t_0(\in j) \). Therefore \( \varepsilon(t) = t + n\pi \text{sgn} \ \alpha' \) even for \( t \in j \) and from \( \alpha(t + \pi) = \alpha(t) + n\pi \text{sgn} \ \alpha' \) and (iv) we have \( \varphi_n(t) = t + \pi \). This completes the proof.

Remark 1. There is \( q' \neq 0 \) on \( j \) in the assumptions of the Theorem. If \( q \) is a constant \( (= k) \), then we can easily see \( \varphi_n(t) = t + \pi(t \in j) \) for a positive integer \( n \) if and only if \( k = -n^2 \).

Remark 2. The integral conditions (2) may be formulated in terms of \( q \). Then these are more complicated.

Remark 3. A general form of the carrier \( q \) of (q) having the basic dispersion equal to \( t + \pi \) has been found in [1] and [3].

References


\[
y'' = q(t)y.
\]
Tensor, N. S. Vol. 26 (1972), 121—128.

В работе исследуются линейные дифференциальные уравнения второго порядка вида (q): \( y'' = q(t)y \), \( q \in C(0, -\infty) \). Указаны необходимые и достаточные условия при выполнении которых центральная дисперсия с индексом \( n \) дифференциального уравнения (q) ровна \( t + \pi \). Эти условия представлены при помощи функции \( q \) и первой фазы дифференциального уравнения (q).